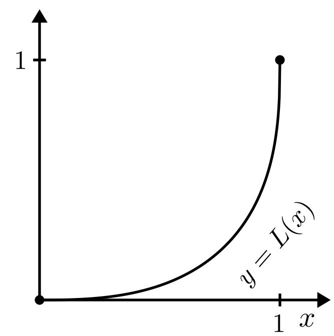
Application: The Wealth of Families

In *Rich and Poor*, we looked at the distribution of wealth as described by the Lorenz curve. In this article, we use the Lorenz curve to determine the wealth of a single family.

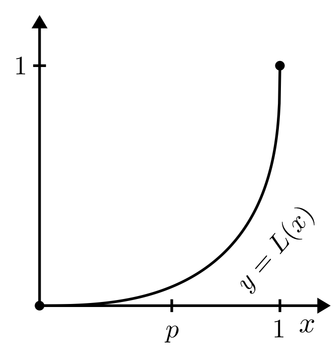
Recall that the Lorenz function for wealth in the United States L is defined as the fraction of the nation's wealth owned by the poorest x of all families. A typical Lorenz curve looks like this:



We're going to use the Lorenz function to answer the following question:

> *Suppose a certain family's rank is p in family wealth. How much does that family own?*

We'll say that such a family is p "from the bottom." A family that is, for example, .43 "from the bottom" is one which is richer than 43% of all families and poorer than 57% of all families.



We start by considering $L(p)$. This is the fraction of the nation's wealth held by the bottom p. Let N be the total wealth of all families. Then $L(p)$ is the fraction of N owned by the poorest p

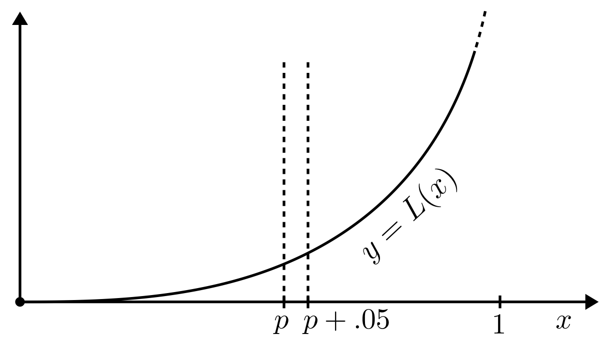
families. So, in total, those families own

$$L(p) \cdot N.$$

Then we can also say that

$$(L(p+.05) - L(p)) \cdot N$$

is the total owned by all the families in the p to $p + 5\%$ bracket.



That's 5% of all families, so if F is the total number of families in the United States, then there are $.05F$ families in this bracket, so the average owned by a family in the bracket is

$$\frac{(L(p+.05) - L(p)) \cdot N}{.05F}.$$

We're going to rewrite this as:

$$\frac{N}{F} \cdot \frac{L(p+.05) - L(p)}{.05}.$$

But this is not the answer to the question. This isn't the wealth of a family p from the bottom. It's the average wealth of a family in the p to $p + 5\%$ bracket. We'd get a better answer if we narrowed the bracket. If we look at the p to $p + 1\%$ bracket, we get for the average wealth:

$$\frac{N}{F} \cdot \frac{L(p+.01) - L(p)}{.01}.$$

This is better, but why stop here? We could, in theory, look at the p to $p + .01\%$ bracket. Then the average wealth of a family in the bracket is

$$\frac{N}{F} \cdot \frac{L(p+.0001) - L(p)}{.0001}.$$

You may see what's happening now. You may recognize in the expression above the difference quotient for the function L. As we take smaller and smaller brackets, we get closer and closer to the exact wealth of a family p from the bottom. The exact answer, then, is the limit of the expression:

$$\lim_{\Delta p \to 0} \frac{N}{F} \cdot \frac{L(p+\Delta p) - L(p)}{\Delta p} = \frac{N}{F} \cdot L'(p).$$

This is the answer to our question. This is the exact wealth of a family p from the bottom. In general, for any x, the wealth of a family x from the bottom is $\frac{N}{F} \cdot L'(x)$. We'll call this value $W(x)$. Note that since

$$W(x) = \frac{N}{F} \cdot L'(x),$$

we have

$$L'(x) = \frac{F}{N} W(x),$$

and so

$$L = \int \frac{F}{N} W \, dx.$$

Thus, for any value p,

$$L(p) = \int_0^p \frac{F}{N} W(x) \, dx.$$

Odd-numbered solutions begin on page ??

Exercises: Pennies From Heaven

1. Prove what we stated in *Rich and Poor*, that if everyone's wealth suddenly doubled, then the Lorenz curve wouldn't change. [*Hint:* Let L be the Lorenz function before doubling and let L_{new} be the Lorenz function after doubling. Show that $L_{\text{new}} = L$.]