Sinusoidal Input: Phasors & Impedance

EGR 220, Chapter 9
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Switch/Step input
Second Order Circuit

\[ x(t) = Ae^t + Be^{-t} \]
\[ x(t) = (A + Bt)e^{-at} \]
\[ x(t) = e^{-at}(A\cos\omega t + B\sin\omega t) \]

AC input
Sinusoidal Steady State

Third Part of Course

- AC (alternating current) circuits
  - Voltage and current sources alternate the polarity (sine and cosine input signals)
  - Introduce frequency and time as considerations in circuit analysis
- Review complex math from PHY 210
  - Transform to and from ‘phasor domain’
- Single frequency: Electric power systems
  - Sinusoidal Steady-State & Phasors
- Multiple frequencies: audio, etc.
Sinusoids

• Amplitude?
• Frequency?
• Phase angle?
  o (always relative to a reference signal)
• \( V_2 = ? \)
• Which signal “leads” the other?

• Which are the most useful mathematical tools to add, subtract, multiply and divide sinusoids?

Phasor Domain:
Phasor \( \approx \) Vector

Phasors: Euler’s Formula for the complex exponential

\[ re^{j\phi} = r \cos \phi \pm jr \sin \phi \]

\[ \cos \phi = \text{Re}(e^{j\phi}) \]

\[ \sin \phi = \text{Im}(e^{j\phi}) \]
Phasors: Using Euler

- Using Euler’s formula, express the following sinusoidal function as phasor

\[ v_1(t) = V_m \cos(\omega t + \phi) \]

\[ v_1(t) = \text{Re} \left( V_m e^{j(\omega t + \phi)} \right) = \text{Re} \left( V_m e^{j\omega t} e^{j\phi} \right) \]

Define a phasor: \[ V = V_m e^{j\phi} = V_m \angle \phi \]

\[ v_1(t) = \text{Re} \left( V e^{j\omega t} \right) \]

- Note: drop the frequency term, \( e^{j\omega t} \)
- Note: use \( \cos() \) not \( \sin() \) (… to ensure real functions, or signals)

Simplify math … with complex numbers

- Express the complex number in
  - Rectangular form: \( x + jy \)
  - Polar form:
  - Exponential form:
  - Using Euler’s formula:

- Important conversions (plot these numbers)
  - \( j' = \ldots^{\circ} \) (degrees)
  - \( 1/j = \ldots^{\circ} \)
  - Note that \( 1/j = -j \)

Complex Math Examples

- Simplify and express in rectangular form

\[ \frac{2 + \frac{3 + j4}{5 - j8}}{4\angle -10^\circ} + \left( \frac{1 - j2}{3\angle 6^\circ} \right) \]
**Phasor Problem**

- Write the sinusoid form for the following
  \[ V = 60 \angle 15^\circ \ V, \ \omega = 377 \]

- \[ V = 2.8e^{-j\pi/3} \ A, \ \omega = 10^3 \]

**Phasor Problem**

- Find a single sinusoid for the following
  \[ V = -30 \angle 10^\circ + 50 \angle 60^\circ \]

**Phasor Problem**

- Find a single phasor for the following
  \[ 3\cos(20t + 10^\circ) - 5\sin(20t - 30^\circ) \]

**Leading & lagging phasors**

- Plot the voltage and current phasors in the complex plane
  a) \[ V = V_m \angle \varphi; \ I = I_m \angle \delta \]
  b) \[ V = 10 \angle 30^\circ; \ I = 5 \angle -90^\circ \]
Time v. Phasor Domain

- A phasor is a special vector
  - Be familiar with Euler’s formula: the basis for converting between time and phasor domains
- \( v(t) = (A_1e^{s_1t} + A_2e^{s_2t}) \) V
  - IS time dependent
  - Is always a real number
- \( V \) (phasor)
  - IS NOT time dependent
  - Typically is a complex number

Complex Algebra (for Phasors)

- Polar form
  - Easy to multiply and divide
- Rectangular form
  - Easy to add and subtract
  - Tedious to multiply and divide
- Complex conjugates
  - If \( z = x + jy \), which also = \( r \angle \phi = re^{j\phi} \)
  - Then \( z^* = \)

Today’s Concepts

- AC input signals to circuits & circuit behavior
- Sinusoids
  - Euler’s identity and sinusoids
- Complex numbers in all their forms
  - Converting between representations
- Phasors (~ vectors)
  - Simple representation of sinusoids
- Phasor ‘domain’ and transformation

For Reference: Properties page 373

- \( \sin(A \pm B) \) and \( \cos(A \pm B) \)
- \( \cos(\omega t \pm 90^\circ) = -/+ \sin(\omega t) \)
  - Or \(-/+ \sin(\omega t) = \cos(\omega t \pm 90^\circ) \)
  - So \(- \sin(\omega t) = \cos(\omega t + 90^\circ) \)
  - \( \sin(\omega t) = - \cos(\omega t + 90^\circ) \)
  - \(+ \sin(\omega t) = \cos(\omega t - 90^\circ) \)
- Also \( \sin(\omega t \pm 90^\circ) = \pm \cos(\omega t) \)
- And \( \cos(\omega t \pm 180^\circ) = - \cos(\omega t) \)
Images for the Series RLC Step-Response Lab
(Lab 7, Part 1)

Note that Part 2 uses phasors – collect voltage magnitude and phase data.

Notes for Series RLC lab circuit

- Series RLC circuit
  - L becomes a short circuit so in steady-state $V_L = 0$
  - C becomes an open circuit so $V_C = V_S$
  - R becomes what? $V_R = \text{what?}$

![Series RLC circuit diagram]
Examples of truncating the natural response in a time (seconds) which leads to interesting shapes as the superposition of the natural response + next step input add together.
Given position of truncating the natural response, compare the overshoot of this versus next slide.
Note at resonance, the ‘overshoot’ is all we see, and with the
+/- summing of VC + VL, can get huge amplitude. Need to rescale V/div
on vertical scale

Note: VL settles to 0V since L becomes a short circuit (previous slides)
Images from the Series RLC Frequency Response (Lab 7, Part 2)

Chapter 14 focuses on this topic. Need concept of phase shifting for midterm 2, but not full ‘Bode Plot’ from Lab 7.

Series RLC Frequency Response

- Series RLC circuit with \( \cos() \) input
  - \( L \) is a short circuit in DC conditions (low frequency) but with a high frequency signal, it behaves as ... ?
  - \( C \) is an open circuit in DC conditions, but with a high frequency signal... ?

Sinusoid Signals in a Circuit

- Voltage signals across energy storage circuit elements will often be phase shifted with respect to the input (source) (As well as have a different amplitude)
Plotting Lab 7 Data

- **magnitude Vc**
  - ![Graph](image1)

- **phase Vc**
  - ![Graph](image2)

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**Plotting Lab 7 Data**

- ![Graph](image3)

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**Plotting Lab 7 Data**

- ![Graph](image4)