Nimrod

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Abstract

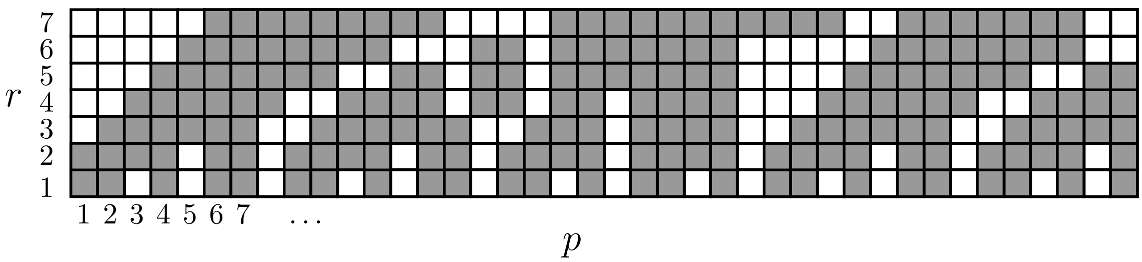
We present a new Nim-type game. The description is simple but the strategy is complex. There is a well-defined core outside of which the game is neatly structured. Inside the core the game appears chaotic. We have partial results for the core, a strategy outside the core, and an overall strategy that might be described as "promising."

Nimrod[1] is inspired by the partly mythic teenage duel in which participants race at each other in cars (hotrods). The loser is the first to swerve away (both lose if neither swerves). Imagine two players facing each other in cars with velocities in the set of natural numbers. By turns, each player elects either to maintain speed, to accelerate 1 or to decelerate 1. After the choice is made, one unit of time passes and the cars move toward each other according to their velocities. It then becomes the other player's turn. The player during whose move the cars crash is the loser.

In this paper, we recast Nimrod in the equivalent but less confrontational context of a pile of sticks. A game position is an ordered pair, (p, r), where p is the size of the pile and r is the rate at which sticks are being removed. Players may increase r by 1, decrease r by 1 or leave it alone, then they must remove r sticks from the pile. If this is not possible the player loses.

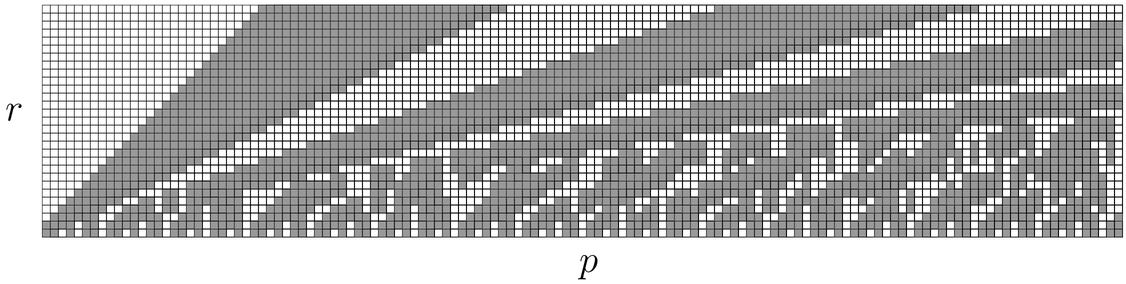
Nimrod is complex

For each (p, r) either the player to move next has a winning strategy or else the other player does. In the chart below the square at (p, r) is shaded gray if the player to move has the winning strategy.



[1]The authors thank Fred Henle for suggesting this name.

The white triangle at the left consists of the pairs (p, r) where $p < r - 1$ and the player to move next has no legal move. That triangular sector gives rise to the alternating gray and white sectors to the right.



The region of the sectors can be analyzed and we will do this shortly, but more interesting is the part below the sectors which seems frankly chaotic. The pattern of white squares on the row $r = 1$,

translated into natural numbers,

$$3, 5, 8, 11, 13, 17, 20, 22, 25, 27, 30, 32, 35, 38, 40, 43, \ldots$$

is a sequence previously unknown to the On-line Encyclopedia of Integer Sequences [1].

Very partial results

We have a host of partial results. We include a sample in this section.

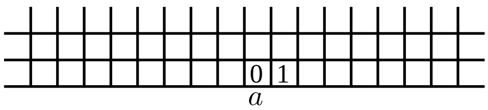
Definition 1 *For $p, r > 0$, let $w(p, r) = 1$ if the next player to move at game position (p, r) has a winning strategy. Let $w(p, r) = 0$ if the second player to move has a winning strategy.*

Proposition 1 *For $p > 0$,*

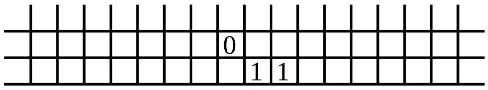
1. *If $w(a, 1) = 0$ then $w(a + 1, 1) = 1$.*
2. *If $w(a, 1) = 1$ and $w(a + 1, 1) = 1$ then $w(a + 2, 1) = 0$.*

Proof:

1. If $w(a, 1) = 0$ then the player to move at $(a + 1, 1)$ can take 1 stick, leaving the opposing player with $(a, 1)$, a losing position.



2. If $w(a,1) = 1$ and $w(a+1,1) = 1$ then a winning move for a player at $(a+1,1)$ can't be to take a single stick, since $(a,1)$ is a winning position. The only other move is to take two sticks, hence it must be that $w(a-1,2) = 0$.



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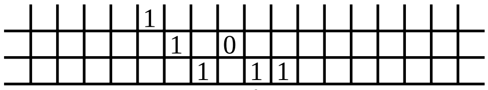
This in turn tells us that all positions a player can move to from $(a-1,2)$ are winning positions, that is, $w(a-2,1) = w(a-3,2) = w(a-4,3) = 1$.



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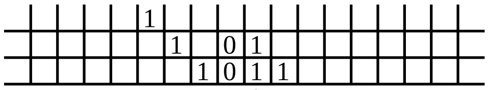
From this we learn that $w(a-1,1) = 0$ since only winning positions are accessible. That gives us that $w(a,2) = 1$, since a player at $(a,2)$ can move to $(a-1,1)$, a losing position.



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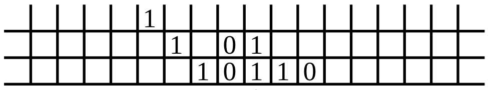
Finally we see that $(a+2,1)$ is a losing position since a player there can only move to $(a+1,1)$ and $(a,2)$, winning positions.



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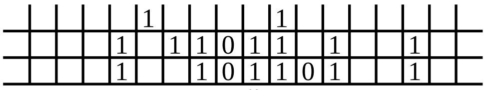
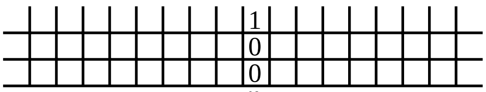
This is just a sample. From hypothesis 2 we can further deduce:



p

Something as simple as this configuration,



p

determines the value of w at at least 30 positions:

		1									1											
			1				0				1											
				1				1			1		1			1						
				1			1		1	1	0	1	1	0	1	1		1			1	
							1			1	0	1	1	0	1	1	0	1			1	
											p											

Definition 2 *Let $\{n_k\}$ enumerate those numbers i such that $w(i, 1) = 0$. Let $f(k)$ the number of 0's at the bottom of the n_kth column, that is, $w(n_k, j) = 0$ for all $j \leq f(k)$ and $w(n_k, f(k) + 1) = 1$.*

We state the following without proof.

Proposition 2 *For all $k > 0$,*

1. *If $f(k) = 1$ or 2 then $f(k + 1) \geq 2$.*
2. *If $f(k) = 3$ then $f(k + 1) = 1$.*
3. *If $f(k) = 4$ then $f(k + 2) \neq 1$ or 3.*
4. *If $f(k) \geq 5$ then $f(k + 2) = 2$ or 3.*

Nimrod can be (partly) explained

Outside of the region at the bottom, Nimrod is reasonably well-behaved.

Proposition 3 *For all $k \geq 0$, $p, r > 0$,*

- *A_k: If $(2k - 1)r - k - 1 < p < 2kr - 3k$ then (p, r) is a winning position and taking $r - 1$ sticks from the pile is a winning move.*
- *B_k: If $(2k - 1)r + k - 2 < p < 2kr - k$ then (p, r) is a winning position and taking r sticks from the pile is a winning move.*
- *C_k: If $(2k - 1)r + 3k - 3 < p < 2kr + k$ then (p, r) is a winning position and taking $r + 1$ sticks from the pile is a winning move.*
- *D_k: If $2kr + k - 1 < p < (2k + 1)r - k - 1$ then (p, r) is a losing position.*

Proof: We prove these simultaneously by induction on k. Note first that A_0, B_0, and C_0 are vacuously true and that D_0 states what we observed earlier, that if $p < r - 1$, there is no legal move.

Now suppose that proposition holds for a given $k - 1$.

- A_k: Suppose that
$$(2k-1)r - k - 1 < p < 2kr - 3k. \tag{1}$$
Since $k \geq 1$, $p > r-2$, taking $r' = r-1$ sticks from the pile is a legal move leaving $p' = p - r'$ sticks in the pile. If we substitute $r = r'+1$, $p = p'+r'$, and $k = k'+1$ in (1), we have
$$(2(k'+1)-1)(r'+1) - (k'+1) - 1 < p' + r' < 2(k'+1)(r'+1) - 3(k'+1)$$
which yields
$$2k'r' + k' - 1 < p' < (2k'+1)r' - k' - 1.$$
Then by D_{k-1}, (p', r') is a losing position.

- B_k: If
$$(2k-1)r + k - 2 < p < 2kr - k, \tag{2}$$
$k \geq 1$, then $p > r-1$ so taking $r' = r$ sticks from the pile is a legal move. If we substitute $r = r'$, $p = p'+r'$, and $k = k'+1$ in (2), we have
$$(2(k'+1)-1)r' + (k'+1) - 2 < p' + r' < 2(k'+1)r' - (k'+1)$$
which again yields
$$2k'r' + k' - 1 < p' < (2k'+1)r' - k' - 1,$$
so (p', r') is a losing position by D_{k-1}.

- C_k: If
$$(2k-1)r + 3k - 3 < p < 2kr + k, \tag{3}$$
$k \geq 1$, $p > r$ so taking $r' = r+1$ sticks from the pile is a legal move. Substituting $r = r'-1$, $p = p'+r'$, and $k = k'+1$ in (3), we have
$$(2(k'+1)-1)(r'-1) + 3(k'+1) - 3 < p' + r' < 2(k'+1)(r'-1) + (k'+1)$$
again yielding
$$2k'r' + k' - 1 < p' < (2k'+1)r' - k' - 1.$$
As before, (p', r') is a losing position by D_{k-1}.

- D_k: Suppose that
$$2kr + k - 1 < p < (2k+1)r - k - 1 \tag{4}$$
and consider the possible moves of a player in position (p, r). Taking $r-1$ sticks from the pile gives us $r = r'+1$ and $p = p'+r'$. Substituting these values in (4), we have
$$2k(r'+1) + k - 1 < p' + r' < (2k+1)(r'+1) - k - 1$$

which yields

$$(2k-1)r' + 3k - 1 < p' < 2kr' + k.$$

This implies $(2k-1)r' + 3k - 3 < p' < 2kr' + k$ so by C_k (established above), (p', r') is a winning position.

Suppose instead that r sticks are taken from the pile. Substituting $r = r'$ and $p = p' + r'$ into (4), we have

$$2kr' + k - 1 < p' + r' < (2k+1)r' + 1 - k - 1$$

giving us

$$(2k-1)r' + k - 1 < p' < 2kr' - k.$$

This in turn gives us $(2k-1)r' + k - 2 < p' < 2kr' - k$ so by B_k, (p', r') is a winning position.

Finally, if $r+1$ sticks are taken from the pile, we substitute $r = r' - 1$ and $p = p' + r'$ into (4) to get

$$2k(r'-1) + k - 1 < p' + r' < (2k+1)(r'-1) - k - 1$$

giving us

$$(2k-1)r' - k - 1 < p' < 2kr' - 3k - 2.$$

This implies that $(2k-1)r - k - 1 < p < 2kr - 3k$ so by A_k, (p', r') is a winning position.

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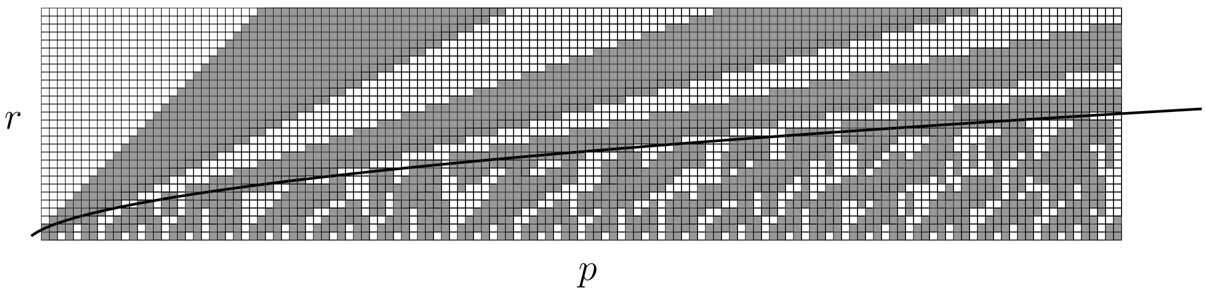
Coverage by the conditions A_k, B_k, C_k, D_k breaks down when the intervals drift apart. This can occur, for example, when the right endpoint for A_k matches the left endpoint for B_k. If

$$p = 2kr - 3k = (2k-1)r + k - 2 \tag{5}$$

then p is not in any of the intervals. Solving (5) for p and r gives us $p = \frac{1}{2}r^2 + \frac{1}{4}r - \frac{3}{2}$.

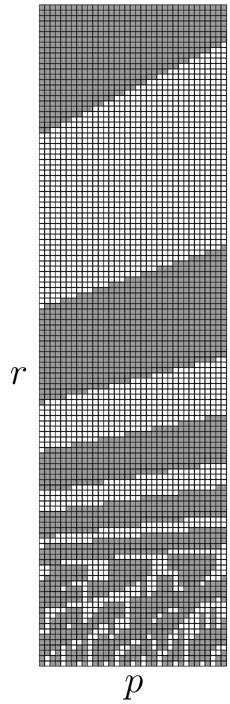


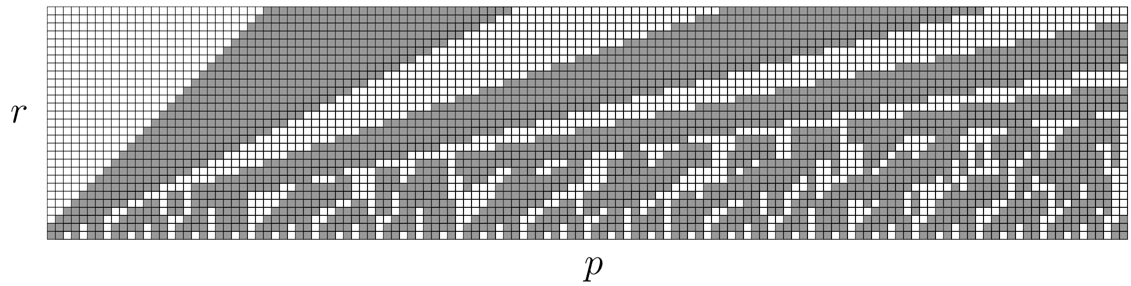
A faulty but promising strategy

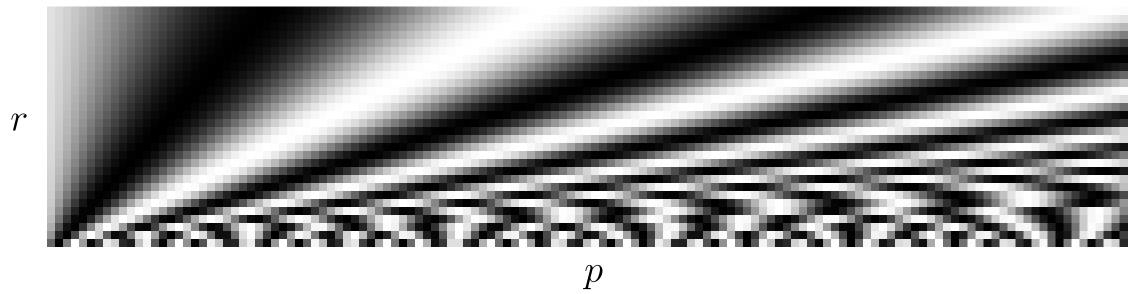
Above the chaotic region of the game plot, for a fixed p, the picture looks a bit like a periodic function of r with a period that diminishes as r gets smaller. This suggests that we might mimic the plot using a trigonometric function. We tried various forms, eventually settling on

$$G(r,p) = \cos\left(\frac{ar+bp}{cr+d}\right).$$

Using this function to assign shades to squares on the quarter plane gives us a plot resembling the game plot.







Let "Model" stand for the strategy of taking $r-1$, r, or $r+1$ sticks from the pile depending on which of $G(r-1, p-(r-1))$, $G(r, p-r)$, $G(r+1, p-(r+1))$ is largest, if $r > 1$, and similarly, r, or $r+1$ depending on which of $G(r, p-r)$ and $G(r+1, p-(r+1))$ is largest. Model is smart enough to know and use Proposition 3. Let "Random" be the strategy that moves at random. Starting inside the chaotic region, with $r = 1$, Model defeats Random rather decisively, over 80% of the time. This is true for p as high as a million sticks.

For better competition, we define "Proposition" to be the strategy that plays Random unless $p > \frac{1}{2}r^2$ in which case it plays Model. A game between Model and Proposition tests how well Model deals with the chaotic region. These games are more evenly matched but Model still dominates. We ran 2000 games each of pile sizes 1000, 10000, 100000, and a million sticks. Model beat Proposition over 60% of the time at each level.

References

[1] Sloane, Neil and AT&T, *The On-line Encyclopedia of Integer Sequences*, http://www.research.att.com/ njas/sequences/