Inequality is the cause of all local movements.

Leonardo da Vinci

Essay: Rich and Poor

We're often told that the gap between the rich and the poor is widening (or less often, narrowing). As citizens, we can have a range of reactions to this, depending on our political beliefs. As students of mathematics, however, we're instantly curious about how such a conclusion is reached.

The 'gap' is sort of an elusive concept. Imagine, for a moment, that suddenly each person's fortune was doubled. Did the gap increase? The relative wealth of the rich and the poor doesn't really change. Person X with $500,000 before the doubling would have $1,000,000 afterwards. Person Y with $50,000 before would have $100,000 after. X had ten times as much as Y had before doubling and the ratio is the same after doubling. But by another measure, the gap did widen. The personal gap between X and Y went from $450,000 to $900,000.

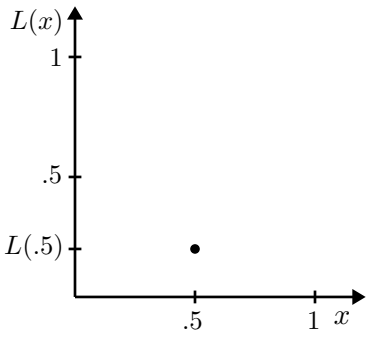
How can we assess the fairness of a society? In particular, is there a measure which doesn't change if everyone's wealth doubles?

There *is* such a measure, and it's neatly connected to the calculus. It's based on the Lorenz curve, invented by statistician Conrad Lorenz. There are Lorenz curves for all sorts of quantities: income, debt, infant mortality, and so on. Each represents the distribution of a quantity over a population. We'll be concerned with the Lorenz curve for the distribution of wealth among U.S. families.

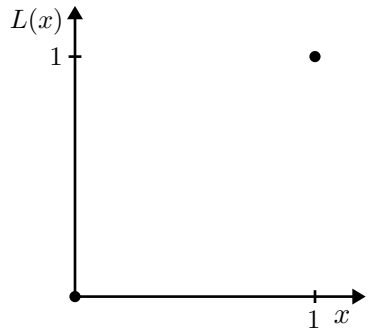
Mathematically, a Lorenz curve is the graph of a function L from $[0, 1]$ to $[0, 1]$. The Lorenz curve for wealth is defined by:

$L(x)$ is the fraction of the nation's wealth owned by the poorest x of all families.

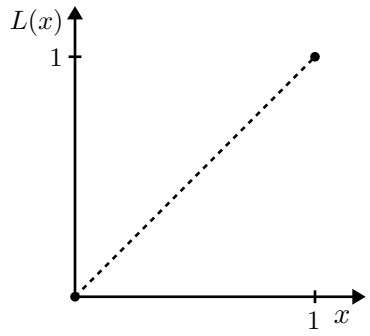
For example, $L(\frac{1}{3})$ is the fraction owned by the poorest third of all families. $L(.5)$ is the fraction owned by the bottom 50% of all families. Note that since the top 50% inevitably own more than the bottom 50%, $L(.5)$ will be less than .5.



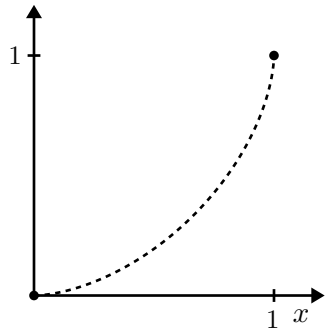
For any Lorenz function, $L(0) = 0$ (the wealth of nobody is nothing). Similarly, $L(1) = 1$ (altogether, everyone owns everything).



In a perfectly egalitarian society, where everyone has just as much as everyone else, $L(x) = x$ for all x— the bottom third owns one third of the wealth, and so on.

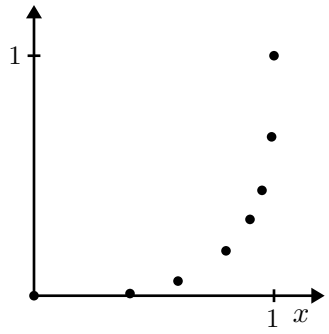


Of course, this is never the case, and the typical Lorenz curve $L(x)$ is below x.



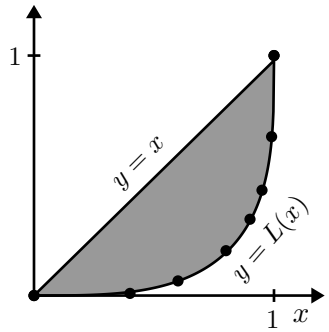
Notice that the Lorenz curve doesn't change if everyone's net assets double. The fraction of wealth held by a given fraction of U.S. families doesn't change. This should make sense to you, but we'll prove it later (see *The Wealth of Families*).

Here are points on the Lorenz curve for the wealth of U.S. families in 1983:



Source: "Recent Trends in Wealth Ownership, 1983-1998" by Edward N. Wolff, Jerome Levy Economics Institute.

So how can we measure the unfairness of wealth distribution? One attractive idea is to measure the area of the difference between the Lorenz curve and the line $y = x$.



Since the area under the straight line is $\frac{1}{2}$, the area between the line and the curve is $\frac{1}{2} - \int_0^1 L(x)\,dx$. This area represents in a very direct way the difference between the real distribution of wealth and

the theoretical optimum distribution. Since the Lorenz curve doesn't change if everyone's net worth doubles, the area between the curves doesn't change either.

Economists multiply the area by two so that the answer is always a number between 0 and 1. The result is 0 if the distribution is perfectly fair and 1 if the distribution is perfectly unfair. The number has a name. It's called the "Gini coefficient" after its inventor, Corrado Gini.

For a Lorenz curve L, the **Gini coefficient**, G, is defined as

$$G = 2\left(\frac{1}{2} - \int_0^1 L(x)\,dx\right).$$