Problem 1: Circuits

For \( t < 0 \) the circuit is

For \( t = 0 \) we have

For the transient period between \( t < 0 \) and \( t = 0 \) we have
Series RLC source-free response

\[ t < 0 \text{ find } V_0 = 15.4 \times 60V \quad \text{since } i_0(0) = 0 \]

\[ t = 0 \quad V_0 = 0V \]

also \( t < 0 \)

\[ \frac{dV(t)}{dt} \bigg|_{t=0} = \frac{i_c(0)}{C} = \frac{i_l(0)}{C} = 0 \text{ V/s} \]

Now for \( t > 0 \), find \( \alpha + \omega_0 \)

Series \( \alpha = \frac{R}{2L} = \frac{10}{2 \times 4} = 2.5 \quad \omega_0 = \sqrt{\frac{1}{L} \cdot \frac{C}{400}} = 10 \)

\( \alpha > \omega_0 \) so \( s_1, s_2 = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \)

\[ -2.5 \pm \sqrt{2.5^2 - 10^2} = -3.73, -2.68 \]

for an overdamped response soln is of form \( A_1 e^{s_1t} + A_2 e^{s_2t} = v(t) \)

Complete soln

\[ v(t) = \ldots + A_1 e^{-3.73t} + A_2 e^{-2.68t} \]

\[ v(0) = 60 = A_1 + A_2 \quad A_1 = 60 - A_2 \]

\[ \frac{dv}{dt} \bigg|_{t=0} = 0 = -3.73A_1 - 2.68A_2 \]

\[ 0 = -3.73(60 - A_2) - 2.68A_2 \]

\[ 238 = 34.6A_2 \quad \text{so } A_2 = 6.90 \]

\[ A_1 = -4.68 \]

So \( v(t) = -4.68e^{-3.73t} + 64.68e^{-2.68t} \text{ V} \)

This is the natural response, which is also the complete response since this circuit has no forcing function for \( t > 0 \)
Problem 1 Graph

HW 6 #1: RLC Circuit Overdamped Response

\[-4.68e^{37.7t} + 64.68e^{2.68t}\ V\]
This 2nd order diff eq defines, or describes, the behavior of a 2nd order circuit, \( \Rightarrow \) a circuit w/ 1-R, 1-L + 1-C

Assume \( R = 2k\Omega \). Find \( L + C \)

**Solution**

From §8.6, eq'n (8.47) we see that

\[
\frac{d^2 i}{dt^2} + \frac{1}{RC} \frac{di}{dt} + \frac{1}{LC} i = \frac{Is}{LC} \\
\text{substitute } i(t) = Ae^{st}
\]

Or from §8.4 eq'n (8.29)

\[
\frac{d^2 v}{dt^2} + \frac{1}{RC} \frac{dv}{dt} + \frac{1}{LC} v = 0 \Rightarrow Ae^{st}(s^2 + \frac{1}{RC} s + \frac{1}{LC}) = 0
\]

Matching the coefficients from either of these equations to the eq'n for our design circuit we see

\[
\frac{1}{RC} = 100 \quad \text{and} \quad 10^6 = \frac{1}{LC}
\]

\[
R = 2k \quad \text{so} \quad C = \frac{1}{100 \cdot 2k} = \boxed{5\mu F} \quad \text{and} \quad L = \frac{1}{10^6 \cdot 5 \times 10^{-6}} = \boxed{0.2H}
\]

**Circuit** — This circuit is a source-free circuit. To make it not have zero-energy for all time, assume there is a charging source (for \( t < 0 \))

---

**Diagram**

![Circuit Diagram](image)
Now for plotting again we need to assume some initial stored energy, which is provided by the source, \( I_s u(t) \), that charged the \( E \)-field for the capacitor and the \( B \)-field for the inductor. This is so we can have a circuit response that is not simply zero for all time.

Now you have 2 decisions

1. Will the circuit response expression you develop be for \( i(t) \) or \( v(t) \). Either will work. § 8.4 uses \( v(t) \), § 8.6 uses \( i(t) \).

2. What source value (for charging \( C + L \)) will you select, and will it be a current source or a V-source?

I will select \( I_s = 2A \). Note that choosing a voltage source will be more problematic, so don’t make things harder than they need to be.

\( t < 0 \) : L is a short circ, so all 2A flow through it (\( i_L(0^+) = 2A \))

Since L shorts out C, \( v_c(0) = 0V \)

Now we need either \( \frac{di_L(0^+)}{dt} \) or \( \frac{dv_c(0^+)}{dt} \)

\( @ t = 0^+ \)

\( v_L = v_c = L \frac{di_L}{dt} \) so \( \frac{di_L(0^+)}{dt} = \frac{1}{L} v_c(0^+) = (0 \text{A}) \)

\( 0 = i_L + i_R + i_c \); \( \frac{dv}{dt} = \frac{1}{C} i_c = \frac{1}{C} (-i_L - i_R) = \frac{1}{5 \mu} (-2 - 0) = -4 \times 10^5 \text{V/s} \)
Find roots, \( s_1 + s_2 \), so find \( \alpha + \omega_0 \)

For parallel RLC, \( \alpha = \frac{1}{2RC} = 50 \)

\[ \omega_0 = \frac{1}{\sqrt{LC}} = 10^3 \]

\( \omega_0 > \alpha \) so this is an underdamped system

\[ \omega_d = \sqrt{\omega_0^2 - \alpha^2} = 998.7 \]

For current response, note source-free so \( i(0) = 0 \)

\[ i(t) = i(0) + e^{-\alpha t} (A_1 \cos \omega_0 t + A_2 \sin \omega_0 t) \]

\( i(0) = 0 = e^0 (A_1 \cos 0 + A_2 \sin 0) \Rightarrow A_1 = 0 \)

\[ \frac{di}{dt} = 0 = -\alpha e^{-\alpha t} (A_1 \cos + A_2 \sin) + e^{-\alpha t} (\omega_0 (A_1) \sin + \omega_0 A_2 \cos) \]

(at \( t = 0 \)) \(-50 (A_1 + 0) + (1)(0 + 998.7 A_2) \)

So \( 50A_1 = 998.7 A_2 \) or \( A_2 = 0.1 \)

\[ i(t) = e^{-50t} \left( 2 \cos 998.7 t + 0.1 \sin 998.7 t \right) \]

For the voltage response, again no source-free so \( V(0) = 0 \)

\[ v(t) = e^{-\alpha t} (A_1 \cos \omega_0 t + A_2 \sin \omega_0 t) \]

\( v(0) = 0 = A_1 \)

\[ \frac{dv(0)}{dt} = -4 \times 10^5 = -50 A_1 + 998.7 A_2 \Rightarrow A_2 = -400 \]

So \( v_c(t) = e^{-50t} (-400 \sin 998.7 t) \)

\[ v_c(t) = -400 e^{-50t} \sin 998.7 t \]
Problem 2 Graphs:

For these two graphs notice:

- The frequency of oscillation is the same for both \( i(t) \) and \( v(t) \)
- The initial values and amplitudes for \( i(t) \) and \( v(t) \) are 2 orders of magnitude different!!
- One of these graphs is a sine wave and the other is a cosine wave.
- The outer 'envelope' of the exponential decay of the oscillations is the same for both current and voltage, and is equal to \( e^{-\alpha t} = e^{-50t} \)
Circuits for problem 3

$t < 0$

$t = 0$

transient period we have
3) Parallel RLC Circuit - src-free

\( v_c(0) = 0 \), \( i_c(0) = 12A \)

\[ \left. \frac{di_c(t)}{dt} \right|_{t=0} = \frac{v_c(0)}{L} = \frac{0}{L} \]

\[ \left. \frac{dv_c(t)}{dt} \right|_{t=0} = \frac{i_c}{C} = \frac{12}{10m} = 1.2K \frac{V}{S} \]

\[ i_c = i_r + i_c = \frac{v_c}{R} + i_c \text{ so } i_c = 12 \]

\( V_c = V_r \) \( v_c \left|_{t=0} = 0 \right. \)

\( t > 0 \)

Find \( i(t) \): src free so \( v_c = 0 \)

\( \alpha + \omega_0 \) : \( \alpha = \frac{1}{2RC} = \frac{1}{2 \times 10 \times 10m} = 5 \)

\( \omega_0 = \sqrt{\frac{1}{25 \times 10m}} = 20 \)

\( \alpha < \omega_0 \) so is underdamped

solution of form \( i(t) = e^{-\alpha t} (A_1 \cos \omega dt + A_2 \sin \omega dt) \)

\( \omega_d = \sqrt{\omega_0^2 - \alpha^2} = 19.4 \)

\[ i(t) = e^{-5t} (A_1 \cos 19.4t + A_2 \sin 19.4t) + A_0 \]

Solve for \( A_1 + A_2 \):

\( i(0) = 12 = (A_1 \cdot 1 + 0) \) so \( A_1 = 12 \)

\[ \left. \frac{di(t)}{dt} \right|_{t=0} = 0 = (1)(19.4A_2) + (-5)(A_1) \) so \( A_2 = +3.09 \)

\[ i(t) = e^{-5t} (12 \cos 19.4t + 3.09 \sin 19.4t) \]
Problem 3 graph:

See if you can determine the expression for the exponential decay envelope in this graph.
Circuits for problem 4

for $t < 0$

\[ 12V \quad 4\Omega \quad 5\Omega \quad 3A \]

for $t = 0$

\[ 5\Omega \quad 3A \]

for the transient period

\[ 5A \quad \frac{1}{20}F \quad 5\Omega \quad 3A \]
4) Step response for parallel RLC circuit

\( t < 0 \)
\[
\begin{align*}
\left. i(t) \right|_{t=0} &= \frac{12}{7} + 3 = 6A \\
\left. \frac{di}{dt} \right|_{t=0} &= \frac{V_i}{L} = \frac{V_c}{L} = 0
\end{align*}
\]
\( u(0) = 0 \) (because \( C \) is shorted out by \( L \))
\[
\begin{align*}
\frac{dv}{dt} \bigg|_{t=0} &= \frac{i_c}{C} = \frac{i_s - i_r - i_L}{C} = \frac{3 - \frac{0}{5} - 6}{\frac{V_0}{5}} \\
i_s &= i_r + i_c + i_L = \frac{V_0}{R} + i_c + i_L = \frac{0}{R} + i_c + 6 \Rightarrow
\end{align*}
\]
so \( i_c = -3 \)

\( t = \infty \)
\[
\left. i(t) \right|_{t=\infty} = 3A
\]

\( t > 0 \)

find \( \alpha + \omega \)
\[
\alpha = \frac{1}{2RC} = \frac{1}{2 \cdot 5 \cdot \frac{V_0}{5}} = 2 \quad \omega = \frac{1}{\sqrt{\frac{L}{m}}} = 2
\]
\[
\alpha = \omega \Rightarrow \text{Critical damping} \quad \Rightarrow S_1, S_2 = -\alpha \pm \sqrt{\alpha^2 - 4 \alpha} = -2
\]

so \( i(t) = I_0 + e^{-\alpha t} (A_1 + A_2 t) \)

solve for \( A_1 + A_2 \)
\[
\left. \frac{di}{dt} \right|_{t=0} = 0 = (1)A_2 + (-2)(A_1) \quad \Rightarrow A_1 = 3
\]
\[
\frac{di}{dt} \bigg|_{t=0} = 0 = (1)A_2 + (-2)(A_1) \quad \Rightarrow A_2 = 6
\]

\[
i(t) = 3 + e^{-2t} (3 + 6t) A
\]
Problem 4:

For \( t = 0^+ \), \( i(0) = 3 + 12/4 = 6 \) and \( v(0) = 0 \).

For \( t > 0 \), we have a parallel \( RLC \) circuit with a step input.

\[
\alpha = \frac{1}{2RC} = \frac{1}{2 \times 5 \times 0.05} = 2
\]

\[
\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{5 \times 0.05}} = 2
\]

Since \( \alpha = \omega_0 \), we have a critically damped response.

\[
s_{1,2} = -2
\]

Thus,

\[
i(t) = I_s + [(A + Bt)e^{-2t}], \quad I_s = 3
\]

\[
i(0) = 6 = 3 + A \quad \text{or} \quad A = 3
\]

\[
v = L\frac{di}{dt} \quad \text{or} \quad v/L = \frac{di}{dt} = [Be^{-2t}] + [-2(A + Bt)e^{-2t}]
\]

\[
v(0)/L = 0 = di(0)/dt = B - 2 \times 3 \quad \text{or} \quad B = 6
\]

Thus,

\[
i(t) = \{3 + [(3 + 6t)e^{-2t}]\} A
\]
Problem 4 Circuit Response Graph:

Examine this graph and see if you can identify what makes it different from a simple exponential decay.
5) **Initial & Final Conditions**

- **t < 0**
  - C open
  - L short
  - 2A src is off

Ponder the energy that is in this circuit. There is no energized path for current to flow, yet the 10V source is active → it is very actively maintaining the electric field that charges the capacitor. The + terminal of this 10V source is connected to the - terminal for \( V_c \).

Therefore

\[
I_c(0^-) = 0A \neq I_c(0^+)
\]

\[
V_c(0^-) = -10V = V_c(0^+)
\]

To find \( V_R(0) \) we must advance to time \( t > 0 \). Though we could find \( V_c(0^-) \), this tells us nothing about what we really want, which is \( V_c(0^+) \) so don’t bother with \( V_c(0^-) \).

\[ t > 0 \]

![Diagram]

At time \( t = 0^+ \) we know:

1. \( V_c(0^+) = -10V \) and \( V_s = +10V \)
2. \( V_c + V_s = 0 \) and \( (V_c + V_s) \) is in parallel with \( V_R \)

So ……. \( V_R(0^+) = 0V \)
Problem 5 cont’d ⇒ on to the initial values for the first deriv.

\[
\left. \frac{di_c}{dt} \right|_{t=0^+} = \left. \frac{V_e}{L} \right|_{t=0^+} \quad \text{so we need } V_e(0^+) \]

All branches are in parallel, so again

\[ V_e(0^+) = V_c(0^+) + V_s(0^+) = -10 + 10 = 0 \text{V} \]

Therefore

\[
\frac{V_c(0^+)}{L} = \frac{0}{L} = 0 \text{A/s} = \frac{di_c(0^+)}{dt} \]

\[
\left. \frac{dV_c}{dt} \right|_{t=0^+} = \frac{i_c}{C} \left. \right|_{t=0^+} \quad \text{so we need to find } i_c(t) \text{ at time } t=0^+ \]

In seeking a current, KCL will probably help

\[
\text{KCL} \quad \Sigma I_{in} = \Sigma I_{out} \]

\[ 2 = I_{10A} + I_c + I_L \quad \text{note } I_L = I_{4A} \quad R \]

we know \( i_L(0^+) \) and want to find \( i_c(0^+) \). Can we find \( I_{10A} \) at \( t=0^+ \)? One thing we know for certain about \( I_R \) is Ohm’s Law ⇒ \( V_R = IR \)  Hmm, can we find \( V_R \)?

As above, we know for the middle branch \( V = V_c + V_s \) AND we know \( V_R = V_c + V_s = 0 \text{V} \) so we have \( V_R(0^+) = 0 \text{V} \) (as above)

Back to \( i_R(0^+) = \frac{V_R(0^+)}{R} = \frac{0}{R} = 0 \)

And further back to finding \( i_c(0^+) = 2 - i_R(0^+) - i_L(0^+) \)

\[ = 2 - 0 - 0 = 2 \text{A} \]

Wow! This means that at the moment of time \( t=0^+ \)

all the 2A from the current source are flowing through the capacitor ... so forcing its way through the 10V source as well.

\[
\text{Finally } \frac{dV_c}{dt} \left. \right|_{t=0^+} = \frac{i_c}{C} \left. \right|_{t=0^+} = \frac{2}{14} = \frac{8}{5} = \frac{dV_c(0^+)}{dt} \]
HW problem 5

Now to \( \frac{dv_R(t)}{dt} \)

What do we know, for certain, about \( V_R \)?
We know this resistor is in parallel with the \( V_C + V_S \) branch:
\[ V_R = V_C + V_S \]

Basic calculus tells us, therefore
\[ \frac{dv_R}{dt} = \frac{d}{dt} (V_C + V_S) = \frac{dv_C}{dt} \]

So this is true for all time, including time \( t = 0^+ \)
\[ \frac{dv_C(t)}{dt} = 8 V/S \]

Find Final Conditions at time \( t = \infty \)

@ \( t = \infty \), Cap = open
inductor = short

Superposition seems like a great approach, because we already have \( V_C, V_R + i_L \) from \( 10V \) source, from \( t = 0^+ \)
Now, add in the effect from the \( 2A \) source that turned on at \( t = 0^+ \)
Use current divider to find \( i_R + i_L \)
\[ i_L = 2 \left( \frac{10}{50} \right) = \frac{2}{5} A \]
\[ i_R = 2 \left( \frac{40}{50} \right) = \frac{8}{5} A \quad \text{and so} \quad V_R = i_R R = \left( \frac{8}{5} \right) 10 = \frac{80}{5} = 16V \]

As we have used repeatedly \( V_{\text{total}} = V_C + V_S \) \( \therefore V_C = 16 - 10 = 6V \)
Combining the energy from both sources at \( t = \infty \)

\[ i_i(\infty) = 0 + \frac{2}{5} = 0.4 \text{ A} \]

\[ V_c(\infty) \Rightarrow V_c(\infty) \text{ from } 2A = V_K(\infty) = 16 \]

\[ V_c(\infty) = -10 + 16 = 6 \text{ V} \]

\[ V_K(\infty) = 0 + 16 = 16 \text{ V} \]