Excerpts from Chapter 3:

3.1 The Difference Quotient

The idea behind estimating the value of a derivative is the old formula for movement at a constant speed:

$$\text{rate} = \frac{\text{distance}}{\text{time}}.$$

To be very precise, in this formula for rate, "distance" really means "change in distance" (how far did you go?) and "time" really means "change in time" (how long did it take?). Mathematicians denote change in a variable by putting the Greek letter Δ (capital delta) in front of the variable. So we can rewrite the rate formula as the ratio of a *change* in distance corresponding to a *change* in time:

$$\text{rate} \quad = \frac{\Delta\text{distance}}{\Delta\text{time}}.$$

We also know that in the language of calculus "rate" means derivative, which we can write as distance$'$. So we can write our rate equation as:

$$\text{distance}' = \frac{\Delta\text{distance}}{\Delta\text{time}}.$$

We emphasize that this rate equation is valid only for *constant* rates. Most things—falling books, rising populations, cooling bodies—do not change at constant rates. For that reason for most cases this rate equation can't be used over large time intervals to calculate derivatives. To be precise, it can't be used over *any* time interval to get an exact derivative. What we get when we compute $\frac{\Delta\text{distance}}{\Delta\text{time}}$ is the *average* rate over the time interval Δt. For example, if a 100-mile trip takes 2 hours, then the average rate over the 2-hour period is $\frac{100}{2} = 50$ miles per hour. At any instant t during the two-hour interval, the exact derivative distance$'(t)$, may or may not be 50 miles per hour. It might be much faster (you might be passing a line of trucks) or much slower (you might be stopped at a toll booth). Over a very short time interval, however, at every instant the instantaneous rate (the derivative) is generally quite close to the average rate over the interval.

What we'll do in this chapter is *approximate* derivatives by calculating average rates over time intervals of tiny size. In mathematics the symbol $\approx$ stands for "is approximately equal to", so we'll write

$$\text{distance}' \approx \frac{\Delta\text{distance}}{\Delta\text{time}}.$$

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Exercises: Estimating Derivatives at Specific Points

In the following exercises you'll be finding approximate values of derivatives at certain points with difference quotients. In each of the following we give you a function f, a specific point for t, and a value for Δt. We ask you to approximate $f'(t)$ by computing the difference quotient $\frac{f(t+\Delta t)-f(t)}{\Delta t}$.

1. $f(t) = 2t + 1$, $t = 3$, $\Delta t = 1$
2. $f(t) = 2t + 1$, $t = 3$, $\Delta t = -1$
3. $f(t) = 2t + 1$, $t = 3$, $\Delta t = .1$
4. $f(t) = 2t + 1$, $t = 3$, $\Delta t = .01$

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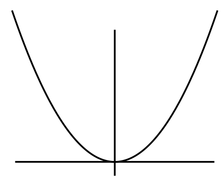
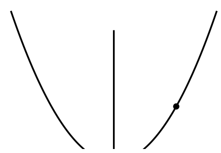
	algebraic view	geometric view
Given a function	$f(t) = t^2$	
and a point	$t = 2$	
we examine the difference quotient	$\frac{(t+\Delta t)^2 - t^2}{\Delta t}$	 slope of chord is $\frac{(t+\Delta t)^2 - t^2}{\Delta t}$
and let Δt go to zero to get the derivative	$f'(2) = 4$	slope of tangent line is 4

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3.5 Euler's Method

When we computed derivatives, we were given information about a function information about the *rate of change*, f', can we find the *change* in f itself? That is, if we know how fast someting is changing, can we compute how much it changes over some time interval? In this section, we'll come close. Given a description of the rate of change, f', we'll approximate the change in f. That is, we'll approximate the integral of f'.

As with the derivative, we start our discussion based on the formula for when rate is constant

$$\text{rate} = \frac{\text{distance}}{\text{time}},$$

or in the language of calculus,

$$\text{distance}' = \frac{\Delta\text{distance}}{\Delta\text{time}}.$$

As with the derivative, we account for the cases where rate isn't constant by writing

$$\text{distance}' \approx \frac{\Delta\text{distance}}{\Delta\text{time}},$$

which we write for functions f in general as

$$f' \approx \frac{\Delta f}{\Delta t}.$$

Finally, we rewrite this last expression to emphasize that instead of computing f', given Δf, we wish to compute Δf, given f':

$$\boxed{\Delta f \approx f'\Delta t.}$$

That's the expression we'll build upon to estimate integrals.

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Exercises: Integrating

To **integrate** means to find the integral. Solving an initial value problem is the same as integrating. (In the example above, our initial value problem is to find $w(50) = w(0) + \int_0^{50} w'$. Since we're given $w(0)$, the problem is to find the integral $\int_0^{50} w'$, the total change in w as t goes from 0 to 50.)

1. Consider the following initial value problem: Find $y(2)$, given

$$\begin{aligned} y' &= 2t - y \\ y(0) &= 1. \end{aligned}$$

We're going to ask you to find increasingly accurate estimates for $y(2)$.

a. Estimate $y(2)$ using a single step (time interval $\Delta t = 2$). Do this by filling in the table.

t	y	y' $(= 2t - y)$	Δy $(\approx y' \cdot \Delta t)$
0	1		
2			

b. Then estimate $y(2)$ again, this time using two steps (each step size is $\Delta t = 1$). Do this by filling in the table.

t	y	y' $(= 2t - y)$	Δy $(\approx y' \cdot \Delta t)$
0	1		
1			
2			

c. Now estimate $y(2)$ using 10 steps (each step size is $\Delta t = .2$). Do this by filling in the table.

t	y	y' $(= 2t - y)$	Δy $(\approx y' \cdot \Delta t)$
0	1		
.2			
.4			
.6			
.8			
1			
1.2			
1.4			
1.6			
1.8			
2			

d. Now compare your work with a computer. Find the value of $y(2)$ using a computer integrator, as you did in Chapter 2.

Solve the following initial value problems:

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2. $\begin{bmatrix} z' & = & 2 \\ z(0) & = & 1 \\ z(2) & \approx & ? \end{bmatrix}$ $\Delta t = .5$

3. $\begin{bmatrix} z' & = & 2 \\ z(0) & = & 1 \\ z(2) & \approx & ? \end{bmatrix}$ $\Delta t = .25$

4. $\begin{bmatrix} f' & = & \frac{t}{5} \\ f(-4) & = & 10 \\ f(6) & \approx & ? \end{bmatrix}$ $\Delta t = 2$