Application: **The Age of the Universe.**

In *The Size of the Universe* we discussed Hubble's discovery of an expanding universe. Hubble's equation governing the rate of expansion was $d' = H_0 d$, where d is the distance of an object from the Earth and H_0 is Hubble's constant.

In *Did the Universe have a Beginning?* we mentioned that in truth, Hubble's constant is really a variable. A more correct equation is

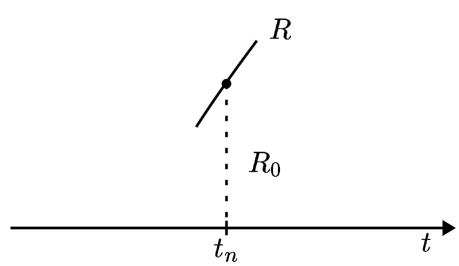
$$d' = Hd,$$

where H is a quantity that varies with time. Let t_n be the time now, and let $H_0 = H(t_n)$, the current value of Hubble's constant. In this section, we'll attempt to find the age of the universe.

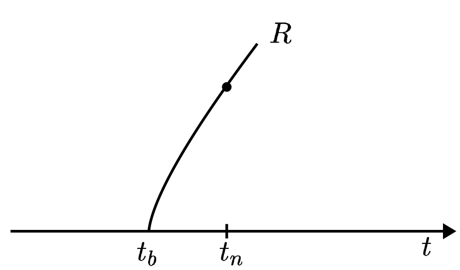
Let R be the radius of the universe and let M be its mass. A point at the edge of the universe is subject to gravitational acceleration equal to

$$R'' = -\frac{GM}{R^2},$$

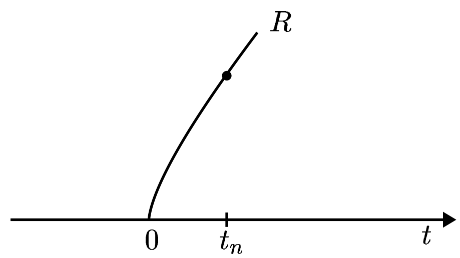
where G is the universal gravitational constant (see *Mechanics*, section 6.12. This is a classic "Friedmann" model of the universe.



We're using $R_0 = R(t_n)$ for the radius of the universe *now* (not a known length). With a little integration and one more assumption, we can use the equation above to show that the universe has a finite age, i.e., that at some time t_b in the past, R was zero. That was the time of the "big bang."



We're using t_b for the value of t at the beginning of the universe. If we take t_b to be zero, then t_n will be the age of the universe.



We're going to find the age of the universe—we're going to calculate t_n. Here we go:

Let $v = R'$ be the velocity of expansion, so $R'' = v'$. Then

$$\begin{aligned} v' &= -\tfrac{GM}{R^2} \\ \tfrac{dv}{dt} &= -\tfrac{GM}{R^2} \\ \tfrac{dR}{dt}\tfrac{dv}{dR} &= -\tfrac{GM}{R^2} \qquad \text{(the Chain Rule)} \\ v\tfrac{dv}{dR} &= -\tfrac{GM}{R^2} \\ v\,dv &= -\tfrac{GM}{R^2}dR. \end{aligned}$$

Now we integrate both sides:

$$\begin{aligned} \int v dv &= -GM\int \frac{1}{R^2}dR \\ \int v dv &= -GM\int R^{-2}dR \\ \tfrac{1}{2}v^2 &= -GM\left(\tfrac{1}{-1}R^{-1}\right) + C \\ \tfrac{1}{2}R'^2 &= \tfrac{GM}{R} + C. \end{aligned}$$

This is where we need one more assumption. The value of the constant C depends on M, the mass of the universe, an unknown. If we take C to be zero, we get the so-called "flat" or "De Sitter-Einstein" universe. It's such an attractive universe, that cosmologists have felt intuitively that it's correct and have been searching the heavens for enough mass to make C zero.

Our additional assumption is that we live in a De Sitter-Einstein universe. In this universe, we can calculate t_n.

Laboratory: The Age of the Universe

We ask you to complete the calculation of t_n, the age of the universe. We outline the steps for you:

1. Solve the differential equation,
$$\frac{1}{2}R'^2 = \frac{GM}{R}$$
using the method of separation of variables. Don't forget the constant of integration.

2. Evaluate the constant of integration by substituting in 0 for t and 0 for R (at $t = 0$, the radius of the universe is zero).

3. Now substitute t_n for t and R_0 for R (at $t = t_n$, the radius of the universe is R_0) and solve for t_n.

4. Show from previous equations that $H_0 R_0 = \sqrt{\frac{2GM}{R_0}}$. and find the age t_n of the universe in terms of H_0.