CHAPTER 6 or INV (formerly PB)

INVARIANT MANIFOLDS

10/17/99

I just gutted this chapter, formerly PB, of the proof of Poincaré-Birkhoff . I intend to put survey sections on KAM and separatrices and their splitting (first and last sections).Sections will have to be revised.This could include proofs that KAM tori are Lagrangian, and orbits on Lagrangian graphs are minimizers. I might need to move the "ratchet" proposition to Chapter AM

35. The Theory of Kolmogorov–Arnold–Moser

Will contain the precise statement of the theorem, an idea of the scheme of proof, and proofs that these tori are Lagrangian and made of minimizers.

KAM theory, which proves the existence of many invariant tori for systems close to integrable, is one of the greatest achievements in Hamiltonian dynamics. It has historical roots going back to Weierstrass who, in 1878, wrote to S. Kovalevski that he had constructed formal power series for quasi-periodic solutions to the planetary problem. The denominators of the coefficients of these series involved integer combinations of the frequencies of rotation of the planets around the sun, which could be close to zero and hence impeded the convergence of the series. Weierstrass advised Mittag-Leffler to make this problem of convergence a question for a prize sponsored by the king of Sweden. In the 271 pages work (Poincaré (1890)) for which he won the prize, Poincaré does not solve the problem completely, and his tentative answer to the convergence is negative. In Poincaré (1899) , he speculates on the possibility of such a convergence, given appropriate number theoretic conditions, but still deems it improbable. It was therefore a significant event when Arnold (1963) (in the analytic, Hamiltonian context) and Moser (1962) (in the differentiable twist map context) gave, following the ideas of Kolmogorov (1954) a proof of existence of quasi-periodic orbits on invariant tori filling up a set of positive measure in the phase space. We can only give here a very limited account of this complex theory, and refer to Moser (1973) and de la Llave (1993) for introductions as well as Bost (1986) for an excellent survey and bibliography. There are many KAM theorems, the most applicable ones being often the hardest ones to even state. We present here a relatively simple statement, cited in Bost (1986).

Theorem 35.1 (KAM for symplectic twist maps) *Let f_0 be an integrable symplectic twist map of $\mathbb{T}^n \times \mathbb{D}^n$ of the form:*

$$f_0(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{q}+\omega(\boldsymbol{p}),\boldsymbol{p})$$

where $\mathbb{D}^n$ is a disk in $\mathbb{R}^n$ and $\omega : \mathbb{D}^n \mapsto \mathbb{R}^n$ is C^∞ (since f_0 is twist, $D\omega$ is invertible). Let $\boldsymbol{p}_0$ be an interior point of $\mathbb{D}^n$. Suppose that the following condition is satisfied:
Diophantine condition: *there are positive constants τ and c such that:*

$$\text{(35.1)} \qquad \forall k \in \mathbb{Z}^{n+1}\backslash\{0\}, \quad \left|\sum_{j=1}^{n} k_j\omega_j(\boldsymbol{p}_0) + k_{n+1}\right| \geq c\left(\sum_{j=1}^{n+1}|k_j|\right)^{-\tau}$$

Then there is a neighborhood W of f_0 of C^∞ exact symplectic maps such that, for each $f \in W$, there exists an embedded invariant torus $\mathbb{T}_f \simeq \mathbb{T}^n$ in the interior of $\mathbb{T}^n \times \mathbb{D}^n$ such that:
(i) $\mathbb{T}_f$ is a C^∞ Lagrangian graph over the zero section
(ii) $f\big|_{\mathbb{T}_f}$ is C^∞ conjugated to the rigid translation by $\omega(\boldsymbol{p}_0)$
(iii) $\mathbb{T}_f$ and the conjugacy depend C^∞ on f.
Moreover the measure of the complement of the union of the tori $\mathbb{T}_f(\boldsymbol{p}_0)$ goes to 0 as $\|f - f_0\|$ goes to 0.

Remark 35.2
1) The diophantine condition (35.1) is shared by a large set of vectors in $\mathbb{R}^n$. As an example, when $n = 1$, the set of real numbers $\mu \in [0, 1]$ such that $|\mu - p/q| > K/q^3$ for some K is dense in $[0, 1]$ and has measure going to 1 as K goes to 0.
2) The most common versions of KAM theorems concern Hamiltonian systems with a Legendre condition. In Chapter 6 we show the intimate relationship of such Hamiltonian systems with symplectic twist maps . It therefore comes as no surprise that KAM theorems have equivalents in both categories of systems. Note that there are *isoenergetic* versions of the KAM theorem for Hamiltonian systems, where the existence of many invariant tori is proven in a prescribed energy level.
3) One important contribution in Moser (1962) was his treatment of the finitely differentiable case: he was able to show a version for $n = 1$ (twist maps) where f_0 and its perturbation are $C^l, l \geq 333$ instead of analytic. This was later improved to $l > 3$ and in higher dimension n, to $l > 2n + 1$ (at least if the original f_0 is analytic).
4) There is a version of the KAM for *non symplectic* perturbations of completely integrable maps of the annulus, called the *Theorem of translated curves*, due to Rüssmann (1970). It states that, around an invariant circle for f_0 whose rotation number ω satisfies the diophantine condition (35.1) (only one j in this case), there exists a circle invariant by $t_a \circ f$ for a perturbation f of f_0 and $t_a(x, y) = (x, y + a)$ which has same rotation number as the original (i.e. the map f has flux $-a$).
5) One may wonder if, among all invariant tori of a symplectic twist map close to integrable, the KAM tori are typical. KAM theory says that in measure, they are. However Herman (1992a) (see also Yoccoz (1992)) shows that, for a generic symplectic twist map close to integrable, there is a residual set of invariant tori on which the (unique) invariant measure has a support of Hausdorff dimension 0. Things get even worse when the differential $D\omega$ in Theorem 35.1 is not positive definite: there may be many invariant tori that project onto, but are not graphs over the 0–section, and this for maps arbitrarily close to integrable (see Herman (1992 b)).

6) KAM theory implies the stability of orbits on the KAM tori, hence stability with high probability. But in "real situations" it is impossible to tell whether motion actually takes place on a KAM torus. Nekhoroshev (1977) provides an estimate of how far a trajectory can drift in the momentum direction over long periods of time: If $H(\boldsymbol{q},\boldsymbol{p}) = h(\boldsymbol{p}) + f_\varepsilon(\boldsymbol{q},\boldsymbol{p})$ is a real analytic Hamiltonian function on $T^*\mathbb{T}^n$ with $f_\varepsilon < \varepsilon$ (a small parameter) and $h(\boldsymbol{p})$ satisfies a certain condition (steepness) implied by convexity, then there exist constants ε_0, R_0, T_0 and a such that, if $\varepsilon < \varepsilon_0$, one has:

$$|t| \leq T_0 \exp[(\varepsilon_0/\varepsilon)^a] \Rightarrow |\boldsymbol{p}(t) - \boldsymbol{p}(0)| \leq R_0(\varepsilon/\varepsilon_0)^a.$$

With a (quasi) convexity condition instead of the steepness condition, Lochak (1992) and Pöshel (1993) showed that the optimal a is $\frac{1}{2n}$. Delshams & Gutiérrez (1996a)presents unified proofs of the KAM theorem and Nekhoroshev estimates for analytic Hamiltonians.

Whereas we cannot give a proof of the KAM theorem in this book, the following theorem (Arnold (1983)) offers a simple model in a related situation in which the KAM method can be applied in a less technical way.This will allow us to sketch very roughly the central ideas of the method.

Theorem 35.3 *There exists $\varepsilon > 0$ depending only on K, ρ and σ such that, if a is a 2π –periodic analytic function on a strip of width ρ, real on the real axis with $a(z) < \varepsilon$ on the strip and such that the circle map defined by*

$$f : x \mapsto x + 2\pi\mu + a(x)$$

is a diffeomorphism with rotation number μ satisfying the diophantine condition:

$$|\mu - p/q| > \frac{K}{q^{2+\sigma}}, \quad \forall\ p/q \in \mathbb{Q}$$

then f is analytically conjugate to a rotation R of angle $2\pi\mu$

Sketch of proof: We seek a change of coordinates $H : \mathbb{S}^1 \to \mathbb{S}^1$ such that:

(35.2) $$H \circ R = f \circ H$$

write $H(z) = z + h(z)$, with $h(z + 2\pi) = h(z)$. Then (35.2) is equivalent to

(35.3) $$h(z + 2\pi\mu) - h(z) = a(z + h(z)).$$

Since $a(z) < \varepsilon$, h must be of order ε as well and thus, in first approximation, (35.3) is equivalent to:

(35.4) $$h(z + 2\pi\mu) - h(z) = a(z)$$

Decomposing $a(z) = \sum a_k e^{i2\pi kz}$, $h(z) = \sum b_k e^{ikz}$ in their Fourier series and equating coefficients on both sides of (35.4) we obtain:

$$b_k = \frac{a_k}{e^{i2\pi k\mu} - 1}$$

where we see the problem of small divisors arise: the coefficients b_k of h may become very big if μ is not sufficiently rational.

It turns out that, assuming the diophantine condition and using an infinite sequence of approximate conjugacies given by solutions of (35.4) , one obtains sequences h_n, a_n and corresponding $H_n, f_n = H_n^{-1} \circ f \circ H_n$ which converge to H, R for some H. The domain of h_n and f_n is a strip that shrinks with n but in a controllable way. This iterative process of "linear" approximations to the conjugacy can be interpreted as a type of Newton's method for the implicit equation $\mathcal{F}(f, H) = H^{-1} \circ f \circ H = R$ (given f, find H) and inherits the quadratic convergence of the classical Newton's method: $R - \mathcal{F}(f_n, H_n) = O(\varepsilon^{2n})$ (see Hasselblat & Katok (1995) Section 2.7.b). □

36. Properties of Invariant Tori

The previous section showed the existence of many invariant tori for symplectic twist maps close to integrable. These tori are Lagrangian graphs with dynamics conjugated to quasi–periodic translation. In dimension 2, the Aubry-Mather theorem gives an answer to the question of what happens to these tori when they break down, in large perturbation of integrable maps. In higher dimension, Mather's theory of minimal measure also provides an answer to that question (see Chapter AMG). In this section, we look for properties that invariant tori may have whether they arise from KAM or not. We will see that certain attributes of KAM tori (*eg.* graphs with recurrent dynamics) imply their other attributes (*eg.* Lagrangian), as well as other properties not usually stated by the KAM theorems (minimality of orbits).

Recurrent Invariant Toric Graphs are Lagrangian

Theorem 36.1 (Hermann (1990)) *Let T be an invariant torus for a symplectic twist map f of $T^*\mathbb{T}^n$ and suppose $f\big|_T$ is conjugated by a diffeomorphism h to a an irrational translation R on $\mathbb{T}^n$. Then T is Lagrangian.*

Proof. Since the 2-form $\omega\big|_T$ is invariant under $f\big|_T$ and since $R = h^{-1} \circ f\big|_T \circ h$, the 2-form $h^*\omega\big|_T$ is invariant under R. Since R is recurrent, $h^*\omega\big|_T = \sum_{i,j} a_k j dx_k \wedge dx_j$ must have constant coefficients a_k. Integrating $h^*\omega\big|_T$ over the x_k,x_j subtorus yields on one hand a_{kj}, on the other hand 0 by Stokes' theorem since $h^*\omega\big|_T = dh^*\lambda\big|_T$ is exact. Hence $h^*\omega\big|_T = 0 = \omega\big|_T$ and the torus T is Lagrangian. □

Orbits on Lagrangian Invariant Tori as Minimizers

The following theorem is attributed to Herman by MacKay & al. (1989), whose proof we reproduce here.

Theorem 36.2 *Let T be torus, C^1 graph over the zero section of $T^*\mathbb{T}^n$ which is invariant for a symplectic twist map f which satisfies the convexity condition (29.1) :*

$$\langle \partial_{12} S_k(\boldsymbol{q}, \boldsymbol{Q}).\boldsymbol{v}, \boldsymbol{v} \rangle \leq -a \left\| \boldsymbol{v} \right\|^2, \quad \forall \boldsymbol{q}, \boldsymbol{Q}, \boldsymbol{v} \in \mathbb{R}^n, k \in \{1, \dots, N\}.$$

Then any orbit on T is minimizing.

Proof. Since T is Lagrangian, it is the graph of the differential of some function plus a constant 1-form: $T = dg(\mathbb{T}^n) + \beta$ (see SGexolagraph). Let $\psi(\boldsymbol{q}) = \pi f(\boldsymbol{q}, dg(\boldsymbol{q}) + \beta)$ and

$$R(\boldsymbol{q}, \boldsymbol{Q}) = S(\boldsymbol{q}, \boldsymbol{Q}) + g(\boldsymbol{q}) - g(\boldsymbol{Q}) + \beta(\boldsymbol{q} - \boldsymbol{Q}).$$

We now show that R is constant on T, where it attains its minimum. Following Mather, we first note that:

$$\partial_1 R(\boldsymbol{q}, \boldsymbol{Q}) = \partial_1 S(\boldsymbol{q}, \boldsymbol{Q}) + dg(\boldsymbol{q}) + \beta = 0 \Leftrightarrow \boldsymbol{p} = dg(\boldsymbol{q}) + \beta \Leftrightarrow \boldsymbol{Q} = \psi(\boldsymbol{q})$$
$$\partial_2 R(\boldsymbol{q}, \boldsymbol{Q}) = \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}) - dg(\boldsymbol{Q}) - \beta = 0 \Leftrightarrow \boldsymbol{P} = dg(\boldsymbol{Q}) - \beta \Leftrightarrow \boldsymbol{Q} = \psi(\boldsymbol{q})$$

Hence $R(\boldsymbol{q}, \psi(\boldsymbol{q})) = R_0$ is constant , and $DR(\boldsymbol{q}, \boldsymbol{Q})$ is non zero if $\boldsymbol{Q} \neq \psi(\boldsymbol{q})$. In Lemma STMPlemconvquad, we proved that, when (29.1) holds, the generating function satisfies the following quadratic growth:

$$S(\boldsymbol{q}, \boldsymbol{Q}) \geq \alpha - \beta \|\boldsymbol{q} - \boldsymbol{Q}\| + \gamma \|\boldsymbol{q} - \boldsymbol{Q}\|^2 . \tag{36.1}$$

where γ is given by $a/2$ in the convexity condition. Since $\partial_{12} R = \partial_{12} S$, the same kind of quadratic estimate holds for R which is thus bounded below. Since R has all its critical points on T, it must attain its minimum there. It is now easy to see that the $\boldsymbol{q}$ coordinates $\boldsymbol{q}_n, \ldots, \boldsymbol{q}_k$ of any orbit segment on T must minimize the action. Let $r_n, \ldots, r_k$ another sequence of points of $\mathbb{T}^n$ with $\boldsymbol{q}_n = r_n, \boldsymbol{q}_k = r_k$. Then:

$$\begin{aligned} W(r_1, \ldots, r_k) &= \sum_{j=n}^{k-1} R(r_j, r_{j+1}) + g(\boldsymbol{q}_k) - g(\boldsymbol{q}_n) + \beta(\boldsymbol{q}_k - \boldsymbol{q}_n) \\ &\geq (k-n) H_0 + g(q_k) - g(q_n) + \beta(\boldsymbol{q}_k - \boldsymbol{q}_n) = W(q_1, \ldots, q_k) \end{aligned}$$

□

Remark 36.3 Arnaud (1989) (see also Hermann (1990)) has interesting examples which show that the condition that the graph be Lagrangian is essential in Theorem 36.2. Consider the Hamiltonians on $T^*\mathbb{T}^2$ is given by:

$$H_\varepsilon(q_1, q_2, p_1, p_2) = \frac{1}{2}(p_1 - \varepsilon \cos(2\pi q_2))^2 + \frac{1}{2} p_2^2.$$

The torus $\{(q_1, q_2, \varepsilon \cos(2\pi q_2), 0)\}$ is made of fixed points for the corresponding Hamiltonian system, but it is not Lagrangian (exercise). A further perturbation $G_{\varepsilon,\delta}(\boldsymbol{q}, \boldsymbol{p}) = H_\varepsilon(\boldsymbol{q}, \boldsymbol{p}) + \delta \sin(2\pi q_2), 0 < \delta \leq \varepsilon$ of these Hamiltonians also provide counterexamples to the strict generalization of the Aubry-Mather theorem to higher dimensions: such systems have no minimizers of rotation vector 0. All the fixed points for the time 1 map have non trivial elliptic part.

Graph Theorem

Theorem 36.4 (Birkhoff) *Let f be a twist map of the cylinder $\mathcal{A}$. Then:*

(1) **(Graph Theorem)** *Any invariant circle which is homotopic to the circle $C_0 = \{y = 0\}$ is a (Lipschitz) graph over C_0.*

(2) If two invariant circles C_- and C_+ homotopic to $\{y = 0\}$ bound a region without other invariant circles, for any ϵ, there are (uncountably many) orbits going from ϵ-close to $C_\pm$ to ϵ-close to $C_\mp$.

This theorem was proved as two independent theorems by Birkhoff (1920).

Proof. For both (1) and (2), we can assume the existence of an invariant circle, say C_+. Take any circle C which is a graph over C_0 and which lies under C_+ . The image $f(C)$ of this circle may not be a graph anymore, but one can make a *pseudo-graph* $UF(C)$ by *trimming* it: take all the points of $F(C)$ that can be "seen" vertically from above. This set forms the graph of a function which is continuous except for at most countably many jump discontinuities. Because of the positive twist condition, these jumps must always be downward as x increases: if C is given the rightward orientation, a vector tangent to C must avoid the cone Θ_v^+, by the ratchet phenomenon. Make a circle out of this graph by adjoining vertical segments at the jumps. This is $UF(C)$. We call such a curve a *right pseudograph*: a curve made of the graph of a function $y = h(x)$ which is continuous except for downward jump discontinuities (the limit to the right $h(x^+)$ and the left $h(x^-)$ exist at each point and $h(x^-) \geq h(x^+)$), and by adjoining to this graph vertical segments to close the jumps.

We can apply F to a pseudograph C and trim it as we did for a graph. Because of the positive twist condition, the horizontal part of $UF(C)$ is made of images under F of horizontal parts of C. Given a (right pseudo) graph C, we obtain a sequence of curves $C_n = (UF)^n C$.

Lemma 36.5 *$C_\infty = \limsup C_n$ is an f-invariant graph, where $\limsup$ is taken in the sense of functions $y = h(x)$ with the obvious allowance for vertical segments.*

Proof. After one iteration of $U \circ F$ on a (right pseudo) graph C, we get a pseudograph with a *downward modulus of continuity*: the ratchet phenomenon and the vertical cuts implies that, for any pair of points z and z' in the lift of $U \circ F(C)$, $z' - z$ is in a cone V of vectors (x, y) with $y \geq \delta x$ if $x \leq 0$ and $y \leq \delta x$ if $x > 0$ (see Figure 36. 1). This implies that C_∞ also has this modulus of continuity, and hence is a pseudograph.

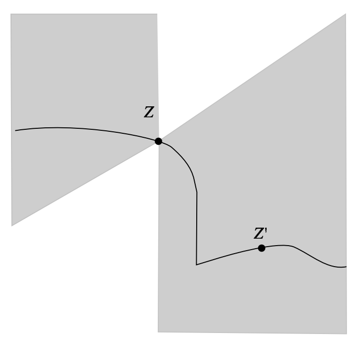

Fig. 36. 1. The cone defining the modulus of continuity at a point z of $U \circ F(C)$.

There is a partial order on circles homotopic to $\{y = 0\}$: we say that $C \preceq C'$ if C' is in the closure of the upper component of $\mathcal{A} \backslash C$, which we denote by $\mathcal{A}_+(C)$. Clearly F and U preserve this order, and $C \preceq U(C)$ for any circle C homotopic to $\{y = 0\}$. This implies that $F^n(C) \preceq UF^n(C) \preceq C_\infty$ for all n, and hence $F(C_\infty) \preceq UF(C_\infty) \preceq C_\infty$. By area preservation $F(C_\infty) = UF(C_\infty) = C_\infty$.

If C_∞ were not a graph, its vertical segments would be mapped by F inside $\mathcal{A}_-(C_\infty) = \mathcal{A}_-(UF(C_\infty))$, and $\mathcal{A}_-(C_\infty)$ would contain $\mathcal{A}_-(F(C_\infty))$ as a proper subset. This contradicts the fact that f has zero flux. Hence C_∞ is an f-invariant graph. □

We now finish the proof of Birkhoff's theorems. Suppose that f admits an invariant circle C_0 homotopic to the boundaries. We show that it is a (Lipschitz) graph. The region below C_0 is invariant. Let C_{max} be the supremum of the invariant *graphs* in this region (under the partial order $\prec$). By continuity, C_{max} is an invariant circle and by Proposition 12.3, it is a Lipschitz graph. If $C_{max} \neq C_0$, then there exist a (not invariant) graph C with $C_{max} \prec C \prec C_0$. Applying the trimming iteration process to C, we get an invariant (Lipschtitz) graph C_∞ with $C_{max} \prec C_\infty \prec C_0$. This contradicts the maximality of C_{max}. Hence $C_0 = C_{max}$ is a Lipschitz graph.

If f does not admit any other invariant circle homotopic to the boundaries than the boundaries themselves, the iteration process performed on any (right) pseudograph must converge to the upper boundary: we have $C \prec UF(C)$. Since $C_\infty \subset closure(\cup f^n(C_0))$, on any graph ϵ close to the lower boundary, there is a point whose ω-limit set is in the upper boundary. We could have defined a trimming L of curves homotopic to the boundaries by taking their lower envelope (the points seen from below) instead of U. Then $L(C)$ is a *left pseudograph* and L preserves the order of circles and $L(C) \prec C$ for any curve C homotopic to the boundaries. Using $\liminf$ instead of $\limsup$ in the argument above, we get an iteration process $L \circ F$ which converges to an invariant graph, which must be the lower boundary this time. And on any graph ϵ close to the upper boundary, there is a point whose ω-limit set is in the lower boundary. □

Remark 36.6 Performing both the $U \circ F$ and $L \circ F$ trimming processes on the same curve C yields points that come arbitrarily close to both boundaries in *forward time*. This fact was proven by Mather (1993) variationally and Hall (1989) topologically. See also LeCalvez (1990) . The results of Mather and Hall are actually sharper as they find orbits whose α-limit set is in one boundary, the ω- limit set in the same or the other boundary (??? check this!). Moreover they find orbits "shadowing" any prescribed sequence of Aubry-Mather sets in a region of instability. It would be interesting to find a new proof of these results based on the trimming technique used above. It would be interesting to generalize the trimming process to Lagrangian pseudographs in higher dimensions.

*. AUBRY-MATHER THEOREM VIA TRIMMING

The above proof of Birkhoff's theorems appears as an aside in Katznelson-Ornstein's paper. They also recover the Aubry-Mather theorem with their trimming method. For this they define, abstractly, a new type of trimming operator, that they call *proper trimming*: one which is such that the area below a curve is preserved under trimming. The main difficulty is to show the existence of such an operator. Once the existence is established, one takes limits of iterations under the map and the trimming operator. The limit is a pseudograph whose horizontal parts are *forward* invariant under f. The Aubry-Mather sets are the intersection of all the forward images of these horizontal parts. Finally, they show the existence of Aubry-Mather sets of all rotation numbers by applying this trimming procedure simultaneously to all the horizontal circles in the annulus. Fathi (???am)offers some relatively distant analog to this in higher dimension, by considering a certain flow on graphs of differentials on cotangent bundles, and recovering the generalized Aubry-Mather sets in the limit.

*. Generalizations of Birkhoff's Graph Theorem to Higher Dimensions

This section surveys (in an all too brief manner) the important work of Bialy, Polterovitch and, indirectly Herman, on invariant Lagrangian tori. It will require from the reader knowledge of material dispersed throughout the book, and more. Bialy & Polterovitch (1992) prove the following generalization to Birkhoff's Graph Theorem. We explain the terminology in the sequel.

Theorem 36.7 *Let F be the time one map of an optical Hamiltonian system of $T^*\mathbb{T}^n$, and let L be a smooth invariant Lagrangian torus for F which satisfies the following conditions:*
1) L is homologous to the zero section of $T^*\mathbb{T}^n$.
2) $F\big|_L$ is either chain recurrent or preserves a measure which is positive on open sets. Then L is a smooth graph (i.e. a section) over the 0-section.

Optical (see Chapter 6) means that the Hamiltonian H is time periodic and convex in the fiber: H_{pp} is positive definite. Homologous to the zero section means that both the invariant torus and the 0-section, seen as homology cycles (which they are because they have empty boundaries) bound a chain of degree $n+1$, presumably some smooth manifold of dimension $n+1$ in our case. As for Condition 2), it suffices here to say that either chain recurrence or existence of an invariant Borel measures are satisfied when the invariant torus is of the type exhibited by the KAM theorem, where the map $F\big|_L$ is conjugated to an irrational translation. In their paper, the authors use a more general condition than 2), which we show at the end of this section is implied by it: *2') the suspension of $F\big|_L$ admits no transversal codimension 1 cocycle homologous to zero.*

This theorem is a culmination of efforts by these authors, as well as Hermann (1990) who gives a perturbative version of this result as some important *a priori* Lipschitz estimates for invariant Lagrangian tori. We now give a very rough idea of the proof of Theorem 36.7. First reduce the theorem to the case of an autonomous Hamiltonian on $T\mathbb{T}^{n+1}$ by viewing time as an extra 1 dimension, with the energy as its conjugate momentum (extended phase space). Assume by contradiction that the invariant torus L is *not* a graph. Consider the set $S(L)$ of critical points of the projection $\pi\big|_L$. Generically, $S(L)$ consists of an $n-1$ dimension submanifold of L whose boundary is of dimension no more than $n-3$. Assume we are in the generic case. Then $S(L)$ can be cooriented by the flow: the Hamiltonian vector field is transverse to it. This makes $S(L)$ a cocycle, i.e. a representent of a cohomology class. It turns out that this cohomology class is dual to the Maslov class of the torus L. The *Maslov class* of L is the pull-back of the generator of $H_1(\Lambda(n))$ by the Gauss map, where $\Lambda(n)$ is the (Grassmanian) space of all Lagrangian planes in $\mathbb{R}2n$. Prosaically, this means the following: *the oriented intersection of $S(L)$ with any closed curve on L counts how many "turns" the Lagrangian tangent spaces of L makes along the curve*. We explain that a little. The number of turns can be made quite precise because $\Lambda(n)$ has one "hole" around which Lagrangian spaces can turn ($H_1(\Lambda(n)) = \mathbb{Z}$). $S(L)$ is the set of points on L where the Lagrangian tangent space becomes vertical in some direction. The tangent space, seen as a graph over the vertical fiber, is given by a bilinear form which is degenerate at points of $S(L)$ and, thanks to the optical condition, decreases index (*i.e.* the dimension of the positive definite subspace increases) when following the flow at those points.

The authors refer to Viterbo (1989) who proves that tori homologous to the zero section have Maslov class zero. Condition 2') now concludes: since it is homologous to zero, the cocycle $S(L)$ must be empty, $i.e.$ there are no singularity in the projection $\pi\big|_L$ and the torus is a graph. The non generic case follows by making a limit argument using uniform Lipschitz estimates for invariant tori proven by Hermann (1990).
Finally, let us show how the fact that $F\big|_L$ is measure preserving implies Condition 2'). Assume F is the time 1 map of an autonomous Hamiltonian system on $T^*\mathbb{T}^n$, L is an invariant torus and Ω is the volume form on L preserved by the Hamiltonian vector field X_H. The Homotopy Formula SGformhomotopy $L_{X_H}\Omega = di_{X_H}\Omega + i_{X_H}d\Omega$ implies that $di_{X_H}\Omega = 0$. Assume X_H is transversal to S, a codimension 1 cocycle homologous to zero and let C be an n-dimensional chain that S bounds. Transversality implies $\int_S i_{X_H}\Omega \neq 0$. On the other hand, Stokes' Theorem yields $\int_S i_{X_H}\Omega = \int_C di_{X_H}\Omega = 0$. This contradiction implies that $S = \emptyset$. □

Remark 36.8 As noted by the authors, it is not clear that Theorem 36.7 is optimal: Condition 2) maybe unnecessary, as is the case in dimension 2. One could imagine a new proof of this theorem using higher dimensional trimming on Lagrangian pseudographs, which would not need this hypothesis...

37. (Un)Stable Manifolds and Heteroclinic orbits

(Un)stable Manifolds

Consider two hyperbolic fixed point $\boldsymbol{z}^* = (\boldsymbol{q}^*, \boldsymbol{p}^*)$, $\boldsymbol{z}^{**} = (\boldsymbol{q}^{**}, \boldsymbol{p}^{**})$ for a symplectic twist map F of $T^*\mathbb{T}^n$. We remind the reader that the *stable and unstable manifolds* at any fixed point $\boldsymbol{z}^*$ are defined as:

$$\mathcal{W}^s(\boldsymbol{z}^*) = \{\boldsymbol{z} \in T^*\mathbb{T}^n \mid F^n(\boldsymbol{z}) = \boldsymbol{z}^*\}, \qquad \mathcal{W}^u(\boldsymbol{z}^*) = \{\boldsymbol{z} \in T^*\mathbb{T}^n \mid F^{-n}(\boldsymbol{z}) = \boldsymbol{z}^*\}$$

Moreover the tangent space to $\mathcal{W}^s$ at $\boldsymbol{z}^*$ is given by the vector subspace $E^s(\boldsymbol{z}^*)$ of eigenvectors of eigenvalue of modulus less than 1, with a similar fact for $\mathcal{W}^u$ and E^u. In our case, the differential DF at the points $\boldsymbol{z}^*$ and $\boldsymbol{z}^{**}$ has as many eigenvalues of modulus less than 1 as it has of modulus greater than 1. Hence the stable and unstable manifolds at these points have both dimension n. The following appears in Tabacman (1993):

Proposition 37.1 *The (un)stable manifolds of a hyperbolic fixed point for a symplectic twist map are Lagrangian. Close to the hyperbolic fixed point, they are graphs of the differentials of functions.*

Proof. Consider a point $\boldsymbol{z}$ on the stable manifold of the hyperbolic fixed point $\boldsymbol{z}^*$, and two vectors $\boldsymbol{v}, \boldsymbol{w}$ tangent to that manifold at $\boldsymbol{z}$. Then:

$$\omega_{\boldsymbol{z}}(\boldsymbol{v}, \boldsymbol{w}) = \omega_{F^k(\boldsymbol{z})}(DF^k(\boldsymbol{v}), DF^k(\boldsymbol{w})) \to \omega^*_{\boldsymbol{z}}(0,0) = 0, \text{as } k \to \infty.$$

which, since it has dimension n in $T^*\mathbb{T}^n$, proves that the stable manifold is Lagrangian. The same argument, using F^{-k}, applies to show that the unstable manifold is Lagrangian. We leave the proof of the second statement to the reader (Exercise 37.2). □

In Exercise INVexoexactstabw, the reader will show a generalization of this fact that makes it applicable to exact symplectic maps (not necessarily twist) of general cotangent bundles.

Variational Approach to Heteroclinic Orbits

As a consequence of Proposition 37.1, we obtain a variational approach to heteroclinic orbits. Let $\boldsymbol{z}^* = (\boldsymbol{q}^*, \boldsymbol{p}^*)$ be a hyperbolic fixed point. Let Φ^u, Φ^s defined on a neighborhood U of $\boldsymbol{q}^*$ be the functions whose differentials define the (un)stable manifolds of $\boldsymbol{z}^*$. We can add appropriate constants to these functions and get $\Phi^s(\boldsymbol{q}^*) = \Phi^u(\boldsymbol{q}^*) = 0$. In the proof of Theorem 36.2, we showed that the function $R(\boldsymbol{q}, \boldsymbol{Q}) = S(\boldsymbol{q}, \boldsymbol{Q}) + g(\boldsymbol{q}) - g(\boldsymbol{Q}) + \beta(\boldsymbol{q} - \boldsymbol{Q})$ was constant on the Lagrangian manifold $Graph(dg + \beta)$. Applying this to $g = \Phi^s$ or $\Phi^u, \beta = 0$, we obtain

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \Phi^s(\boldsymbol{Q}) - \Phi^s(\boldsymbol{q}) + \text{constant},$$

where $F(\boldsymbol{q}, \Phi^s(\boldsymbol{q})) = (\boldsymbol{Q}, \Phi^s(\boldsymbol{Q}))$ (this makes sense in a subset of U). Applying the equation to $(\boldsymbol{q}^*, \boldsymbol{q}^*)$ shows that the constant is $S(\boldsymbol{q}^*, \boldsymbol{q}^*)$. Hence

$$S(\boldsymbol{q}, \boldsymbol{Q}) - S(\boldsymbol{q}^*, \boldsymbol{q}^*) = \Phi^s(\boldsymbol{Q}) - \Phi^s(\boldsymbol{q})$$

for a point $(\boldsymbol{q}, \boldsymbol{Q})$ on the local stable manifold of $\boldsymbol{z}^*$. We now sum over the orbit $(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ of the point $(\boldsymbol{q}, \boldsymbol{Q}) = (\boldsymbol{q}_0, \boldsymbol{q}_1)$ to get:

$$\sum_{k=0}^{N-1} [S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) - S(\boldsymbol{q}^*, \boldsymbol{q}^*)] = \sum_{k=0}^{N-1} [\Phi^s(\boldsymbol{q}_{k+1}) - \Phi^s(\boldsymbol{q}_k)] = \Phi^s(\boldsymbol{q}_N) - \Phi^s(\boldsymbol{q}_0)$$

As $N \to \infty, \Phi^s(\boldsymbol{q}_N) \to \Phi(\boldsymbol{q}^*) = 0$ and thus the sum converges to $-\Phi(\boldsymbol{q}_0)$:

(37.1) $$\sum_{k=0}^{\infty} [S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) - S(\boldsymbol{q}^*, \boldsymbol{q}^*)] = -\Phi^s(\boldsymbol{q}_0).$$

Applying the same manipulations to the unstable manifold, using the fact that the generating function for F^{-1} is $-S(\boldsymbol{Q}, \boldsymbol{q})$, this leads to

Proposition 37.2 *Let $\boldsymbol{z}^* = (\boldsymbol{q}^*, \boldsymbol{p}^*), \boldsymbol{z}^{**} = (\boldsymbol{q}^{**}, \boldsymbol{p}^{**})$ be two hyperbolic fixed points for the symplectic twist map F. Let U^* and U^{**} be neighborhoods of $\boldsymbol{q}^*$ and $\boldsymbol{q}^{**}$ on which the differentials of the functions Φ^u and Φ^s respectively give the unstable manifold of $\boldsymbol{z}^*$ and the stable manifold of $\boldsymbol{z}^{**}$. Then critical points of the function*

$$W(\boldsymbol{q}_0, \ldots, \boldsymbol{q}_N) = \Phi(\boldsymbol{q}_0) + \sum_{k=0}^{N-1} S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) - \Phi^s(\boldsymbol{q}_N), \qquad \boldsymbol{q}_0 \in U^*, \boldsymbol{q}_N \in U^{**}$$

are segments of heteroclinic orbits.

Proof. Left to the reader.

With this set-up, Tabacman (1995) shows that, in the 2 dimensional case, any two local minima (*i.e.* fixed points) ξ and η of $\phi(x) = S(x, x)$ such that $\phi(\xi) = \phi(\eta) < \phi(x)$ for all $x \in (\xi, \eta)$, are joined by some trajectory.

Here is a sketch of a numerical algorithm also proposed (and used) by E. Tabacman to find heteroclinic orbits between two given hyperbolic fixed points $\boldsymbol{z}^*, \boldsymbol{z}^{**}$:

(1) Find a basis for the unstable plane E^u of DF at $\boldsymbol{z}^*$, and display the basis vectors as columns of a $2n \times n$ matrix $\begin{pmatrix} A \\ B \end{pmatrix}$

(2) The matrix $M = BA^{-1}$ is symmetric and E^u is the graph of the differential of the quadratic form $\boldsymbol{q} \mapsto \boldsymbol{q}^t M \boldsymbol{q}$. This function is an approximation to Φ^u (see SGexolagsym.)

(3) Perform similar steps to approximate Φ^s at $\boldsymbol{z}^{**}$.

(4) Pick N (large enough) and use your favorite numerical method to search for critical points of the function W defined above, with points $\boldsymbol{q}_0, \boldsymbol{q}_N$ suitably close to $\boldsymbol{z}^*$ and $\boldsymbol{z}^{**}$ respectively.

(5) For more precision, make $\boldsymbol{q}_0$ and $\boldsymbol{q}_N$ closer to $\boldsymbol{z}^*$ and $\boldsymbol{z}^{**}$ (resp.) and increase N.

Splitting of Separatrices and Poincaré–Melnikov Function In Hamiltonian systems, the *Poincaré–Melnikov function* (actually an integral), measures how much the intersecting stable and unstable manifolds of two hyperbolic fixed points split. This kind of function has a long and rich history: Poincaré (1899) introduced it as a way to prove non-integrability in Hamiltonian systems. It has then been used to prove the existence of Chaos (transverse intersections of stable and unstable manifolds often lead to "horseshoe" subsystems), and to estimate the rate of diffusion of orbits in the momentum direction. The discrete, two dimensional case was considered by Easton (1984)???, Gambaudo (1985), Glasser & al. (1989), Delshams & Ramírez-Ros (1996). Here, following Lomeli (1997), we give a formula for a Poincaré–Melnikov function for a higher dimensions symplectic twist map in terms of its generating function. A more general treatment, valid in general cotangent bundles, and which does not assume that the separatrix is a graph over the zero section, is given in Delshams & Ramírez-Ros (1997).

Theorem 37.3 *Let F_0 be an symplectic twist map of $T^*\mathbb{T}^n$ with hyperbolic fixed points $\boldsymbol{z}^* = (\boldsymbol{q}^*, \boldsymbol{p}^*), \boldsymbol{z}^{**} = (\boldsymbol{q}^{**}, \boldsymbol{p}^{**})$ such that $\mathcal{W}^u(\boldsymbol{z}^*) = \mathcal{W}^s(\boldsymbol{z}^{**}) = \mathcal{W}$ is the graph $\boldsymbol{p} = \psi(\boldsymbol{q})$ of a function ψ over some open set. Let S_0 be the generating function of F_0. Consider a perturbation F_ε of F_0 with generating function $S_\varepsilon = S + \varepsilon P$ such that $P(\boldsymbol{q}^*, \boldsymbol{q}^*, 0) = P(\boldsymbol{q}^{**}, \boldsymbol{q}^{**}, 0) = 0$ and $\frac{d}{dq}\big|_{q=q^*} P(\boldsymbol{q}, \boldsymbol{q}, \varepsilon) = 0 = \frac{d}{dq}\big|_{q=q^{**}} P(\boldsymbol{q}, \boldsymbol{q}, \varepsilon)$. Then the function $L : \mathcal{W} \to \mathbb{R}$:*

$$L(\boldsymbol{z}) = \sum_{k \in \mathbb{Z}} P(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}, 0) \quad \text{where } \boldsymbol{q}_k = \pi \circ F^k(\boldsymbol{q}, \psi(\boldsymbol{q})) \tag{37.2}$$

is well defined and differentiable. If L is not constant then, for ε small enough, the (un)stable manifolds of the perturbed fixed points of F_ε split. Their intersection is transverse at nondegenerate critical points of L.

Proof. Work in the covering space $\mathbb{R}2n$ of $T^*\mathbb{T}^n$. Let $\Phi : U \subset \mathbb{R}^n \to \mathbb{R}$ and $\psi = d\Phi$ be such that $Graph(\psi) = \mathcal{W}$. Change coordinates so that $\mathcal{W}$ lies in the zero section: $(\boldsymbol{q}, \boldsymbol{p}') = (\boldsymbol{q}, \boldsymbol{p} - \psi(\boldsymbol{q})$. If $F_0(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$, then, in the coordinates $(\boldsymbol{q}, \boldsymbol{p}')$, we have $\boldsymbol{q} = \boldsymbol{q},\ \boldsymbol{p}' = \boldsymbol{p} - \psi(\boldsymbol{q}), \boldsymbol{Q}' = \boldsymbol{Q}, \boldsymbol{P}' = \boldsymbol{P} - \psi(\boldsymbol{Q})$. Thus the generating function becomes:

$$S_{new}(\boldsymbol{q}, \boldsymbol{Q}) = S_{old}(\boldsymbol{q}, \boldsymbol{Q}) + \Phi(\boldsymbol{q}) - \Phi(\boldsymbol{Q}).$$

Note that P remains the same under this change of coordinates, since we only added terms which are independent of ε. For ε small enough, the (un)stable manifolds $\mathcal{W}^u_\varepsilon, \mathcal{W}^s_\varepsilon$ of the perturbed fixed points $\boldsymbol{z}^*_\varepsilon, \boldsymbol{z}^{**}_\varepsilon$ (respectively) will be graphs of the differentials $\psi^{u,s}_\varepsilon = d\phi^{u,s}_\varepsilon$ for some functions $\Phi^{u,s}_\varepsilon$ of the base variable $\boldsymbol{q}$. Clearly, the manifolds $\mathcal{W}^{u,s}_\varepsilon$ split for ε small enough whenever the following *Poincaré–Melnikov* function:

$$M(\boldsymbol{q}) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\Phi^u_\varepsilon(\boldsymbol{q}) - \Phi^s_\varepsilon(\boldsymbol{q}))$$

is not constantly zero, and their intersection is transverse if the DM is invertible at the zeros. We will now show that:

$$M(\boldsymbol{q}) = \frac{\partial L}{\partial \boldsymbol{q}}$$

where $L(\boldsymbol{q})$ is the function defined in (37.2) , expressed in our new coordinates. Formula (37.1) gives us an expression of $\Phi^{u,s}_\varepsilon$:

$$\Phi^u_\varepsilon(\boldsymbol{q}) = \sum_{k<0} \left[S_\varepsilon(\boldsymbol{q}^u_k(\varepsilon), \boldsymbol{q}^u_{k+1}(\varepsilon)) - S_\varepsilon(\boldsymbol{q}^{**}, \boldsymbol{q}^{**})\right], \quad \Phi^s_\varepsilon(\boldsymbol{q}) = -\sum_{k\geq 0} \left[S_\varepsilon(\boldsymbol{q}^s_k(\varepsilon), \boldsymbol{q}^s_{k+1}(\varepsilon)) - S_\varepsilon(\boldsymbol{q}^*, \boldsymbol{q}^*)\right]$$

where $\boldsymbol{q}^u_k(\varepsilon)$ (resp $\boldsymbol{q}^s_k(\varepsilon)$) is the $\boldsymbol{q}$ coordinate of $F^k_\varepsilon(\boldsymbol{q}, \psi^u_\varepsilon(\boldsymbol{q}))$ (resp. of $F^k_\varepsilon(\boldsymbol{q}, \psi^s_\varepsilon(\boldsymbol{q}))$). We can change order of differentiation:

$$M(\boldsymbol{q}) = \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \frac{\partial}{\partial \boldsymbol{q}}(\psi^u_\varepsilon(\boldsymbol{q}) - \psi^s_\varepsilon(\boldsymbol{q})) = \frac{\partial}{\partial \boldsymbol{q}} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} (\psi^u_\varepsilon(\boldsymbol{q}) - \psi^s_\varepsilon(\boldsymbol{q})),$$

and compute one of these terms:

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{q}} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \psi^u_\varepsilon(\boldsymbol{q}) \\
&= \frac{\partial}{\partial \boldsymbol{q}} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \sum_{k<0} \left[S_\varepsilon(\boldsymbol{q}^u_k(\varepsilon), \boldsymbol{q}^u_{k+1}(\varepsilon)) - S_\varepsilon(\boldsymbol{q}^{**}, \boldsymbol{q}^{**})\right] \\
&= \sum_{k<0} \left[\partial_1 S(\boldsymbol{q}^u_k(0), \boldsymbol{q}^u_{k+1}(0)) \frac{\partial}{\partial \varepsilon}\boldsymbol{q}^u_k(\varepsilon) + \partial_2 S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \frac{\partial}{\partial \varepsilon}\boldsymbol{q}^u_{k+1}(\varepsilon) + P(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})\right] \\
&= \sum_{k<0} P(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),
\end{aligned}$$

where in the last line we took advantage of $\partial_1 S(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = 0$: these are the $\boldsymbol{p}$ coordinates of an orbit on the zero section in our new coordinate. In the line before the last, the terms involving $S_\varepsilon(\boldsymbol{q}^{**}, \boldsymbol{q}^{**})$ disappeared because of our assumption on P. The same computation shows that $\frac{\partial}{\partial \boldsymbol{q}} \left.\frac{\partial}{\partial \varepsilon}\right|_{\varepsilon=0} \psi^s_\varepsilon(\boldsymbol{q}) = -\sum_{k\geq 0} P(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$. The proof that $\frac{\partial L}{\partial \boldsymbol{q}} = M(\boldsymbol{q})$ follows. □

Remark 37.4 We have only touched the surface of a vast subject here. Once a Melnikov function is found, one has to be able to show that it is non zero on specific examples. This is usually hard, even in dimension 2. Explicit computations often utilizes the fact that, in good situations, the complexified Melnikov function (think of $\boldsymbol{q}$ as complex in the above) is an elliptic functions. As a result of such computations, one often finds (*eg.* for standard like maps) that the angle of splitting of the separatrices are *exponentially small in the perturbation parameter* ε, making numerical methods inapplicable. We let the reader consult Delshams & Ramírez-Ros (1996b), Delshams & Ramírez-Ros (1998), Glasser & al. (1989), Gelfriech & al. (1994).

Exercise 37.5 a) Prove that the local (un)stable manifold of a hyperbolic fixed point $\boldsymbol{z}^*$ for a symplectic twist map F is a graph over the zero section (*Hint.* use the formula for the differential of F given in 26.5, and the twist condition det $(\partial_{12}S) \neq 0$ to show that the (un)stable subspace of $DF_{\boldsymbol{z}^*}$ cannot have a vertical vector. To do this, expend $\omega_{\boldsymbol{z}^*}(DF\boldsymbol{w}, \boldsymbol{w})$ assuming $\boldsymbol{w} = (0, w)$ and show that necessarily $w = 0$.)
b) Deduce from this that the (un)stable manifolds are graphs of differentials of functions Φ^u, Φ^s defined on a neighborhood of $\pi(\boldsymbol{z}^*)$ in the zero section.

Exercise 37.6 Let F is an exact symplectic map (not necessarily twist) of the cotangent bundle T^*M of some manifold: $F^\lambda - \lambda = dS$ for some function $S : M \to \mathbb{R}$ (λ is the canonical 1 form on T^*M). In Appendix 1 or SG, it is shown that any Hamiltonian map is exact symplectic, and any composition of exact symplectic map is exact symplectic.
a) Show that the (un)stable manifolds $\mathcal{W}^{s,u}$ of a fixed point are *exact Lagrangian* (immersed) submanifolds, *i.e.* $i\mathcal{W}^{s,u}\big|_\lambda = dL$ for some functions $L^{s,u} : \mathcal{W}^{s,u} \to \mathbb{R}$
b) Show that if and $\mathcal{W}$ is an exact Lagrangian manifold invariant under the exact symplectic map F, then:

$$S(z) + constant = L(F(z)) - L(z), \quad \forall p \in \mathcal{W}$$

c) Conclude that $L^u(z^u) = \sum_{k<0} S(F^k(z^u)$ and $L^s(z^s) = -\sum_{k\geq 0} S(F^k(z^s)$.
For more on this approach, see Delshams & Ramírez-Ros (1997).

*. INSTABILITY, TRANSPORT AND DIFFUSION

Part (2) of Birkhoff's Theorem 36.4 is responsible for the name *region of instability* for a region located between two invariant circles, and which does not contain any other invariant circle. This is better understood in the light of the twist maps that appear as normal forms around elliptic fixed points (see Section 28.0): In this example, the lower boundary of the annulus corresponds to the fixed point, and the drifting from the lower boundary to the upper one reflects instability of the fixed point. Mather (1993) and Hall (1989) show that the dynamics in the regions of instability can be quite complicated: given any (infinite) sequence of Aubry-Mather sets in such a region, they find an orbit that shadows it, *i.e.* stays at a prescribed distance from each one for a prescribed amount of time (the transition time is not controlled). In particular, for twist maps of the cylinder without any invariant circles, there exist orbits that are unbounded on the cylinder. To find these orbits, it suffices to take an orbit that shadows an unbounded sequence of Aubry-Mather sets. Note that Slijepčević (1999a) has recently given a proof of these results using the gradient flow of the action methods of GCchapter.

Another approach to instability uses *partial barriers*: invariant sets made of stable and unstable manifolds of hyperbolic periodic orbits or Cantori. The *theory of transport* seeks to study the rate at which points cross these barriers. This theory was initiated by MacKay, Meiss & Percival (1984). The survey Meiss (1992) is beautifully written and encompasses the theory of twist maps of the annulus and transport theory. For other developments, see Rom-Kedar & Wiggins (1990) and Wiggins (1990). MacKay suggested that (the projection in the annulus of) ghost circles could be used as partial barriers.

Mather has announced a striking result for Hamiltonian systems on $T^*\mathbb{T}^2$: For a C^r ($r \geq 2$) generic Riemannian metric g on $\mathbb{T}^2$ and C^r generic potential V periodic in time, the classical Hamiltonian system $H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{2}\|\boldsymbol{p}\|_g^2 + V(\boldsymbol{q}, t)$ possesses an unbounded orbit. Mather's proof brings together beautifully the constrained variational methods developed in Mather (1993) , the theory of minimal measures of Mather (1991b) as well as hyperbolic techniques. Delshams, de la Llave & Seara (1998)have given recently an alternate proof to this result, using methods of geometric perturbation. Finally, de la Llave just (fall 1999) announced a generalization of this theorem to cotangents bundles of arbitrary compact manifolds. His method uses a generalizations of Fenichel's theory of perturbation of normally hyperbolic sets. Interestingly, the orbits found start at high energy levels, where the system is close to integrable, marking a clear distinction with the two dimensional case where invariant circles would block the escape of orbits. In higher dimensions, KAM tori do not topologically obstruct the passage to higher energy levels.

These results offer a very significant contribution to the problem of diffusion first encountered by Arnold. (???? complete this)

Theorem INVthmkam is 35.1, PBremarkcompactam is 11.4, Section PBsecpbsymp is 12, Theorem PBthmbirkhoff is 36.4, Theorem INVthmgraphmin is 36.2, Exercise INVexoexactstabw is 37.6