From Chapter 5:

5.1 Sequences

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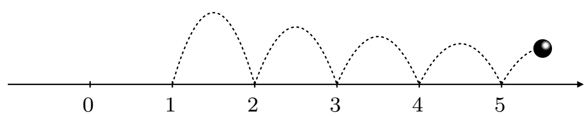
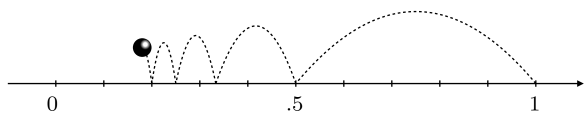
Think of a sequence as the footprints of a bouncing ball.

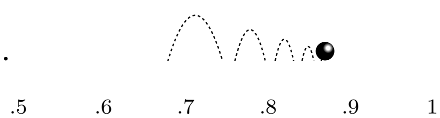
$1,\ 2,\ 3,\ 4,\ 5,\ \ldots$



$\frac{1}{1},\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \ldots$



$\frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6},\ \frac{6}{7},\ \ldots$

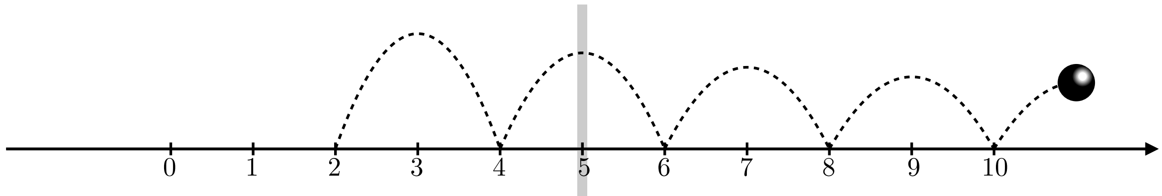


We can talk about sequences almost the way we talk about numbers. We can write equations and inequalities with them and these have a special meaning. For example, if we say

$$\boldsymbol{c} > 2,$$

we mean that the terms of $\boldsymbol{c}$ are greater than 2 *from some point on.*

Example: For the sequence $\boldsymbol{a} = 2,\ 4,\ 6,\ 8,\ \ldots,$ we can write $\boldsymbol{a} > 5$.



The inequality is true because from the third term of the sequence on every term is greater than 5. We don't care that the first two terms are less than 5. All we care about is what happens from some point on. It is unusual in mathematics to say that we can ignore some terms. Usually everything matters. But in sequences, as in life, a few indiscretions early on can be overlooked when dealing with the sequence as a whole.

Notice that $\boldsymbol{a} > 13$ is also true since the sequence continues: $\ldots\ 10,\ 12,\ 14,\ 16,\ \ldots.$ For the same reason, $\boldsymbol{a} > 38{,}194{,}545{,}802$. Actually, $\boldsymbol{a}$ is ultimately greater than *any* real number.

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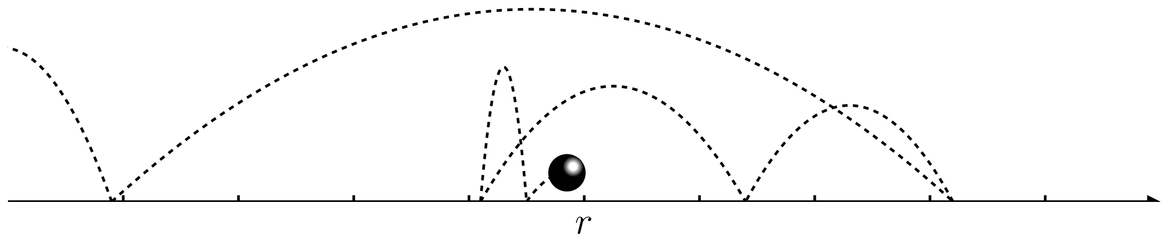
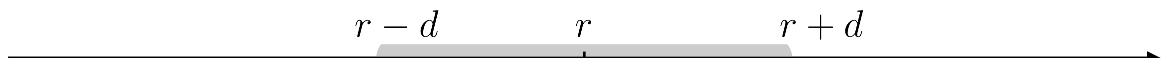
5.2 Limits

We have taken up the study of sequences in order to define "limit." We'll make that definition later in this section. We build up to it by defining when a sequence "approaches" a number. The picture of the bouncing ball is useful in defining "approach."

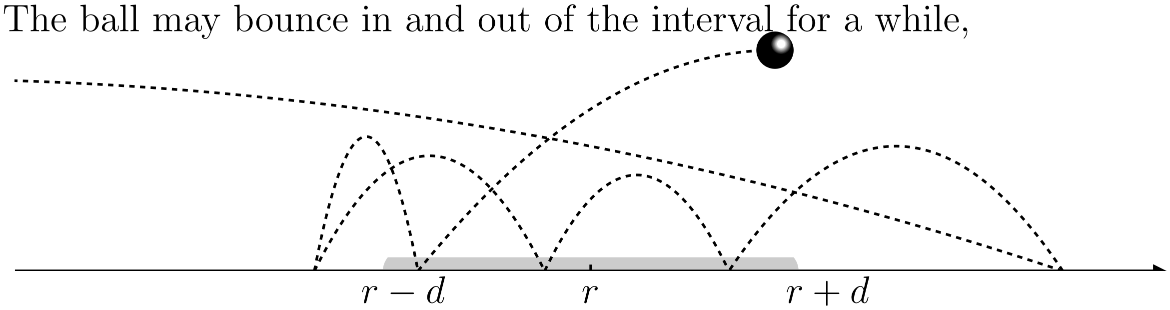


When should we say that the bounces "approach" a number r?

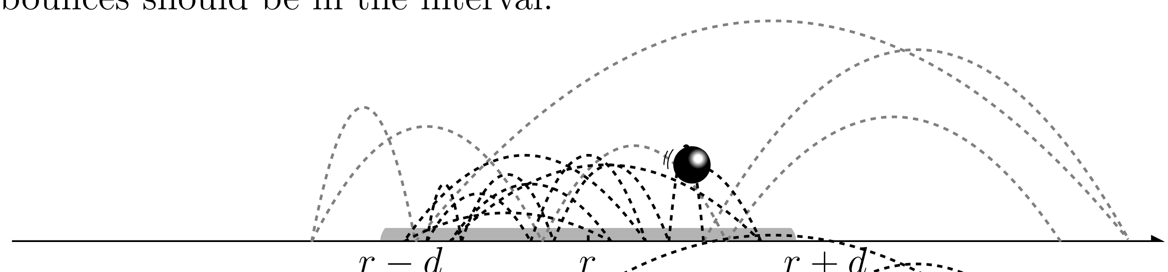
Suppose we observe the bounces in an interval $(r-d, r+d)$ around r.



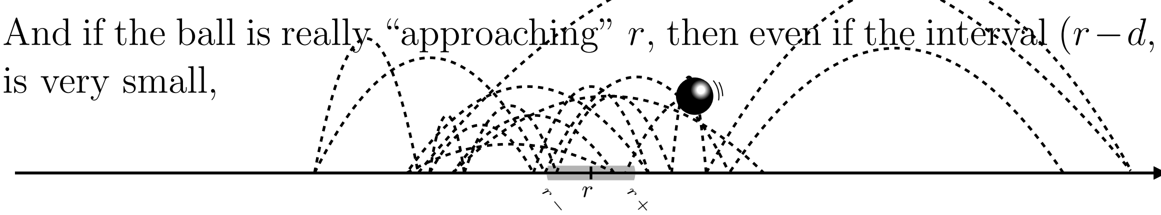
The ball may bounce in and out of the interval for a while,



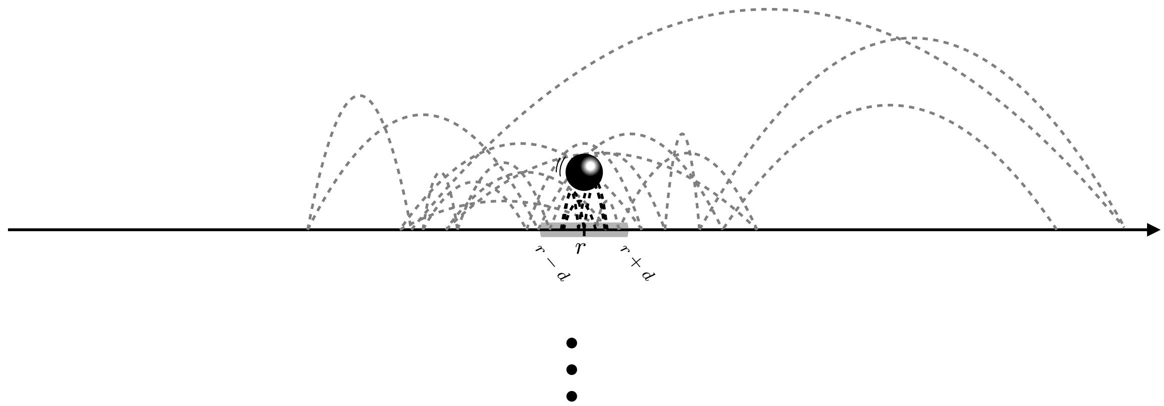
but if the ball is really approaching r, then ultimately it should bounce into the interval and never bounce out again. That is, from some point on, all the bounces should be in the interval.



And if the ball is really "approaching" r, then even if the interval $(r-d, r+d)$ is very small,



the ball should be in the interval (ultimately).


$r-d$
r
$r+d$