Part 1: Multiple Choice and Short Answers  (3 points each)

CIRCLE THE SINGLE BEST ANSWER

1) In a second order circuit with a step input, in order to minimize oscillatory behavior, and reach the steady-state value as quickly as possible, you should build:

a) A circuit with an overdamped response.
b) A circuit with two capacitors.
c) A circuit with a critically damped response.
d) A circuit that maximizes resistance
e) A circuit with an underdamped response.
f) none of the above

2) The following expression characterizes the response for what type of circuit?

\[ i_L(t) = i_L(\infty) \left(1 - e^{-\frac{R}{L}} \right) \]

a) First order circuit natural response
b) [First order circuit forced response]
c) Second order circuit step response
d) First order circuit step response
e) Second order circuit forced response
f) Sinusoidal steady state response for a dynamic circuit
g) none of the above

3) The following expression characterizes the response for what type of circuit?

\[ i(t) = e^{-2t}(5.3\cos(15t) + 22.1\sin(15t)) + 15.7A \]

a) First order circuit natural response
b) First order circuit forced response
c) [Second order circuit step response]
d) Second order circuit natural response
e) Second order circuit forced response
f) Sinusoidal steady state response for a dynamic circuit
g) none of the above
4) The following graph best represents the output of what type of circuit?

![Graph of current vs. time]

a) First order circuit natural response  
b) Sinusoidal steady state response for a dynamic circuit  
c) First order circuit step response  
d) Second order circuit natural response  
e) First order circuit forced response  
f) none of the above

SHORT ANSWER (credit will be deducted for long answers.)

5) Given the simple form for the complete response of a first order circuit

\[ v(t) = v_{\text{natural}}(t) + v_{\text{forced}}(t) \]

state briefly (~2 or 3 phrases each) what \( v_{\text{natural}}(t) \) and \( v_{\text{forced}}(t) \) represent, in terms of energy and how to characterize the underlying circuit behavior.

\( v_{\text{natural}}(t) \): The response of the circuit to discharging energy stored in a capacitor and/or an inductor. It is governed by an exponential function. It may or may not oscillate, given 2 or more C or L. A source-free response leads to all energy dissipated out of the circuit, resulting in a final voltage and current value of 0.

\( v_{\text{forced}}(t) \): This is the response of a circuit to a non-zero DC input energy source. It will always be governed by an exponential function, and may or may not oscillate, given 2 or more energy storage (C or L) elements. This causes the capacitor and/or inductor to charge to a new steady-state value.

6) Energy Storage:

a) An inductor stores ___ magnetic ___ energy.

b) A capacitor stores ___ electric ___ energy.
7) The following figure represents what type or category of response from what type of circuit?

a) type of response: ___ underdamped step response ___

b) type of input: ___ step input __________________

c) type of circuit: ___ 2\textsuperscript{nd} order circuit __________

8) Which of the following are ALWAYS true for capacitors and inductors? **EXPLAIN** why in 1 or 2 sentences.

a) \[ v_c(0^-) = v_c(0^+) \] 

b) \[ v_L(0^-) = v_L(0^+) \]

c) \[ i_c(0^-) = i_c(0^+) \]

d) \[ i_L(0^-) = i_L(0^+) \]

**BRIEF Explanation** (**MUCH** more than you needed to write)
- These are the very important \textit{continuity relationships} for capacitors and inductors.
- They derive from the law of conservation of energy, and relate to the fact that capacitors and inductors are energy storage elements.
- Injecting or extracting energy (\textit{aka} charging or discharging) cannot occur instantaneously. Which is to say that it must proceed over a finite time.
- We can think about this by recalling that a capacitor stores electric energy, which relates to and determines the voltage across the element, and that an inductor stores magnetic energy, which relates to current flow.
- Alternatively, to see this through mathematical expressions for the constituent relationships of these elements
  - Capacitor: \[ i_c = C \frac{dv}{dt} \]
  - Inductor energy: \[ v_L = L \frac{di}{dt} \]
- We see that the capacitor voltage must be a continuous function, and the inductor current must be a continuous function, else there would be infinite energy implied.
- Mathematically, we express the requirement that \( v_c \) and \( i_L \) be continuous by means of the continuity relationships (above, parts (a) and (d)).
9) State the two “Golden Rules” used for analyzing ideal op amp circuits

\[ V_1 = V_2 \quad \text{or} \quad V_- = V_+ \]

b) \( i_1 = i_2 = 0 \text{A}. \) The current IN is equal to zero \((I_i = 0)\). This means that the current in to both input nodes is equal to zero. (Note that the current OUT is typically non-zero (and it may be negative, indicating that the current is flowing into the op amp from the output node.).)

10) In one or two sentences, state what a PHASOR is, and why (or for what) we use them.

a) A phasor is: A vector. A quantity with a magnitude and phase. A way to simplify writing the complex exponential form of circuit signals (voltage and current). This is a frequency dependent, and time independent representation of the circuit signals.

b) We use them: … to represent a complex voltage, current and impedance; to easily plot relative magnitudes and relative phases of circuit values; to show which values lead (i.e., reach their maximum value first) or lag other circuit values;
Problem 2: First order circuit  (24 points)

a) For the circuit below, find an expression for $v_c(t)$ for all time.
b) Graph $v_c(t)$, and label all important values and parameters on the graph.
c) Briefly state why your expression for $v_c(t)$ and the graph make sense.

c) My expression for $v_c(t)$ and the graph make sense because:

For $t<0$ find initial condition:
- $3A$ source is on
- $12V$ source is off

For $t>0$ find $T$:
- $T = \frac{R_{eq} \cdot C}{3.4} = \frac{2}{4} = \frac{1}{2} \text{s}$

For $t=\infty$ use voltage divider $v_c = \frac{v_6 \cdot R}{R + 3}$

Thus $v_c(t) = v(\infty) + [v(\infty) - v(\infty)]e^{-t/T} = 8 - 2e^{-2t} \text{V}$

$V_p$ is 63% of the way between $6V + 8V = (0.63)(8-6)+6 = 7.3V$
Problem 3: Second order circuit

a) Design, sketch and label – provide numerical values for all circuit elements – a circuit that could produce underdamped behavior.

i) Write a mathematical expression for the circuit’s response.

ii) State how you know the behavior will be underdamped. Refer to specific parameter values (e.g., the natural frequencies, the resonant frequency, etc.).

iii) Sketch and label a possible graph of your circuit response, including the input signal.

The main concept behind this question is to know that an underdamped circuit is identified via $\alpha < \omega_0$.

The simplest circuits are series or parallel RLC circuits.

$\omega_0 = \frac{1}{\sqrt{LC}}$; \hspace{1cm} $\alpha_{\text{series}} = \frac{R}{2L}$ \hspace{1cm} Note: $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$

$\alpha_{\text{parallel}} = \frac{1}{2RC}$

As we did in class, simply choosing $R=L=C=1$ will produce an underdamped circuit.

As in HW, the graph should include $e^{-\alpha t}$, $\sin$ with $\omega_d$, ($\omega$: oscillations), steady-state value consistent with your source.

As you do not need to solve for constants $A_1, A_2$

\[ V_c(t) = 1 + e^{-\frac{t}{2}} (A_1 \cos \frac{1}{\sqrt{3}} t + A_2 \sin \frac{1}{\sqrt{3}} t) \]
b) Design, sketch and label – *provide numerical values* for all circuit elements – a circuit that could produce OVERDAMPED behavior.

(EXTRA CREDIT of 5 POINTS: Do this part of the problem simply by rearranging your circuit from part (a))

i) Write a mathematical expression for the circuit’s response.

ii) State how you know the behavior will be overdamped. Refer to specific parameter values (e.g., the natural frequencies, the resonant frequency, etc.).

iii) Sketch and label a possible graph of your circuit response, including the input signal.

For an overdamped circuit, you need \( \alpha > \omega_0 \).

You could perhaps select, \( L=C=1 \) but \( R=4\Omega \), for a series RLC circuit

\[
\alpha = \frac{4}{2\cdot 1} = 2; \quad \omega_0 = \frac{1}{\sqrt{1}} = 1 \quad s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} = -2 \pm \sqrt{3}
\]

\( v_c(t) = 1 + A_1 e^{s_1 t} + A_2 e^{s_2 t} \text{ V}, \) with \( s_{1,2} \) given above

with a step input @ \( t = 0 \) as for part (a)

[Graph of \( v_c(t) \) showing an exponential growth to 1 V]
Problem: Find $v(t)$ for $t > 0$. Also specify whether this response is over-, under- or critically-damped. (25 points)

Notes:
- The initial condition $dv_c(0)/dt = -0.5$ V/s
- There is a switch and a step function ($u(t)$), so take a moment to identify which source(s) are operating/relevant for which time period(s).

\[
t < 0 \text{ find } v_c(0^-) \\
V_c(0^-) = V_{R_{10}} = V_{R_{55}}
\]

using current divider we find

\[
I_{R_{10}} = 1A + I_{55} = 2A \quad \text{so} \quad V_{10} = V_{55} = 10V \quad \therefore \quad V_c(0^-) = 10V = V_c(0^+)
\]

\[
t > 0 \text{ as } t \to \infty \text{ find } v_c(\infty)
\]

Here we see \[V_c = (5\Omega)(4A) = 20V\]

\[
0 < t < \infty \\
\text{Series RLC} \quad \alpha = \frac{R}{2L} = \frac{5}{2} = \frac{5}{2} \\
\omega_0 = \frac{1}{\sqrt{L/C}} = \frac{1}{\sqrt{4.1}} = \frac{1}{2}
\]

\[
\frac{5}{2} > \frac{1}{2} \quad \text{so this is an overdamped natural response}
\]
**Problem:** Find \( v(t) \) for \( t > 0 \). Also specify whether this response is over-, under- or critically-damped. (25 points)

**Notes:**
- The initial condition \( dv_c(0)/dt = -0.5 \) V/s
- There is a switch and a step function \((u(t))\), so take a moment to identify which source(s) are operating/relevant for which time period(s).

\[
\text{complete response } = v(t) = V_0 + \text{natural response} \nonumber = 20 + A_1 e^{-4.9t} + A_2 e^{-0.05t}
\]

\[
\begin{align*}
S_{1,2} &= -\alpha \pm \sqrt{\alpha^2 - \omega^2} = -\frac{5}{2} \pm \sqrt{\left(\frac{2.5}{2}\right)^2 - \frac{1}{4}} = -4.9, -0.05
\end{align*}
\]

**using \( V_c(0^-) \) and \( \frac{dv_c(0^-)}{dt} = -0.5 \), find \( A_1 + A_2 \)

\[
\begin{align*}
V_c(0) &= 10 = 20 + A_1 + A_2 \quad \Rightarrow \quad A_1 + A_2 = -10 \\
\frac{dv_c}{dt} \bigg|_{t=0} &= -0.5 = -4.9A_1 - 0.05A_2
\end{align*}
\]

(solving ... with a calculator or computer) find

\[
\begin{align*}
A_1 &= 0.2 \\
A_2 &= -10.2
\end{align*}
\]

so finally

\[
v(t) = 20 + 0.2 e^{-4.9t} - 10.2 e^{-0.05t} \text{ V}
\]
Problem: Thevenin equivalent

Using Thevenin’s theorem, find and draw the Thevenin equivalent circuit.

Assume \( \omega = 500 \text{ rad/sec.} \)

Notes:
- Terminals \( a-b \) are open circuited as drawn. Do not remove the capacitor
- Source transformation may be helpful in finding \( V_{Th} \).

To find \( Z_{Th} \), travel from node \( a \) to node \( b \), with the voltage source = 0 V

Transform to phasor domain

\[
X_C = \frac{1}{\omega C} = \frac{10}{5} = 2 \quad \Rightarrow \quad Z_C = \frac{1}{2} = 5 \, \Omega
\]

\[
X_L = \omega L = 10 \quad \Rightarrow \quad Z_L = 10 \, \Omega
\]

\[
Z_{Th} = Z_C \parallel (Z_L + 30/j0) = \frac{5(j20 + j10)}{20 + j10 - j5} = \frac{(5 \times 90)(22.4 \times 26.6)}{20.1 \times 14} = 5.4 \times 77.4 \approx 1.2 - j5.3 \, \Omega
\]

\[
V_{Th} = V_c
\]

Strategy: use nodal analysis to find \( V_1 \) and then voltage divider to find \( V_c \) from \( V_1 \).

KCL for \( V_1 \):

\[
\frac{V_5 - V_1}{30} = \frac{V_1 - 0}{60} + \frac{V_1 - 0}{j5} \quad \Rightarrow \quad j60 = j2V_5 + j2V_1 = V_1 (12 + j3)
\]

\[
V_1 = \frac{j2V_5}{12 + j3} = \frac{(2 \times 90)(120 \times 45)}{12.4 \times 14} = 19.4 \times 121^\circ \, V
\]

\[
V_{Th} = V_c = V_5 \left( \frac{-2.5}{j10 + (j5)} \right) = - V_5 = -19.4 \times 121^\circ \, V
\]

So the Thevenin equivalent ckt is...
a) Find \( i(t) \). Think about which circuit analysis tool will yield the easiest solution before starting your work.

b) Draw, in the complex plane, the phasors for (graph next page):
   (i) The current source, \( I_s \)
   (ii) The current found in part (a), \( I \)
   (iii) The circuit equivalent impedance, \( Z_{eq} \) (from the perspective of \( I_s \))
   (iv) Identify which current phasor is leading the other.
   (v) Label the axes and the phasors.

\[ 3 \, \Omega \]
\[ 5 \, \Omega \]
\[ \text{8 sin}(200t + 30^\circ) \, \text{A} \]
\[ 5 \, \text{mH} \]
\[ 1 \, \text{mF} \]

\[ j \omega L = j \frac{1}{\Omega} \]
\[ \frac{-j \omega C}{-j 5 \, \Omega} = \frac{-j 5}{-j 5} = \frac{1}{-1} = -1 \]

Use current divider to find \( i \) from \( I_s \):

\[ \frac{I}{I_s} = 8 \times 30 - 90 = 8 \times -60 \] (to convert to cosine to be able to use phasor notation)

\[ I = I_s \left( \frac{5}{5 + (3 + j(1 - j))} \right) = 8 \times -60 \cdot \left( \frac{5}{8 - j 4} \right) = \frac{(8 \times 60 \cdot (5 \times 0))}{8,992 - 26.6} = 4.5 \times -33.4^\circ \, \text{A} \]

So

\[ i(t) = 4.5 \cos (200t - 33.4^\circ) \, \text{A} \]

Find \( \frac{I}{Z} \) in order to plot it:

\[ Z_{eq} = \frac{5}{(3 - j4)} = \frac{(5)(5 \times 3 - 5 \times 4)}{8,992 - 26.6} = 2.8 \times -26.4^\circ \, \Omega \]

\[ \frac{I}{Z} \text{ is leading } I_s, \]

\[ \theta \]