SQUARING AND NOT SQUARING ONE OR MORE PLANES

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Abstract. A set of natural numbers **tiles the plane** if a square-tiling of the plane exists using exactly one square of sidelength n for every n in the set. In [9] it is shown that $\mathbb{N}$, the set of all natural numbers, tiles the plane. We answer here a number of questions from that paper. We show that there is a simple tiling of the plane (no nontrivial subset of squares forms a rectangle). We show that neither the odd numbers nor the prime numbers tile the plane. We show that $\mathbb{N}$ can tile many, even infinitely many planes.

A square-tiling is **perfect** if no two squares used are the same size. In 1903 Max Dehn [2] asked: Is there a perfect square-tiling of a square? In 1925 Zbigniew Moroń found perfect square-tilings of several rectangles [10]. Dehn's question was ultimately answered affirmatively in 1938 by Roland Sprague [12]. The problem and its solution were the subject of a memorable paper, "Squaring the Square" by Tutte [13], reprinted in Martin Gardner's column in *Scientific American* (see [6]). Papers have continued to appear on the subject ever since (see for example, [5], [3], [4]).

In 1975 Solomon Golomb [7] asked if there was a perfect square-tiling of the infinite plane with every side-length represented. In 1997, Karl Scherer [11] found an imperfect square-tiling of the plane—squares of all integral sides are used, but each size is used multiple times. The number of squares of side n used, $s(n)$, is finite but the function s is not bounded. Golomb's question was ultimately answered affirmatively in "Squaring the Plane." (2008, [9]). The solution opened a host of questions, for example, Which sets tile the plane? Is there a three-colorable tiling? Can the half-plane be tiled?

There are connections between squaring planes and squaring squares (see for example the proof of 5.1). There are also curious disconnects. There is a clever proof that a cube cannot be cubed ([13]). But the technique has not yet shown us that space cannot be cubed.

In section 1, we find a large class of sets, including the set of odd numbers, that do not tile the plane. In section 2, we show that $\mathbb{N}$ can tile many planes at once. In section 3, we show that the prime numbers do not tile the plane.

In trying to square squares, Tutte and his fellow researchers especially prized "simple" tilings, tilings in which no nontrivial subset of the squares forms a rectangle. One question in [9] asked if there is a simple $\mathbb{N}$-tiling of the plane. In section 4, we construct a simple, perfect, square-tiling of the plane.

In section 5, we report on the questions posed in [9], give a $\mathbb{Z}$-tiling of the half-plane, and pose a number of new questions.

Since the results in this paper were obtained, additional research has been done, [1], which deepens the mystery of tiling sets. It is now known, for example, that a set with

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one or three odd numbers may tile the plane but a set with two odd numbers can't. It also known that a set growing faster than the Fibonacci numbers can't tile the plane.

In this paper, "tiling" will mean a perfect square-tiling. If a tiling $\mathcal{T}$ uses all and only the squares in X, we will say $\mathcal{T}$ is an "X-tiling". For simplicity, we will denote the square of side n with the boldface letter '**n**'.

1. A class of sets that do not tile the plane

A simple examination of [9] shows the following:

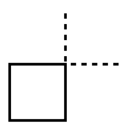
Theorem 1.1. *Suppose for $X \subseteq \mathbb{N}$ that $a + b \in X$ for all $a, b \in X$, $a \neq b$. Then X tiles the plane.*

In [9] we noted that $\mathbb{E}$, the set of even natural numbers, tiles the plane and asked if $\mathbb{O}$, the set of odd natural numbers, tiled the plane. It turns out that closure under addition is important. $\mathbb{E}$ satisfies the hypothesis of . $\mathbb{O}$, on the other hand, is anti-closed. That it fails to tile the plane is a consequence of the following theorem.

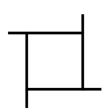
Theorem 1.2. *Suppose for $X \subseteq \mathbb{N}$ that for all $a, b \in X$, $a \neq b$, $a + b \notin X$. Then X does not tile the plane.*

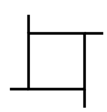
Proof. Suppose there is an X-tiling $\mathcal{X}$. We will derive a contradiction.

First note that at every corner of a square in any tiling there will be a third or fourth edge extending from the corner.

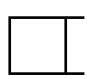


We will say that a square is a **pinwheel** if there are lines at the corners for one of these patterns:





If one side of the square has lines in the same direction,

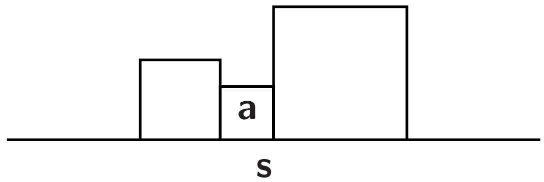


then we will say it has an **integral side** (it will have a whole number, n, of neighbors on that side, $n > 1$). A square may both be a pinwheel and have an integral side, but a square with no integral side must be a pinwheel.

Claim 1.3. *No square in $\mathcal{X}$ can have an integral side.*

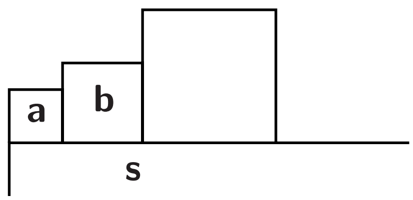
Suppose that the claim is false. Let **s** be the smallest square with an integral side. Square **s** can't have two neighbors on a side because of the assumption on X. Thus, **s** must have an integral side with at least three neighbors.

Along that side, the smallest neighbor **a** must be at one end, since otherwise,

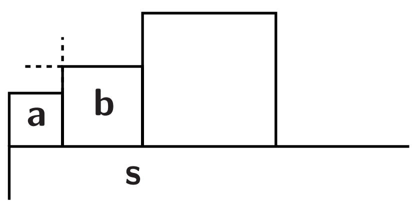


a would have an integral number of neighbors along its top edge, contradicting the choice of **s** as the smallest such square.

But now consider **a**'s neighbor **b** and suppose that **b**'s other neighbor is larger.

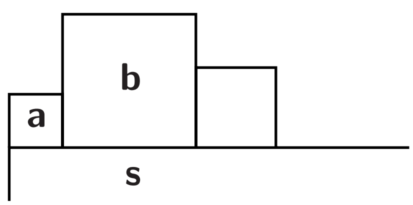


At the corner of **b** above **a** there must either be an edge extending up or an edge extending to the left.

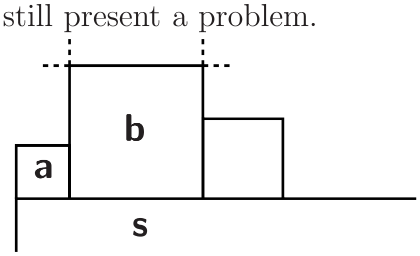


In either case, **b** is forced to have an integral side, a contradiction.

Finally, if **b**'s neighbor is smaller,

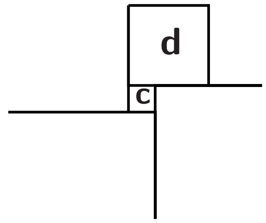


Then the corners of **b** still present a problem.

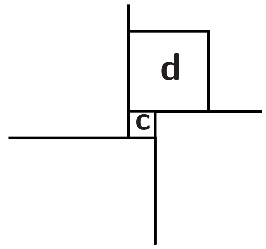


No matter which way they go **b** will have an integral side. This proves 1.3.

Now consider the smallest square **c** in $\mathcal{X}$. Square **c** must be a pinwheel. Let **d** be the smallest of the surrounding squares.

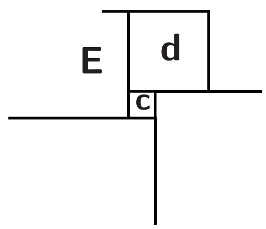


The upper left corner of **d** has an edge going up or to the left. If it goes up,

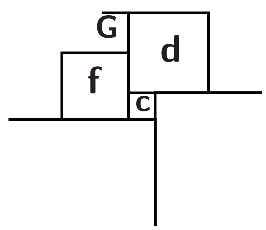


we have a problem because the upper right vertex of **d** will have a third edge coming from it and whichever direction that edge takes, we will have the picture forbidden by the claim.

On the other hand, suppose the edge goes to the left.



There can't be a single square in the region marked **E**, since we would then have two members of X adding to a third member of X.



But square **f** must be larger than **d** (**d** was chosen as the smallest around **c**), so every square in the region marked **G** must be smaller than **c** (chosen as smallest). This is a contradiction and the proof is complete. □

Corollary 1.4. *The set of odd numbers does not tile the plane.*

2. Covering several planes

The fact that the set of even numbers tiles the plane but its complement does not raises the question: is there a set X such that both X and X^c tile the plane?

In fact, there is. Indeed, we can partition $\mathbb{N}$ into any finite number of sets, even infinitely many sets, all of which tile the plane. We'll start with just two.

Theorem 2.1. *There is a set of natural numbers X such that both X and X^c tile the plane.*

We need to review here the proof that $\mathbb{N}$ tiles the plane. The key is the following lemma.

Lemma 2.2. *Given any ell-shaped region formed of squares, it is possible to surround the ell with larger squares in such a way as to form a solid rectangle, with the added squares all larger than those in the ell.*

With this lemma, the proof of the theorem proceeds as follows:

(1) Place two squares in the plane, forming an ell.
(2) Add squares to the ell to create a rectangle.
(3) Add to the rectangle the smallest square not yet used, forming an ell.

Steps 2 and 3 are then repeated *ad infinitum.*

Now the proof of 2.1.

Proof. In effect, we will show that $\mathbb{N}$ can tile two planes, P_0 and P_1. In stages, we will build X, X^c, tiling P_0 with X and P_1 with X^c. The stages will address the planes P_0 and P_1 alternately. At the end of each stage there will be a finite number of numbers each in X and X^c. The squares with side-lengths in X will be arranged in a rectangle in P_0 and the squares with side-lengths in X^c will be arranged in a rectangle in P_1.

We begin by placing two distinct squares, one in each plane.

Suppose we have last addressed X and P_0. Here is how we address X^c and P_1.

1. We choose n larger than all the members yet chosen of X and X^c.
2. We place $\mathbf{n}$, the square of side n, alongside the rectangle in P_1, forming an ell and add n to X^c.
3. Using 2.2, we surround the ell with squares larger than any number in X^c (and hence larger than any number in X) to form a rectangle, adding the squares to X^c.
4. We then choose the smallest number m not in either X or X^c,
5. We place $\mathbf{m}$ alongside the rectangle in P_1, forming an ell and add m to X^c.
6. Again using 2.2, we surround the ell with squares to form a rectangle, adding the side-lengths to X^c.

We address X and P_0 in the same way. At the end of this infinite process P_0 will be tiled by X and P_1 will be tiled by X^c. □

It is easy to extend 2.1 to handle k planes for any finite k. In fact, we can handle a countably infinite number of planes.

Theorem 2.3. *There exists a partition of $\mathbb{N}$ into disjoint sets $\{A_n\}_{n\in\mathbb{N}}$, each of which tiles the plane.*

Proof. We build the sets and the tilings of planes $\{P_n\}_{n\in\mathbb{N}}$ in stages. At the end of every stage, all but finitely many A_n will be empty and for every n with $A_n \neq \emptyset$, the squares with side-lengths in A_n will be arranged in a rectangle in P_n.

Each stage involves work on a single plane. The order in which the planes are addressed,

$$P_1,\ P_2,\ P_1,\ P_2,\ P_3,\ P_1,\ P_2,\ P_3,\ P_4,\ \ldots,$$

ensures that we return to every plane infinitely many times.

At every stage we either address a plane for the first time, or we return to a plane. We discuss these cases separately.

(1) When we address plane P_k for the first time, we identify the smallest i not in any A_j. We put i in A_k, and we place **i** in P_k.
(2) When we address a plane P_k, with $A_k \neq \emptyset$ and the squares in P_k arranged in a rectangle, we choose n larger than all the members of the all the A_j and place **n** next to the rectangle in P_k, forming an ell. We then surround this with squares to form a rectangle, at the same time adding the side-lengths of the squares to A_k.

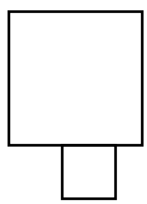
It should be clear that this process uses all the squares, never uses a single tile twice, and tiles all the planes. □

3. The prime numbers

The odd prime numbers satisfy the conditions of 2.1 and so do not tile the plane. Including the number 2 seems a small matter but it seems that an entirely new set of techniques is required to show that the primes do not tile the plane.

Theorem 3.1. *The set of prime numbers doesn't tile the plane.*

Proof. Suppose, to the contrary, that there is a tiling of the plane $\mathcal{P}$, using the set of primes. We will say that a square is a **mother** to a smaller square if in $\mathcal{P}$ an edge of the smaller is contained in an edge of the larger, but the squares do not share a corner.

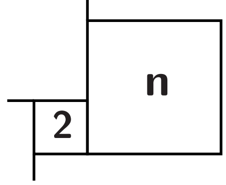


We'll say the smaller square **has a mother** (not true of every square). If a square **s** is smaller than the smallest mother or if there are no mothers, we will say that **s** is **just a kid**.

If a square **n** has an integral side, it either has two neighbors on that side (and one of the neighbors is **2**) or else it has three or more neighbors on that side and is therefore a mother.

If a square **n** has a mother, it is clearly not a pinwheel and so must have an integral side. Consquently, it is either adjacent to **2** or is itself a mother. From this we see that motherhood must continue in a chain of smaller and smaller squares that can only end at **2**.

Now let's examine the neighborhood of **2**. Because **2** is the smallest square, it cannot have an integral side and must therefore be a pinwheel. Let **n** be its smallest neighbor.

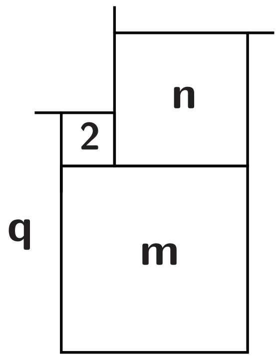


Claim 3.2. **n** *is a pinwheel and just a kid.*

From the discussion above, **n** cannot be a mother because that would lead to a chain of squares of descending size that would continue until it reached **2**—not possible since **n** is the smallest neighbor of **2**. For the same reason, no square smaller than **n** can be a mother, hence **n** is just a kid.

Finally, if **n** is not a pinwheel, then **n** must have an integral side with at least three neighbors. But this would mean **n** is a mother, which it isn't. This proves 3.2.

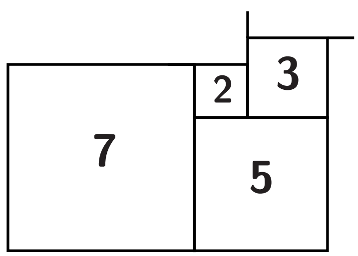
Let **m** be the square sharing a single edge with **2** and **n**.



Since **n** is a pinwheel, $\mathbf{m} = \mathbf{n} + \mathbf{2}$.

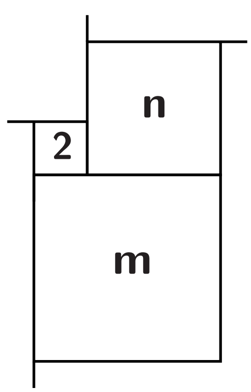
Claim 3.3. *The square* **q** *sharing a single edge with* **m** *and* **2** *has side-length greater than $m + 2$.*

Since **n** is **2**'s smallest neighbor and **m** is **2**'s second smallest neighbor, **q** must have side at least $m+2 = n+4$. One of n, $n+2$, $n+4$ is divisible by 3. If the claim is false, then n, $n+2$, $n+4$ are primes. This is only possible if $n = 3$.

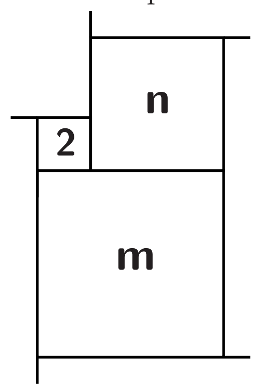


But this is not possible. The remaining neighbor of **2** must be larger than **9**. Any arrangement of the other corners of **7** and **5** produces an interval (7, 12, or 8) that can't be filled with the remaining prime lengths. This proves 3.3.

We now have the picture



and by the reasoning in the previous claim, **m** is just a kid, hence its bottom side is not an integral side and we have this picture.

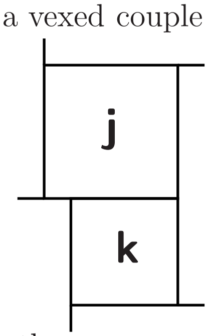


Now **n** and **m** jointly have an integral number of neighbors to the right. There can't be an odd number of them, since we are dealing with primes. There can't be as many as four, or either **n** or **m** would be a mother, which, as we argued earlier, is not possible. Thus there are just two squares to the right.

Definition 3.4. *A* **vexed couple** *is a pair (***j***,***k***) satisfying*
(1) **j** *and* **k** *share a corner and a side,*
(2) $n > j > k > 2$, *and*
(3) **j** *has, on one of its edges, only one neighbor.*

Any square smaller than **n** is just a kid and since it is not a neighbor of **2** it cannot have an integral side and must be a pinwheel. Thus, by condition (2), **j** and **k** are pinwheels.

From condition (1), a vexed couple has a common face.

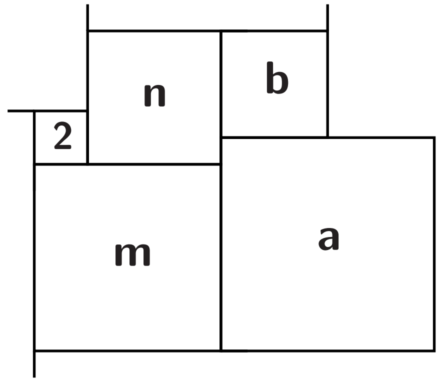


There must be more than one square along this face since $j + k \notin X$. There can't be more than two squares along this face since **j** and **k** are not mothers. Thus **j** and **k** face a pair of squares. We will see shortly that this fact will lead us to another vexed couple in such a way as to give us a contradiction. First, however, we establish the the following.

Claim 3.5. *There is a vexed couple in the tiling $\mathcal{P}$.*

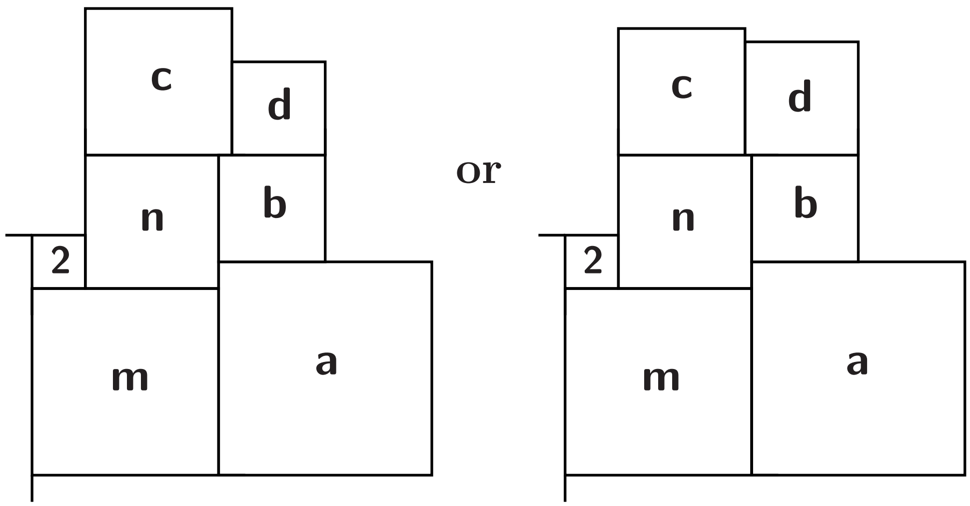
Consider the squares **a** and **b** facing **m** and **n** along their common face, $b < a$. Note that we must have $b < n < n + 2 = m < a$.

<u>Case 1</u> **b** and **n** share a corner.



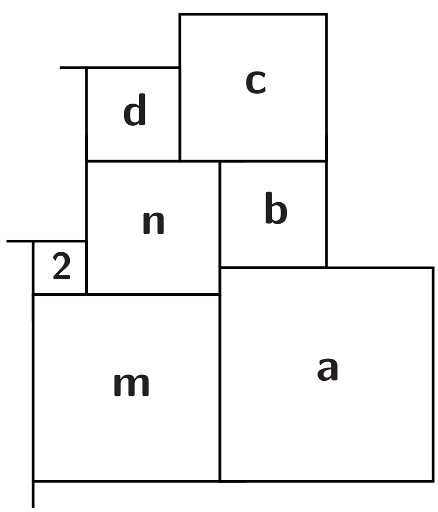
As before, **n** and **b** must have a pair of neighbors, $c > d$ along their common face.

Case 1a **b** and **d** share a corner.



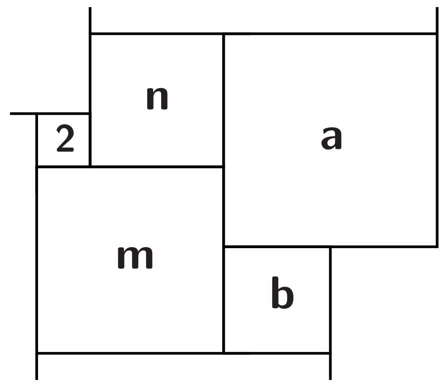
Clearly (**b**,**d**) (or (**d**,**b**)) is a vexed couple. Note that on one edge, **b** and **d** have only one neighbor, satisfying condition (3).

Case 1b **n** and **d** share a corner.



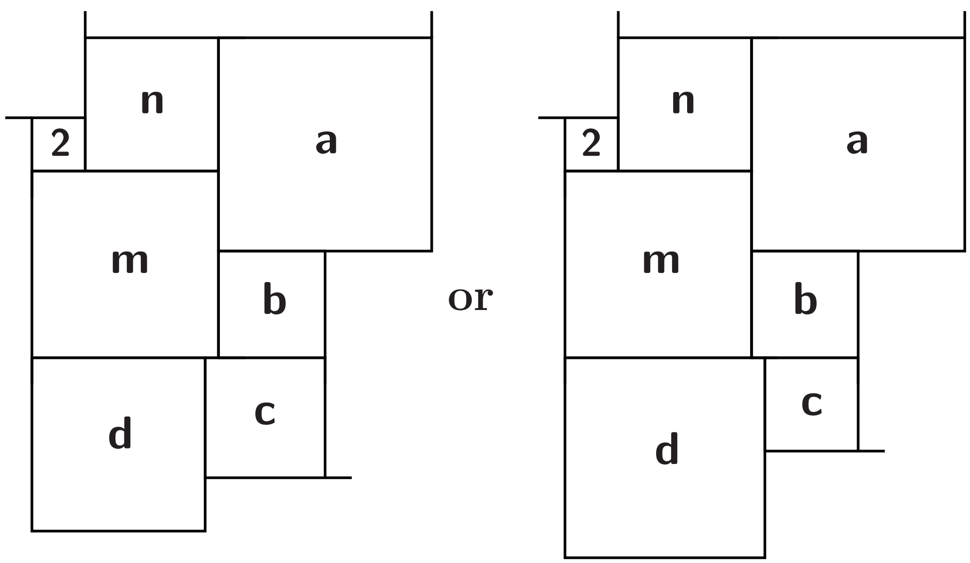
Being smaller than **n**, **d** is a pinwheel. To the left of **n** and **d** there can't be a single square since **n** and **d** are both odd. There can't be three squares or **d** or **n** would be a mother. Thus there are two squares facing **n** and **d** and the smaller of the two, **e**, must be smaller than **d** and share a corner with it since **n** is the smallest neighbor of **2**. We then have that (**d**,**e**) is a vexed couple.

Case 2 **b** and **m** share a corner.



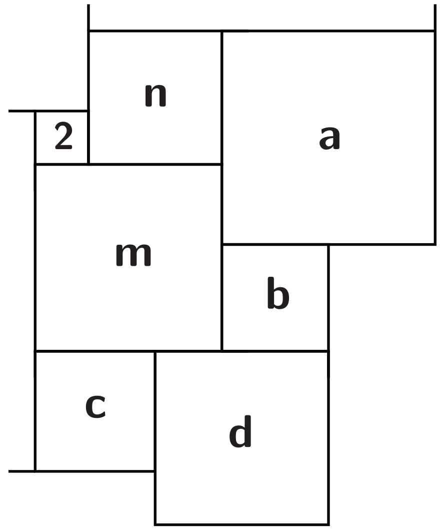
As before, **b** is a pinwheel. Let **c** and **d** be the squares facing **m** and **b** along their common side, $c < d$.

Case 2a **c** and **b** share a corner.

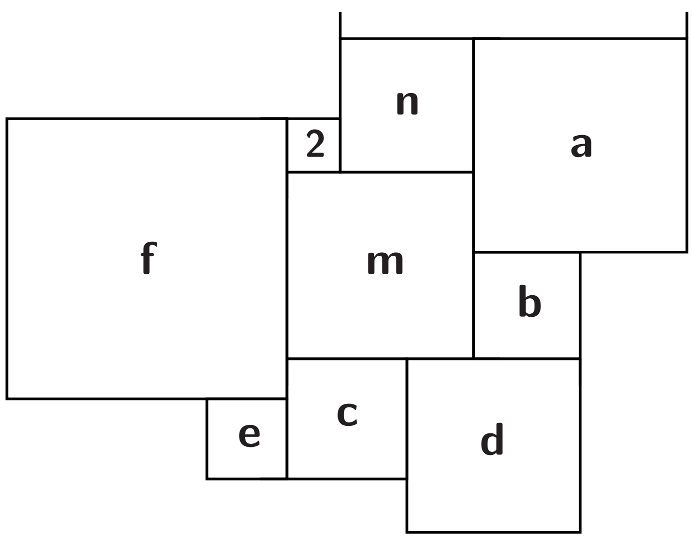


Then **b** and **c** comprise a vexed couple. Note that $c < n$ since
$c < \frac{1}{2}(m+b) < \frac{1}{2}(m+\frac{1}{2}(m+n)) = \frac{1}{2}(n+2+\frac{1}{2}(n+2+n)) = n+1.5$.

<u>Case 2b</u> **d** and **b** share a corner.



As before, $c < n$ so **c** is just a kid. Since $2 + m + c$ is even and neither **m** nor **c** nor **2** is a mother, there must be exactly two squares along the face common to **2**, **m**, and **c**. Furthermore, since **n** and **m** are the two smallest squares adjacent to **2**, we must have the picture:

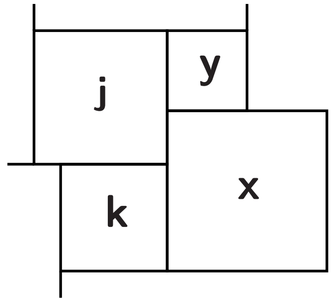


and (**c**,**e**) is a vexed couple. This proves 3.5.

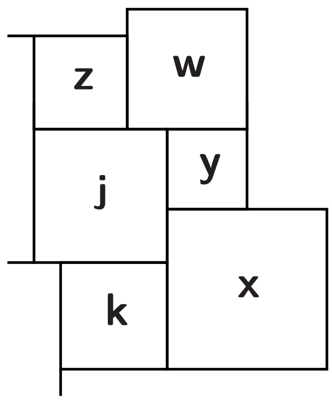
Claim 3.6. *If (*$\mathbf{j}$*,*$\mathbf{k}$*) is a vexed couple then there is another vexed couple (*$\mathbf{u}$*,*$\mathbf{v}$*) with* $j > u$*.*

Let **x** and **y**, $x > y$, be the pair sharing a face with **j** and **k**. If **y** shares a corner with **k**, then (**k**, **y**) or (**y**, **k**) is the promised vexed couple; condition (3) is satisfied either way since **k** has only one neighbor (**j**) along one of its edge and **y** has only one neighbor (**x**) along one of its edges.

Suppose, however, that **y** shares a corner with **j**. The square **y** is, of course, a pinwheel.



Then there will be a pair **z**, **w** facing **j** and **y**, $z > w$. If **w** shares a corner with **y**, then the promised vexed couple is (**w**, **y**) or (**y**, **w**). Suppose, on the other hand, **w** shares a corner with **j**,



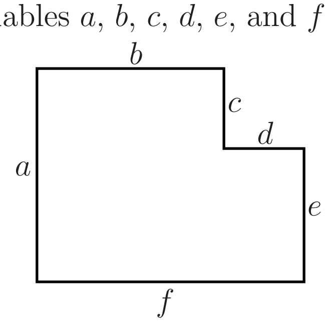
Again there must be a pair of squares, **r** and **s**, $r > s$, facing **j** and **z** along a face and we will now use condition (3). Since **j** has only one neighbor along one of its edges, we must have that **s** shares a corner with **z** and the promised vexed couple is either (**s**, **z**) or (**z**, **s**). This proves 3.6.

We now have a contradiction to the existence of the tiling $\mathcal{P}$. The sequence of vexed couples can't terminate so the sizes of the large squares decreases infinitely. This completes the proof. □

4. A simple tiling

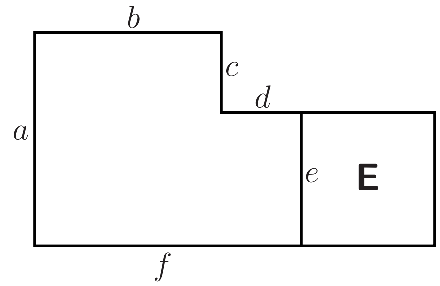
Theorem 4.1. *There is a simple* $\mathbb{N}$*-tiling of the plane.*

Proof. We begin by reviewing the procedure in [9] of 2.2, that every ell can be surrounded by squares to form a rectangle. For ease of reference, we orient our ells as below and assign the variables a, b, c, d, e, and f to represent the lengths as indicated.



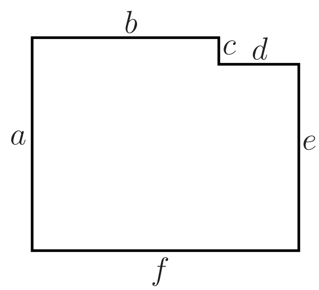
Since the values of b, c, d, and e determine those of a and f, we take $\langle b, c, d, e\rangle$ to represent the unique ell-shaped region shape with those dimensions.

We use the corresponding capital letters to represent the acts of adding squares to the sides. For example **E** represents adding a square of side-length e.

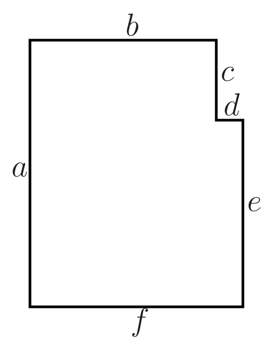


An ell is **regular** if $c > d + e$ and both **BFA**[1] and **ED** can be performed without using squares already in the ell. In outline, the procedure is as follows:

(1) Make the ell regular without changing d by performing **FABAFABA**.
(2) Express $c - e$ as $kd + i$ with $k \geq 1$ and $0 \leq i < d$.
(3) Reduce k to 1 (each application of **BFA** reduces k by 1 without changing d).
(4) Perform **ED**, decreasing c to i (still less than d).



(5) Mentally flip the ell over to produce an ell with $d = i$

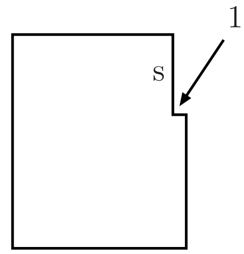


(6) Repeat 1.–5. until $d = 0$ and the ell is squared up.

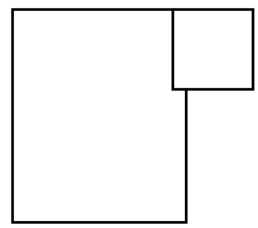
To prove 4.1 we modify this procedure. Assume we are given an ell and that s is the size of the smallest square not yet used. Steps (1)-(5) are as before but we proceed as follows.

(6)′ Reduce d to 1, not 0, so $i = 0$ and $c = k + e$.
(7) Increase k so that it is greater than $s + 1$.
(8) Reduce k so that it equals $s + 1$.
(9) Perform **ED**, decreasing c to s.

[1]This means performing **B** first then **F** and then **A**.



(10) Perform **C**, which adds a square of side s,



and mentally flip the ell. This gives us another ell; we repeat these steps forever.

The difficult part is 6.′ and we save it for last.

7. is straight-forward. Applying **BA** doesn't change the value of d or e, it just raises the value of c. Successive use of **BA** increases k to any desired level.

8. is exactly like 3.

Before launching into 6.′, note that should we succeed we will have created a simple tiling of the plane. The tiling will grow in all four directions. It will use no square more than once (see [9]). It will use every square. And the tiling will not contain a rectangle. This last is not obvious, but if there were a rectangle inside the tiling, there would have to be a smallest such rectangle and one square of the rectangle would have to be the last square added to the rectangle. That square would have to be a corner, since we never add a square with more than two of its sides touching squares in the tiling. But the only time we do add a square with two of its sides touching the tiling is when we add s in step (10) and this clearly does not complete a rectangle.

To show 6.′ we prove the following claim.

Claim 4.2. *If* $\gcd(b, c, d, e) = 1$*, then steps (1)–(5) with slight modifications can lead to a regular ell with* $d = 1$*.*

Steps 1.–5 reduce d irregularly to 0. The worry is that d might go from larger than 0 at one point directly to 0. That is, we might have $d > 1$ at one stage but $c = kd + e + i$ with $i = 0$.

Note first that given an ell with $\gcd(b, c, d, e) = 1$, each of the steps **A**, **B**, etc. leaves $\gcd(b, c, d, e) = 1$. This is easy to check. Note, for example, that the result of applying **A** to $\langle w, x, y, z\rangle$ is to form $\langle w + x + z, x, y, z\rangle$. Clearly, if u divides $w + x + z$, x, y, and z, then it divides w so $\gcd(w + x + z, x, y, z) = 1$.

Now suppose we have $d > 1$, $i = 0$, and having performed (1), our ell is regular. Then $c = kd + e$ and $c - e$ is a multiple of d. Our ell will have dimensions $\langle x, ky + z, y, z\rangle$ for some x, y, z with $\gcd(x, y, z) = 1$.

<u>Case 1</u> x is not a multiple of y.

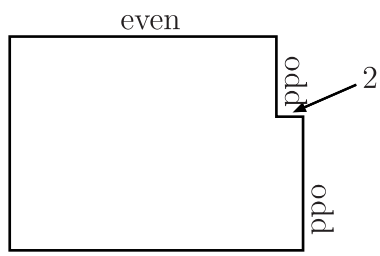
Then performing **BA** doesn't change d or e but makes $c = x + ky + z$. Now $c - e = x + ky$, not a multiple of $d = y$, so the i in $c = kd + e + i$ is no longer 0. Now performing (3)-(5) will give us a new, smaller d not zero.

<u>Case 2</u> x is a multiple of y but $2z$ is not a multiple of y.

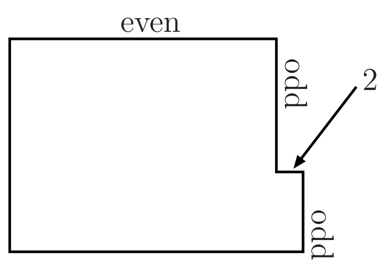
Then performing **BA** twice doesn't change d or e but makes $c = 3x + 2ky + 3z$. Now $c - e = 3x + 2ky + 2z$, not a multiple of y, so again we can perform (3)-(5) to give us a smaller d not equal to zero.

<u>Case 3</u> x and $2z$ are multiples of y.

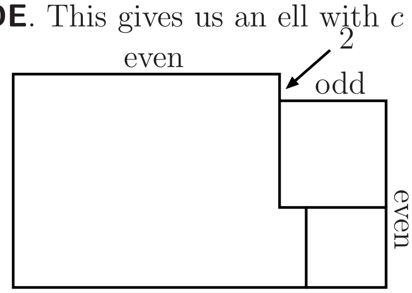
Then y can't have any odd factors since $\gcd(x, y, z) = 1$. Similarly, y can't be a multiple of 4. Since $y = d > 1$, y is 2. That in turn leads to x being even, z being odd, and $c = ky + z$ also odd. We are left with the picture:



Under these circumstances, we successively perform **BFA**, reducing k to 2, that is, $c = 2d + e$. Note that **BFA** doesn't affect $d = 2$. Note also that **BFA** leaves b even and c, e odd.



Now we perform **DE**. This gives us an ell with $c = 2$, d odd, and e even.



When we flip this, we have an ell with the pattern, $\langle$even, odd, 2, even$\rangle$.

To turn this into a regular ell, we perform (1), that is, we perform **FABAFABA** or **AFABAFABA**. **A** alone doesn't change the pattern. The first **FABA** changes it to $\langle$odd, even, 2, even$\rangle$. The second **FABA** changes it to $\langle$odd, even, 2, odd$\rangle$. Then $c - e = 2k + i$ for some k and for some $i < 2$. We must have that $i \neq 0$, since $c - e$ is odd. That makes $i = 1$. Following steps (3)–(5) then gives us an ell with $d = 1$. This proves .

To complete the proof, we note that beginning our construction with one square each of sides 1 and 2 gives us an ell $\langle b, c, d, e\rangle$ with $\gcd(b, c, d, e) = 1$, enabling us to get started. Note also that (6′) takes ell $\langle b, c, d, e\rangle$ to ell $\langle c, c - d, e, b + d\rangle$ and that $\gcd(c, c - d, e, b + d) = 1$, so the procedure may continue. □

5. Questions, answered and unanswered

Most of the questions posed in [9] remain open. We don't know if the half-plane or the quarterplane can be tiled with squares. We don't know if there is a 3-colorable tiling of the plane. We don't know if space can be cubed.

We also haven't found a significantly more efficient algorithm for tiling the plane. Putting the matter more clearly, we have the following question.

Question Is there an $\mathbb{N}$-tiling $\mathcal{T}$ such that for all n, the squares $\mathbf{1}$, $\mathbf{2}$, ..., $\mathbf{n}$ in $\mathcal{T}$ form a connected subset?

That seems unlikely.

Question For an $\mathbb{N}$-tiling, $\mathcal{T}$, let the function $f_{\mathcal{T}}$ be defined by

$$f_{\mathcal{T}}(n) = \begin{array}{l}\text{the perimeter of the smallest}\\ \text{rectangle containing the squares } \mathbf{1},\\ \mathbf{2}, \ldots, \mathbf{n} \text{ in } \mathcal{T}.\end{array}$$

Is there a tiling $\mathcal{T}$ where $f_{\mathcal{T}}$ is bounded by a polynomial?

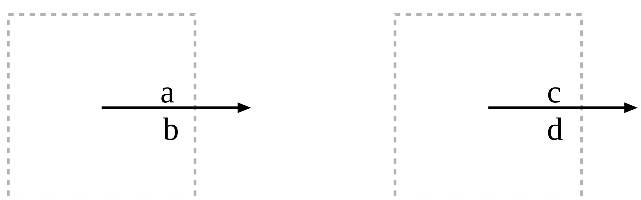
The work of section 3 of this paper suggests the following question:

Question Does the set of odd numbers together with 2 tile the plane?

This seems difficult. The fact that 9 is not prime seems important in the proof of 3.1.

Bill Zwicker[2] suggests the following question.

Question Are there $\mathbb{N}$-tilings of Riemann surfaces? In particular, what about a surface consisting of two planes slit along rays,



where the edge above the slit in the first plane (a in the figure above) is identified with the edge below the slit in the second plane (d) and the edge above the slit in the second plane (c) is identified with the edge below the slit in the first plane (b)?

There were questions in [9] on coloring. We have no results, but our attempts suggest the following interesting question.

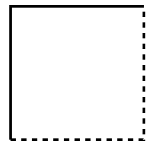
Question Imagine that we place squares of different sizes one at a time in the plane, always, after the first square, placing squares next to previously placed squares (the squares don't overlap, but we aren't trying to tile the plane). Imagine also that we color the squares as they are placed (colors 1, 2, 3, ...) with the "greedy algorithm", that is, always using the least color possible so that adjacent squares have different colors. What is the largest color we might be forced to use?

Finally, while we haven't found an $\mathbb{N}$-tiling of a half-plane, we can prove the following:

Theorem 5.1. *There is a $\mathbb{Z}$-tiling of a half-plane.*

[2]Department of Mathematics, Union College

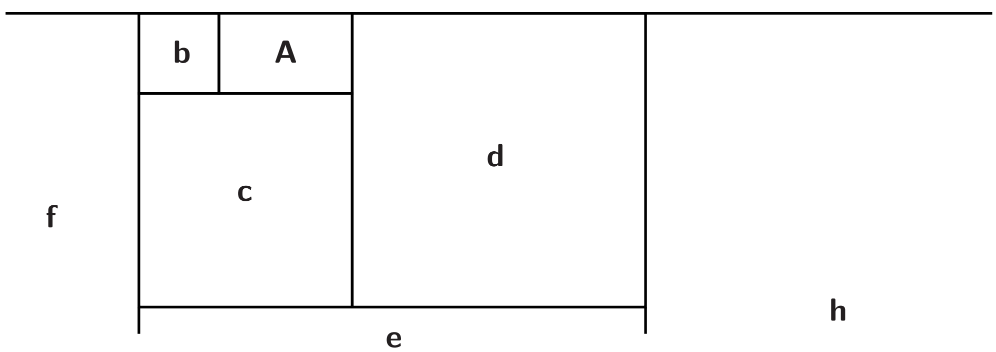
Proof. To be careful, let us use square tiles, of both positive and negative side-length, which include only the top and left edges.



Then by a "$\mathbb{Z}$-tiling of a half-plane," we will mean a tiling of the plane with the properties:

(1) For every point in one half-plane there is a $k \geq 1$ such that the point is covered by k positive squares and $k-1$ negative squares.
(2) For every point in the other half-plane there is a k such that the point is covered by k positive squares and k negative squares.

It is not difficult to tile a half-plane with a subset A of $\mathbb{N}$. You can start with a squared rectangle (**A**), and then add tiles (**b**, **c**, **d**, ...) in a generalized Fibonacci sequence.



Let $\mathcal{S}$ be a B-tiling of a half-plane using $B = \{-n : n \in A\}$. Now use (as described in [9]) the remaining negative tiles plus all the positive tiles to form a tiling $\mathcal{T}$ of the entire plane. Then the combination of $\mathcal{S}$ and $\mathcal{T}$ is a $\mathbb{Z}$-tiling of a half-plane. □

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