Application: **The End of the Universe.**

This article is the last of four on cosmology, the study of the universe. It's the most sophisticated of the four articles and makes significant use of the language and techniques of the calculus.

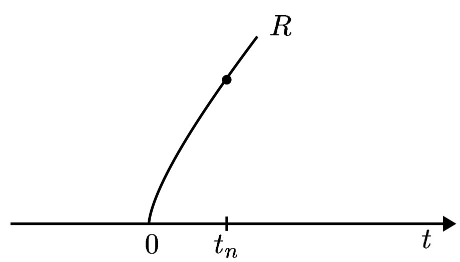
In *The Size of the Universe* we introduced Hubble's discovery of an expanding universe. In *The Age of the Universe* we derived the equation:

$$\frac{1}{2}R'^2 = \frac{GM}{R} + C,$$

where R is the radius of the universe as a function of time, G is the universal gravitational constant, and M is the mass of the universe. With the assumption that C is zero, you used this (if you completed the lab) to calculate the age of the universe at

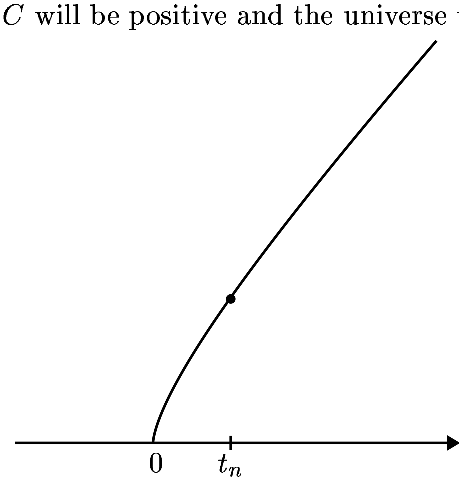
$$t_n = \frac{2}{3H_0},$$

where H_0 is the current value of Hubble's constant. This is what the solution to $\frac{1}{2}R'^2 = \frac{GM}{R} + C$ looks like with $C = 0$.

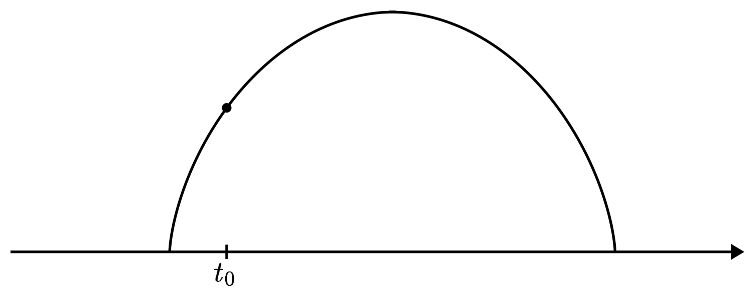


In this section, we look to the future. What is going to happen to the universe, will it expand forever? Will it contract?

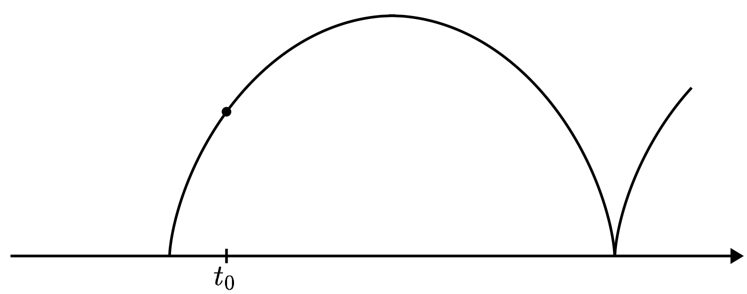
The answer depends on the value of C which in turn depends on M, the mass of the universe. If there isn't too much mass, C will be positive and the universe will continue to expand.



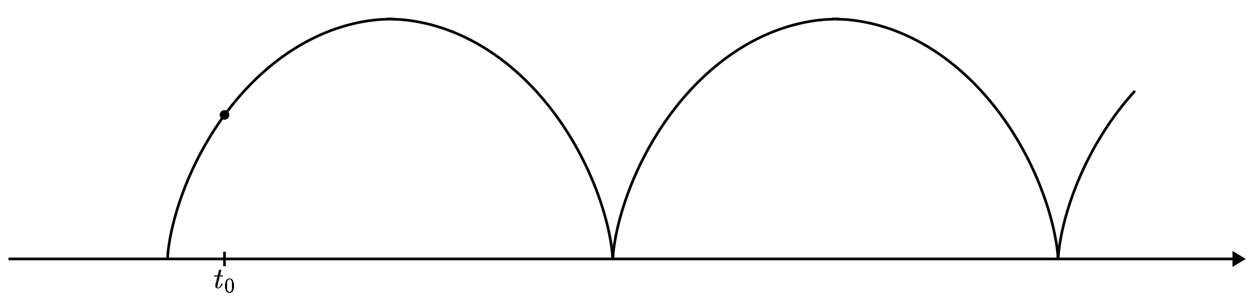
If there's a lot of mass, though, C will be negative and the universe will collapse to nothing.



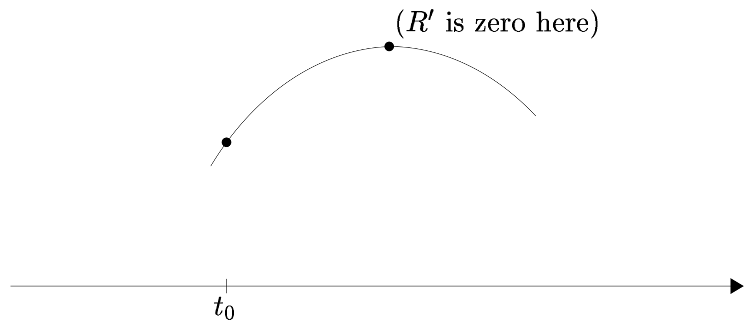
According to the equations, the world will then expand again



and again and again.



It's not hard to see from $\frac{1}{2}R'^2 = \frac{GM}{R} + C$ that if C is positive then R' is never zero. If R' is never zero, then R can't contract.



If C is negative, it's a different story. The initial value problem

$$\begin{aligned}\tfrac{1}{2}R'^2 &= \tfrac{GM}{R} + C \\ R(0) &= 0\end{aligned}$$

can be solved (using separation of variables, integration by substitution, and the integral tables) to get:

$$\sqrt{GM+CR}\sqrt{CR} - GM\ln|\sqrt{GM+CR}+\sqrt{CR}| + GM\ln|GM| = \sqrt{2}c^{3/2}t,$$

which leads to

$$\left(\frac{GM}{\sqrt{GM+CR}+\sqrt{CR}}\right)^{GM} e^{\sqrt{GM+CR}\sqrt{CR}} = e^{\sqrt{2}C^{3/2}t}.$$

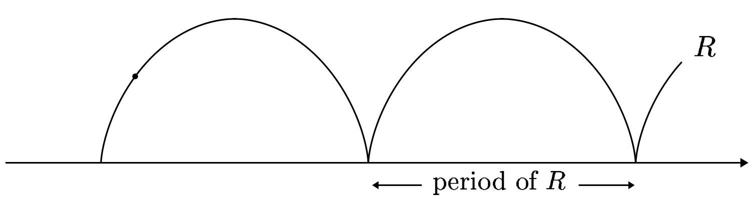
At this point, we can see the effect of a negative C. If $C = -B$, $B > 0$, then

$$\sqrt{2}C^{3/2}t = \sqrt{2}B^{3/2}ti,$$

an imaginary number. From our work in Chapter 11,

$$e^{\sqrt{2}B^{3/2}ti} = \cos(\sqrt{2}B^{3/2}t) + i\sin(\sqrt{2}B^{3/2}t).$$

Sine and cosine are periodic functions; this explains why the universe expands "again and again and again." It's because R is periodic.



We can actually compute the period of R. It's the value of Δt such that

$$e^{\sqrt{2}B^{3/2}i(t+\Delta t)} = e^{\sqrt{2}B^{3/2}it},$$

and that's when

$$\sqrt{2}B^{3/2}\Delta t = 2\pi,$$

the period of sine and cosine. This gives us a period of

$$\Delta t = \sqrt{2}\,\pi B^{-3/2}.$$

How big is this? This is really difficult to say. We can get some idea, though from

$$\frac{1}{2}R'^2 = \frac{GM}{R} + C.$$

We can assume that R_0, the radius of the universe is actually the radius of the observable universe, which we (you, really) calculated in *The Size of the Universe* as

$$R_0 = \frac{c}{H_0},$$

where c is the speed of light. Then substituting this for R in $\frac{1}{2}R'^2 = \frac{GM}{R} + C$ together with c for R', we have:

$$\begin{aligned} \tfrac{1}{2}c^2 &= \tfrac{GMH_0}{c} + C \\ \tfrac{1}{2}c^2 - \tfrac{GMH_0}{c} &= C. \end{aligned}$$

But then, there is some dispute over the value of H_0, and much dispute over the value of M!