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PERIODIC ORBITS FOR HAMILTONIAN SYSTEMS

We present in this chapter some results of existence and multiplicity of periodic orbits in Hamiltonian systems on cotangent bundles. Our main goal is to show the power, and relative simplicity of the method of decomposition by symplectic twist map as presented in Chapter 7, which results into finite dimensional variational problems. Some of the results in this chapter are not optimal. They could probably be improved using methods similar to the ones presented here. In some case, these results have recently been improved by other authors, using usually substantially more complicated methods.

In Section 42, we present two theorems of existence of periodic orbits for Hamiltonian systems in the cotangent bundle of the torus. They are relatively direct applications of Theorem 27.1 of Chapter 5. In Section 43, we prove a theorem of existence of periodic orbits for systems in cotangent bundles of arbitrary compact manifolds. The boundary condition that we impose (that the Hamiltonian flow be the geodesic flow outside a compact set) is inspired by a similar theorem of Conley & Zehnder (1983) for systems on the cotangent bundle of the torus. That theorem was itself inspired by a conjecture of Arnold (1965), where he proposes an entirely topological generalization (the linking of certain spheres) of the boundary twist condition of the theorem of Poincaré-Birkhoff. In Section 44, we explore this linking of sphere condition and prove Arnold's conjecture in the simple case when the map is a symplectic twist map .

42. Periodic Orbits in the Cotangent of the n-Torus

We present here two results of existence and multiplicity of periodic orbits for Hamiltonian systems in $T^*\mathbb{T}^n$. The first one concerns a certain class of optical systems, the second one Hamiltonians that are quadratic nondegenerate outside of a bounded set.

A. Optical Hamiltonians

Assumption 42.1 (Uniform Opticity)

$H(\boldsymbol{q}, \boldsymbol{p}, t) = H_t(\boldsymbol{z})$ is a twice differentiable function on $T^*\mathbb{T}^n \times \mathbb{R}$ which satisfies the following:

(1) $\sup \left\|\nabla^2 H_t\right\| < K$

(2) The matrices $H_{\boldsymbol{pp}}(\boldsymbol{z}, t)$ are positive definite and their smallest eigenvalues are uniformly bounded below by $C > 0$.

Theorem 42.2 *Let $H(\boldsymbol{q}, \boldsymbol{p}, t)$ be a Hamiltonian function on $T^*\mathbb{T}^n \times \mathbb{R}$ satisfying Assumption 42.1. Then the time 1 map h^1 of the associated Hamiltonian flow has at least $n+1$ periodic orbits of type $\boldsymbol{m}, d$, for each prime $\boldsymbol{m}, d$, and 2^n when they are all nondegenerate.*

Proof. We can decompose the time 1 map:

$$h^1 = h^1_{\frac{N-1}{N}} \circ \ldots \circ h^{\frac{k+1}{N}}_{\frac{k}{N}} \circ \ldots \circ h_0^{\frac{1}{N}}.$$

and each of the maps $h^{\frac{k+1}{N}}_{\frac{k}{N}}$ is the time $\frac{1}{N}$ of the (extended) flow, starting at time $\frac{k}{N}$. Proposition 39.11 shows that, for N big enough, such maps are symplectic twist maps. Moreover, we noted in Remark 39.10 that these maps also satisfy a convexity condition which, together with Lemma 27.2 (see (27.5) in its proof) allows us to show that the generating function S is coercive. The result follows from Theorem 27.1. $\square$

B. Asymptotically Quadratic Hamiltonians

We now turn to systems that are not necessarily optical, but satisfy a certain quadratic "boundary condition" which makes them completely integrable outside a compact set:

Theorem 42.3 *Let* $H : T^*\mathbb{T}^n \times \mathbb{R} \to \mathbb{R}$ *satisfy the following boundary condition:*

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle + \boldsymbol{c} \cdot \boldsymbol{p}, \quad A^t = A, \det\, A \neq 0 \text{ when } \|\boldsymbol{p}\| \geq K, \tag{42.1}$$

where A *is an* $n \times n$ *matrix,* $\boldsymbol{c} \in \mathbb{R}^n$ *and* K *is a positive real. Then, for all* $\boldsymbol{m}, d$ *in* $\mathbb{Z}^n \times \mathbb{Z}$*, the time–1* h^1 *map of the Hamiltonian flow has at least* $n+1$ *distinct* $\boldsymbol{m}, d$*–orbits, and at least* 2^n *when they are all nondegenerate (i.e. generically). Furthermore, such an orbit lays entirely in the set* $\|\boldsymbol{p}\| \leq K$ *if and only if the rotation vector* $\boldsymbol{m}/d$ *belongs to the ellipsoid:*

$$\mathcal{E} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \left\| A^{-1}(\boldsymbol{x} - \boldsymbol{c}) \right\| \leq K \right\}.$$

Proof. The boundary condition (42.1) is Assumption 2 preceding Theorem 39.7, in which it is proven that the time ϵ of such Hamiltonians are twist maps. Hence, as remarked in Proposition 39.11, the time 1 map can be decomposed into symplectic twist maps. We now want to apply Theorem 27.1. To insure that these twist maps satisfy the conditions of that theorem, we note that, in the proof of Proposition 39.11, instead of $G(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q}+\boldsymbol{p}, \boldsymbol{p})$, we can take $G(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p} + \boldsymbol{c}, \boldsymbol{p})$, the time 1 map of $H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2} \langle A\boldsymbol{p}, \boldsymbol{p} \rangle + \boldsymbol{c}.\boldsymbol{p}$. This map is clearly a symplectic twist map . With this minor change, outside the set $\|\boldsymbol{p}\| \leq K$, the maps F_{2k}, F_{2k-1} of the decomposition are respectively the time 1 and the time $(\frac{1}{N} - 1)$ of the Hamiltonian flow associated to H_0, that is:

$$F_{2k}(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + A\boldsymbol{p} + \boldsymbol{c}, \boldsymbol{p})$$

$$F_{2k-1}(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + (1/N - 1)(A\boldsymbol{p} + \boldsymbol{c}), \boldsymbol{p}).$$

These maps satisfy the conditions of Theorem (27.2) , which proves the existence of the advertised number of $\boldsymbol{m}, d$ orbits. To localize these orbits, note that an orbit starting in $\|\boldsymbol{p}\| \geq K$ must stay there, and the map h^1 on such an orbit is just G. The rotation number of such an orbit is thus

$$(\boldsymbol{Q} - \boldsymbol{q}) = A\boldsymbol{p} + \boldsymbol{c}$$

from which we conclude that, in this case, $\boldsymbol{m}/d$ is in the complement of $\mathcal{E}$. □

C. Remarks about the Above Results

Periodic Orbits for the Map vs. Periodic Orbits for the Flow. There is a distinction between periodic orbits of h^1 and periodic orbits of the Hamiltonian equations: for a general time dependent Hamiltonian flow, $(h^1)^n \neq h^n$, and hence an $\boldsymbol{m}, d$ periodic orbit for h^1 is not necessarily one for the O.D.E. (which should satisfy $h^{t+d}(\boldsymbol{z}) = h^t(\boldsymbol{z}) + (k\boldsymbol{m}, 0)$ for all $t \in [kd, (k+1)d), k \in \mathbb{Z}$). However, if H is periodic in time, of period 1, the equality $(h^1)^n = h^n$ does hold, and in this case the two notions coincide. In particular, this holds trivially for time independent Hamiltonians. Unfortunately, these cases are degenerate in our setting, since $Dh^d(\boldsymbol{z})$ preserves the vector field X_H, which is thus an eigenvector with eigenvalue one. So in these cases, we can only claim the cuplength estimates for the number of periodic orbits for the Hamiltonian *flow* in either Theorems 42.2 or 42.3. We think that some further argument should yield, even in the time periodic case, the sum of the betti number estimate for the number of flow periodic orbits, when the periodic orbits are nondegenerate *as orbits of the flow*: i.e., when the only eigenvector of eigenvalue one for $Dh^d(\boldsymbol{z})$ is in the direction of the vector field X_H.

Possible Improvements. Note that the full strength of Theorem 27.1 was not brought to bear in the proof of Theorem 42.3: the symplectic twist maps that we obtained in the decomposition of h^1 are linear outside a bounded set, whereas Theorem 27.1 can deal with *asymptotic* linearity. It is very conceivable that one could cover a larger class of systems using this method, including classical mechanical systems on the torus.

Related Results in the Literature. Conley & Zehnder (1983) contains a theorem of existence of multiple *homotopically trivial* periodic orbits, with a boundary condition similar to that of Theorem 42.3. In an impressive and technically difficult piece of work, Josellis (1994), (1994b) gives an improved version of Theorem 42.3 in that the Hamiltonian flow is only asymptotically quadratic. See also Felmer (1992) for related results using a mountain pass lemma. In Benci, V. (1986), it is shown that fiberwise convex, time periodic Lagrangian systems on arbitrary compact manifolds have at least one periodic orbit of any given free homotopy class. This result, which assumes also certain assumptions on the first and second derivative of the Lagrangian implies, via the Legendre transformation, the existence of at least one $\boldsymbol{m}, 1$ orbit for the optical systems we consider in Theorem 42.2. Conversely, via the Legendre transformation, Theorem 42.2 applies to Lagrangian systems whose Lagrangian function satisfies the same conditions as H in our theorem (it is not hard to see that these conditions translate under the Legendre transformation). Hence Theorem 42.2 extends some existing theorems for such systems (see, e.g., Mawhin & Willem(1989), Theorem 9.3).

43. Periodic Orbits in General Cotangent Spaces

We now turn to the study of Hamiltonian systems in cotangent spaces of arbitrary compact manifolds. Our main result, which first appeared in Golé (1994) is:

Theorem 43.1 *Let (M, g) be a compact Riemannian manifold, with associated norm $\|\cdot\|$. Let $F : T^*M \to T^*M$ be the time 1 map of a time dependent Hamiltonian H on B^*M, where H is a C^2 function satisfying the boundary condition:*

$$H(\boldsymbol{q}, \boldsymbol{p}, t) = \frac{1}{2}\|\boldsymbol{p}\|^2 \text{ for } \|\boldsymbol{p}\| \geq C.$$

where C is strictly smaller than the injectivity radius. Then F has at least $cl(M)$ distinct fixed points and at least $sb(M)$ if they are all nondegenerate. Moreover, these fixed points lie inside the set $\{\|\boldsymbol{p}\| < C\}$ and can all be chosen to correspond to homotopically trivial closed orbits of the Hamiltonian flow.

The *injectivity radius* on a Riemannian manifold is defined as

$$\inf_{\boldsymbol{q}\in M} \sup_{r\in[0,+\infty]} \left\{ r \;\middle|\; exp|_{B(\boldsymbol{q},r)} \text{ is injective } \right\}.$$

The rest of this section is devoted to the proof of this theorem. Note that Cielieback (1992) provides a similar theorem, with asymptotically quadratic conditions. His proof uses a version of Floer cohomology.

A. The Discrete Variational Setting

Define

$$B^*M = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*M \mid g(\boldsymbol{q})(\boldsymbol{p}, \boldsymbol{p}) = \|\boldsymbol{p}\|^2 \leq C^2\}.$$

Let π denote the canonical projection $\pi : B^*M \to M$. Let F be as in Theorem 43.1. From Proposition 39.11 we can decompose F into a product of symplectic twist maps :

$$F = F_{2N} \circ \ldots \circ F_1,$$

where F_{2k} restricted to the boundary ∂B^*M of B^*M is the time 1 map h_0^1 of the geodesic flow with Hamiltonian $H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\|\boldsymbol{p}\|^2$. Likewise, F_{2k-1} is $h_0^{\frac{1-N}{N}}$ on ∂B^*M. Let S_k be the generating function for the twist map F_k and $\psi_k = \psi_{F_k}$ the diffeomorphism $(\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{Q})$ induced by the twist condition on F_k. We can assume that ψ_k is defined on a neighborhood U of B^*M in T^*M where

$$U = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*M \mid \|\boldsymbol{p}\| \leq C + \delta\}.$$

Our variational study will take place in the set:

$$O = \{\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_{2N}) \in M^{2N} \mid (\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \in \psi_k(U) \text{ and } (\boldsymbol{q}_{2N}, \boldsymbol{q}_1) \in \psi_{2N}(U)\} \tag{43.1}$$

Proposition 43.2 *The set O can be described as:*

$$O = \{\overline{\boldsymbol{q}} \in M^{2N} \mid Dis(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) < |a_k|(C + \delta), Dis(\boldsymbol{q}_{2N}, \boldsymbol{q}_1) < (C + \delta)\}$$

where

$$a_k = \begin{cases} 1 & \textit{if } k \textit{ is even} \\ \frac{1-N}{N} & \textit{if } k \textit{ is odd.} \end{cases} \tag{43.2}$$

In particular, O contains the set of constant sequences in M^{2N}.

Proof. This is an easy application of the twist condition, using the fact that the map F_k equal the time a_k of the geodesic flow on the boundary of U. □

We note that U and O are independent of the map F, as long as F satisfies the boundary condition of Theorem 43.1. Next, define :

$$W(\overline{\boldsymbol{q}}) = \sum_{k=0}^{2N} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}), \tag{43.3}$$

where we have set $\boldsymbol{q}_{2N+1} = \boldsymbol{q}_1$. Choosing to work in some local coordinates around $\overline{\boldsymbol{q}} \in M^{2N}$, we let $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and $\boldsymbol{P}_k = \partial_2 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$. In other words, $(\boldsymbol{q}_k, \boldsymbol{p}_k) \in T^*_{\boldsymbol{q}_k}M$ is such that $\psi_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ and $(\boldsymbol{q}_{k+1}, \boldsymbol{P}_k) \in T^*_{\boldsymbol{q}_{k+1}}M$ is such that $F_k(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{P}_k)$. We let the reader check that the following proofs can be written in coordinate free notation (see Remark 26.3). By Exercise 26.4, a sequence $\overline{\boldsymbol{q}}$ of O is a critical point of W if and only if the sequence $\{(\boldsymbol{q}_k, \boldsymbol{p}_k)\}_{k\in\{1,\ldots,2N,1\}}$ is an orbit under the successive F_k's, that is if and only if $(\boldsymbol{q}_1, \boldsymbol{p}_1)$ is a fixed point for F: $dW(\overline{\boldsymbol{q}}) = \sum_{k=1}^{2N}(\boldsymbol{P}_{k-1} - \boldsymbol{p}_k)d\boldsymbol{q}_k$ which is null exactly when $\boldsymbol{P}_{k-1} = \boldsymbol{p}_k$, *i.e.* when $F_k(\boldsymbol{q}_{k-1}, \boldsymbol{p}_{k-1}) = (\boldsymbol{q}_k, \boldsymbol{p}_k)$. Now remember that we assumed that $\boldsymbol{q}_{2N+1} = \boldsymbol{q}_1$.

Hence, to prove Theorem 43.1, we need to find enough critical points for W. As before, we will study the gradient flow of W (where the gradient will be given in terms of the metric g) and use the boundary condition to find an isolating block. The main difference with the previous situations on $T^*\mathbb{T}^n$ is that we cannot put W in the general framework of generating phases quadratic at infinity. Nonetheless, thanks to the boundary condition we imposed on the Hamiltonian, we are able to construct an isolating block and use Floer's theorem of continuation (Theorem 63.7in Appendix 2) to get a grasp on the topology of the invariant set, and hence on the number of critical points.

B. The Isolating Block

In this subsection we prove that the set B defined as follows:

$$B = \{\overline{\boldsymbol{q}} \in O \mid \mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \leq |a_k|C\} \tag{43.4}$$

is an isolating block for the gradient flow of W, where O is defined in (43.1) , C is as in the hypotheses of Theorem 43.1, and a_k is defined in (43.2) .

Proposition 43.3 *B is an isolating block for the gradient flow of W.*

Proof. Suppose that the point $\overline{\boldsymbol{q}}$ of U is on the boundary of B. This means that $\mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) = |a_k|C$ for at least one k. Using the boundary condition, this is equivalent to $\|\boldsymbol{p}_k\| = C$, where as usual $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$. We want to show that $\mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ increases either in positive or negative time along the gradient flow of W. The k^{th} component of the gradient vector field is given by:

$$\dot{\boldsymbol{q}}_k = A_k(\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) = \nabla W_k(\overline{\boldsymbol{q}}) \tag{43.5}$$

where $A_k = A(\boldsymbol{q}_k)$ is the inverse of the matrix of coefficients of the metric g at the point $\boldsymbol{q}_k$. We used that, on a Riemannian manifold, the gradient of a function f is given by $g(\boldsymbol{q})(\nabla f, \cdot) = df(\cdot)$, see Exercise 61.9. Remember that we have put the product metric on O, induced by its inclusion in M^{2N}. We compute the derivative of the distance along the gradient flow at a boundary point of B, using Corollary 38.6 and the fact that, on the boundary, $h_0^{a_k}(\boldsymbol{q}_k, \boldsymbol{p}_k) = (\boldsymbol{q}_{k+1}, \boldsymbol{P}_k)$:

$$\begin{aligned} \left.\frac{d}{dt}\right|_{t=0} \mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) &= \partial_1 \mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \cdot \nabla W_k(\overline{\boldsymbol{q}}) \\ &\quad + \partial_2 \mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \cdot \nabla W_{k+1}(\overline{\boldsymbol{q}}) \\ &= sign(a_k) \frac{-\boldsymbol{p}_k}{\|\boldsymbol{p}_k\|} \cdot A_k(\boldsymbol{P}_{k-1} - \boldsymbol{p}_k) \\ &\quad + sign(a_k) \frac{\boldsymbol{P}_k}{\|\boldsymbol{P}_k\|} \cdot A_{k+1}(\boldsymbol{P}_k - \boldsymbol{p}_{k+1}) \end{aligned} \tag{43.6}$$

We now need a simple linear algebra lemma to treat this equation.

Lemma 43.4 *Let $\langle\, , \rangle$ denote a positive definite bilinear form in $\mathbb{R}^n$, and $\|.\|$ its corresponding norm. Suppose that $\boldsymbol{p}$ and $\boldsymbol{p}'$ are in $\mathbb{R}^n$,that $\|\boldsymbol{p}\| = C$ and that $\|\boldsymbol{p}'\| \leq C$. Then :*

$$\langle \boldsymbol{p} \ , \ \boldsymbol{p}' - \boldsymbol{p} \ \rangle \leq 0.$$

Moreover, equality occurs if and only if $\boldsymbol{p}' = \boldsymbol{p}$.

Proof. From the positive definiteness of the metric, we get:

$$\langle \, \boldsymbol{p}' - \boldsymbol{p}, \boldsymbol{p}' - \boldsymbol{p} \, \rangle \geq 0,$$

with equality occurring if and only if $\boldsymbol{p}' = \boldsymbol{p}$. From this, we get:

$$2\langle\, \boldsymbol{p}, \boldsymbol{p}' \,\rangle \leq \langle\, \boldsymbol{p}', \boldsymbol{p}' \,\rangle + \langle\, \boldsymbol{p}, \boldsymbol{p} \,\rangle$$

Finally,

$$\langle\, (\boldsymbol{p}' - \boldsymbol{p}), \boldsymbol{p} \,\rangle = \langle\, \boldsymbol{p}', \boldsymbol{p} \,\rangle - \langle\, \boldsymbol{p}, \boldsymbol{p} \,\rangle \leq \frac{1}{2}\left(\langle\, \boldsymbol{p}', \boldsymbol{p}' \,\rangle - \langle\, \boldsymbol{p}, \boldsymbol{p} \,\rangle\right) \leq 0$$

with equality occurring if and only if $\boldsymbol{p}' = \boldsymbol{p}$. □

Applying Lemma 43.4 to each of the right hand side terms in (43.6) , we can deduce that $\frac{d}{dt}\mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ is positive when k is even, negative when k is odd. Indeed, because of the boundary condition in the hypothesis of the theorem, we have $\|\boldsymbol{P}_k\| = \|\boldsymbol{p}_k\|$ whenever $\|\boldsymbol{p}_k\| = C$: the boundary ∂B^*M is invariant under F and all the F_k's. On the other hand $\overline{\boldsymbol{q}} \in B \Rightarrow \|\boldsymbol{p}_l\| \leq C$ and $\|\boldsymbol{P}_l\| \leq C$, for all l, by invariance of B^*M. Finally, a_k is positive when k is even, negative when k is odd.

The problem is that we have not shown yet that $\frac{d}{dt}\mathrm{Dis}(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ cannot be 0. This problem is confined to the “edges” of ∂B, $i.e.$ the sets of points $\overline{\boldsymbol{q}}$ such that more than one $\boldsymbol{p}_k$ has norm C. The problem at these edges occurs when k is in an interval $\{l, \ldots, m\}$ such that, for all j in this interval, $\left\|\boldsymbol{p}_j\right\| = C = \|\boldsymbol{P}_j\|$ and $\nabla W_j(\overline{\boldsymbol{q}}) = 0$. It is now crucial to note that $\{l, \ldots, m\}$ can not cover all of $\{0, \ldots, 2N\}$: this would mean that $\overline{\boldsymbol{q}}$ is a critical point corresponding to a fixed point of h_0^1 in ∂B^*M. But such a fixed point is forbidden by our choice of C: orbits of our Hamiltonian on the set $\|\boldsymbol{p}\| = C$ are geodesics, but geodesics in that energy level cannot be rest points since $C > 0$, and they cannot close up in time one either since C is less than the injectivity radius. We now let $k = m$ in (43.6) and see that exactly one of the 2 terms in the right hand side of Equation (43.6) is nonzero. Hence the flow must definitely escape the set B at $\overline{\boldsymbol{q}}$ in either positive or negative time, from the m^{th} face of B. □

Remark 43.5 If the Hamiltonian considered is optical and we decompose its time 1 map into a product of N twist maps as in 39.11, all the F_k's coincide with $h_0^{\frac{1}{N}}$ on the boundary of B^*M. In that case, all the a_k's in the above proof are positive, and B is a repeller block.

C. End of Proof of Theorem 43.1

To finish the proof of Theorem 43.1 we use Floer's theorem 63.7 of continuation of normally hyperbolic invariant sets. We consider the family F_λ of time 1 maps of the Hamiltonians:

$$H_\lambda = (1-\lambda)H_0 + \lambda H,$$

where H is as in Theorem 43.1 and $H_0(\boldsymbol{q},\boldsymbol{p}) = \frac{1}{2}\|\boldsymbol{p}\|^2$. Corresponding to this is a family of gradient flows ζ_λ^t, solution of

$$\frac{d}{dt}\overline{\boldsymbol{q}} = \nabla W_\lambda(\overline{\boldsymbol{q}}),$$

where W_λ is the discrete action corresponding to the decomposition in symplectic twist maps of the map F_λ. We take care that this decomposition has the same number of steps, say $2N$, for each λ. As before, the manifold on which we consider these (local) flows is O, an open neighborhood of B in M^{2N}. Each F_λ satisfies the hypothesis of Theorem 43.1, and thus Proposition 43.3 applies to ζ_λ^t for all λ in $[0,1]$: B is an isolating block for each of these flows. Hence the maximum invariant sets G_λ for the flows ζ_λ^t in B are related by continuation. The part of Floer's Theorem that we need to check is that G_0 is a normally hyperbolic invariant manifold for ζ_0^t.

Lemma 43.5 *Let $G_0 = \{\overline{\boldsymbol{q}} \in B \mid \boldsymbol{q}_k = \boldsymbol{q}_1, \forall k\}$. Then G_0 is a normally hyperbolic invariant set for ζ_0^t. G_0 is a retract of O and it is the maximal invariant set in B.*

Proof. The only critical points for W_0 in B are the points of G_0 which correspond to restpoints of the geodesic flow, *i.e.* the zero section. Indeed, critical points of W_0 in B corresponds to periodic points of period 1 for the geodesic flow in B^*M. Our definition of that sets precludes nontrivial periodic geodesics in B^*M. We now show that the maximum invariant set for ζ_0^t in B is included in G_0. Since ζ_0^t is a gradient flow, such an invariant set is formed by critical points and connecting orbits between them. The only critical points of W_0 in B are the points of G_0. If there were a connecting orbit entirely in B, it would have to connect two points in G_0, which is absurd since $W_0 \equiv 0$ on G_0, whereas W_0 should increase along non constant orbits. G_0 is a retract of M^{2N} under the composition of the maps:

$$\overline{\boldsymbol{q}} = (\boldsymbol{q}_1,\ldots,\boldsymbol{q}_{2N}) \to \boldsymbol{q}_1 \to (\boldsymbol{q}_1,\boldsymbol{q}_1,\ldots,\boldsymbol{q}_1) = \alpha(\overline{\boldsymbol{q}})$$

which is obviously continuous and fixes the points of G_0. It remains to show that G_0 is normally hyperbolic. Since $G_0 \cong M$ is an n-dimensional manifold made of critical points, saying that it is normally hyperbolic is equivalent to saying that for $\overline{\boldsymbol{q}}$ in G_0, $ker\nabla^2 W_0(\overline{\boldsymbol{q}})$ has dimension n: indeed, if it is the case, the only possible vectors in this kernel must be tangent to G_0, and thus he differential of the flow is nondegenerate on the normal space to TG_0. In the present situation, the second variation formula of Lemma 29.4 says that the 1-eigenspace of Dh_0^1 is isomorphic to the kernel of $\nabla^2 W_0$. Hence it is enough to check that at a point $(\boldsymbol{q}_1, 0) \in B^*M$ corresponding to $\overline{\boldsymbol{q}}$, 1 is an eigenvalue of multiplicity exactly n for $Dh_0^1(\boldsymbol{q}_1, 0)$. Let us compute $Dh_0^1(\boldsymbol{q}_1, 0)$ in local coordinates. It is the solution at time 1 of the linearized (or variation) equation:

$$\dot{U} = -J\nabla^2 H_0(\boldsymbol{q}_1, 0)U$$

along the constant solution $(\boldsymbol{q}(t), \boldsymbol{p}(t)) = (\boldsymbol{q}_1, 0)$, where J denotes the usual symplectic matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$. An operator solution for the above equation is given by $U(t) = exp\left(-tJ\nabla^2 H_0(\boldsymbol{q}_1, 0)\right)$. On the other hand:

$$\nabla^2 H_0(\boldsymbol{q}_1, 0) = \begin{pmatrix} 0 & 0 \\ 0 & A(\boldsymbol{q}_1) \end{pmatrix}$$

which we computed from $H_0(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}A(\boldsymbol{q})\boldsymbol{p}.\boldsymbol{p}$, the zero terms appearing at $p = 0$ because they are either quadratic or linear in p. From this,

$$Dh_0^1(\boldsymbol{q}_1, 0) = exp\left(J\nabla^2 H_0(\boldsymbol{q}_1, 0)\right) = \begin{pmatrix} I & A(\boldsymbol{q}_1) \\ 0 & I \end{pmatrix}$$

is easily derived. This matrix has exactly n independent eigenvectors of eigenvalue 1 (it has in fact no other eigenvector). Hence, from Lemma 29.4, $\nabla^2 W(\overline{\boldsymbol{q}})$ has exactly n vectors with eigenvalue 0, as was to be shown. □

We now conclude the proof of Theorem 43.1. We have proved that the gradient flow ζ^t, has an invariant set G_1 with $H^*(M) \hookrightarrow H^*(G_1)$. From this we get in particular:

$$cl(G_1) \geq cl(M) \text{ and } sb(G_1) \geq sb(M).$$

Theorem 61.2 and the remark following it tell us that ζ^t must have at least $cl(G_1)$ rest points in the set G_1, and $sb(G_1)$ if all rest points are nondegenerate. But Lemma 29.4 (which, as the reader can readily check, is valid in general cotangent bundles) tells us that nondegeneracy for $\nabla^2 W$ at a critical point is the same thing as nondegeneracy of a fixed point for F (no eigenvector of eigenvalue 1). This proves the existence of the advertised number of fixed points of the map F. In the following section, we will see how to insure that all the fixed points of the time 1 map that we find correspond to homotopically trivial periodic orbits. This concludes the proof of Theorem 43.1. □

D. Periodic Orbits of Different Homotopy Classes

To determine the topological type of a periodic orbit for a map on T^*M, we consider the free homotopy class of a curve built from a given sequence of points on the manifold, in the fashion of broken geodesics.

Free Homotopy Classes. The *free homotopy class* of a curve is an equivalence class of all curves that are homotopic *without a fixed base point*. As a result, free homotopy classes can be seen as conjugacy classes in $\pi_1(M)$ (the conjugacy is by concatenation with a curve that goes from the starting point of the given curve to a given base point, and back). Thus this set of free homotopy classes can not be endowed with a natural algebraic structure. Two elements of a free class give the same element in $H_1(M)$. Hence free homotopy classes form a set smaller than $\pi_1(M)$, bigger than $H_1(M)$. All these sets coincide if $\pi_1(M)$ is abelian.

Construction of the Broken Solutions. For each F_k in the proof of Theorem 43.1 , define φ_k to be the inverse map of the diffeomorphism $\boldsymbol{Q} \to -\partial_1 S_k(\boldsymbol{q}, \boldsymbol{Q})$. That is, fixing $\boldsymbol{q}$, the map that make correspond $\boldsymbol{p}$ to $\boldsymbol{Q}$ according to $F_k(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$. Since each F_k is a symplectic twist map equal to $h_0^{a_k}$ on $\partial B^*_{\boldsymbol{q}} M$ for some positive or negative a_k, the set $\varphi_k(B^*_{\boldsymbol{q}} M)$ is a ball of radius $|a_k|$ centered at $\boldsymbol{q}$ (in the sense of distance induced by the Riemannian metric). In particular $\boldsymbol{q} \in \varphi_k(B^*_{\boldsymbol{q}} M)$. Since $B^*_{\boldsymbol{q}} M \to \varphi_k(B^* M)$ is a diffeomorphism, we can define a path $c_k(\boldsymbol{q}, \boldsymbol{Q})$ between $\boldsymbol{q}$ and a point $\boldsymbol{Q}$ of $\varphi_k(B^*_{\boldsymbol{q}} M)$ by taking the image by φ_k of the oriented line segment between $\varphi_k^{-1}(\boldsymbol{q})$ and $\varphi_k^{-1}(\boldsymbol{Q})$ in $B^*_{\boldsymbol{q}} M$. In the case where $F_k = h_0^1$, φ_k is just the map $exp^{\#}$ and this amounts to taking the unique geodesic between $\boldsymbol{q}$ and $\boldsymbol{Q}$ in $\varphi_k(B^*_{\boldsymbol{q}} M)$. If we look for periodic orbits of period d and in a given free homotopy class, we decompose F^d into $2Nd$ twist maps, by decomposing F into $2N$. Analogously to Equation (43.1) , we then define :

$$O_d = \{\overline{\boldsymbol{q}} = (\boldsymbol{q}_1, \dots, \boldsymbol{q}_{2Nd}) \in M^{2Nd} \mid (\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) \in \psi_k(U) \text{ and } (\boldsymbol{q}_{2Nd}, \boldsymbol{q}_1) \in \psi_{2Nd}(U)\},$$

(U is as before a neighborhood of B^*M). To each element $\overline{\boldsymbol{q}}$ in O_d, we can associate a closed curve $c(\overline{\boldsymbol{q}})$, made by joining up each pair $(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ with the curve $c_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$ uniquely defined as above. This "broken solution" $c(\overline{\boldsymbol{q}})$ is a piecewise differentiable loop, and it depends continuously on $\overline{\boldsymbol{q}}$, and so do its derivatives (left and right). In the case of the decomposition of h_0^1 , taking F_k=h_0^1, this is exactly the construction of the broken geodesics (see Section 38). Now any closed curve in M belongs to some free homotopy class that we denote by $\boldsymbol{m}$. To any d periodic point for F, we can associate a sequence $\overline{\boldsymbol{q}}(x) \in O_d$ of q coordinates of the orbit of this point under the successive F_k's in the decomposition of F^d.

Definition 43.6 Let $\boldsymbol{z}$ be a periodic point of period d for F. Let $\overline{\boldsymbol{q}}$ be the sequence in O_d corresponding to x. We say that x is an $\boldsymbol{m}, d$–point if $c(\overline{\boldsymbol{q}})$ is in the free homotopy class m.

This definition has the advantage to make sense for any map F of T^*M which can be decomposed into the product of symplectic twist maps . If F is also the time 1 map of a Hamiltonian, it agrees with the obvious definition:

Proposition 43.7 *If $\boldsymbol{z}$ is an $\boldsymbol{m}, d$ periodic point, then the projection $\pi(\boldsymbol{z}(t)), t \in [0, d]$ of the orbit of $\boldsymbol{z}$ under the Hamiltonian flow is a closed curve in the free homotopy class $\boldsymbol{m}$.*

Proof. Left as an exercise (*Hint.* Use the geodesic flow to construct the homotopy between $c(\overline{\boldsymbol{q}}(\boldsymbol{z}))$ and $\pi(\boldsymbol{z}(t))$). □

Let

$$O_{m,d} = \{\overline{\boldsymbol{q}} \in O_d \mid c(\overline{\boldsymbol{q}}) \in \boldsymbol{m}\} \tag{43.7}$$

Since $c(\overline{\boldsymbol{q}})$ depends continuously on $\overline{\boldsymbol{q}} \in O_d$, $O_{m,d}$ is a connected component of O_d. The reader who wants to make sure that, in the proof of Theorem 43.1, the orbits found are homotopically trivial, can check that the proof we gave in last section works identically when one replace the space O, by its connected component $O_{e,1}$, where e is the homotopy class of the trivial curve. Another place where one uses this decomposition of O in different homotopy components is the following:

Theorem 43.8 *Let (M,g) be a Riemannian manifold of negative curvature and H be as in Theorem 1. If $\gamma_{\boldsymbol{m}}$ denotes the (unique) closed geodesic of free homotopy class $\boldsymbol{m}$, F has at least 2 $\boldsymbol{m}, d$–orbits in B^*M when length$(\gamma_{\boldsymbol{m}}) < dC$.*

The proof of Theorem 43.8 (see Golé (1994), Theorem 2) has the same broad outline as that of Theorem 43.1. We work in $O_{\boldsymbol{m},d}$ instead of O. The normally hyperbolic invariant set that we continue to in this setting is given by the set G_0 of critical sequences corresponding to the orbits under the $h_0^{a_k}$'s of the points on $\gamma_{\boldsymbol{m}}$. The normal hyperbolicity of G_0 derives this time from the hyperbolicity of the geodesic flow in negative curvature.

44. Linking of Spheres: Toward a Generalization of the Theorem of Poincaré And Birkhoff

This section goes back to the original motivation of Theorem 43.1, namely the following conjecture of Arnold (1965) that generalizes the Theorem of Poincaré-Birkhoff. We will define and explore the notion of linking of spheres in the sequel.

Conjecture 44.1 (Arnold) *State it precisely here ???*

In Banyaga & Golé (1993) we proposed the following generalization of this conjecture:

Conjecture 44.2 *Let M be a compact manifold, and F be a Hamiltonian map of a ball bundle B^*M in T^*M. Suppose that each sphere $\partial B^*_q M$ links with its image by F in ∂B^*M. Then F has at least cl(M) distinct fixed points, and at least sb(M) if they are nondegenerate.*

In Banyaga & Golé (1993) (see also Golé (1994)), we proved the following simple case. We will give the proof in the case of $M = \mathbb{T}^n$.:

Theorem 44.3 *Let F be a symplectic twist map of B^*M which links spheres on the boundary ∂B^*M. Then F satisfies the generalized Arnold Conjecture.*

Linking of Spheres and the Boundary Twist Condition. If you remove a circle C_1 from 3-space, the hole it leaves out creates some topology. In particular, another circle C_2 can "go around that hole" or not. If it does, the circles C_1 and C_2 link. In mathematical terms, the complement of C_1 has a new generator in first homology that the 3-space did not have. The first homology class of C_2 in the complement of C_1 measures the linking of the 2 circles. Likewise, removing an $n-1$ sphere in $\mathbb{R}^{2n-1}$ creates a new generator in H_{n-1}, and the homology class of another sphere in that group measures the linking of the two spheres. We will adapt this notion to the setting where one sphere is the boundary $\partial\Delta_{\boldsymbol{q}}$ of a fiber $\Delta_{\boldsymbol{q}}$ of the ball bundle B^*M, and the other sphere is $F(\partial\Delta_{\boldsymbol{q}})$. We will go into more detail later on these concepts, when we prove that, at least in the case of symplectic twist maps of $T^*\mathbb{T}^n$, the linking of these two $n-1$-spheres in the $2n-1$ dimensional boundary of B^*M is equivalent to:

Definition 44.4 (Fiber Intersection Property) We say that a map $F : B^*M \to B^*M$ satisfies the *Fiber Intersection Property* if each fiber $\Delta_{\boldsymbol{q}} = \pi^{-1}(\boldsymbol{q})$ intersects its image $F(\Delta_{\boldsymbol{q}})$ with a nonzero algebraic intersection number ($i.e.$ the number of intersections counted with orientation).

Note that, in the case of twist maps of the annulus, this property is clearly equivalent to the boundary twist condition of the Poincaré-Birkhoff Theorem 7.1: If points on the two boundary components of the annulus go in opposite directions under F then the vertical fiber $\{x = x_0\}$ and its image by F should have a nonzero algebraic intersection number. Before going through the rigorous definition of sphere linking and its equivalence with the Fiber Intersection Property, we give the proof of Theorem 44.3.

Proof of Theorem 44.3. We assume for now the equivalence of the linking of spheres condition and the Fiber Intersection Property. If F is a symplectic twist map, a fiber $\Delta_{\boldsymbol{q}}$ and its image under F may intersect at most once. Hence the Fiber Intersection Property means in this case that each fiber intersects its image $exactly\ once$. Fixed points of F correspond to critical points of $\boldsymbol{q} \to S(\boldsymbol{q}, \boldsymbol{q})$. This function is well defined since, by the above, the diagonal in $M \times M$ is in the image of B^*M by the embedding ψ_F. Hence F has as many fixed points as the function $\boldsymbol{q} \to S(\boldsymbol{q}, \boldsymbol{q})$ has critical points on M. Morse and Lyusternick-

Schnirelman's theories (See Theorem 61.2 and the remark following it) give the advertised estimates. □

Equivalence of Sphere Linking and Fiber Intersection. We now show that, in the case considered by Arnold, the Fiber Intersection Property is indeed equivalent to linking of boundary spheres. For the case of more general manifolds than $T^*\mathbb{T}^n$, we refer the reader to Banyaga & Golé (1993) or Golé (1994). The reader may already be aware of a connection between linking and intersection: going back to the example of 2 circles in $\mathbb{R}^3$, their linking can be measured by the algebraic intersection number of one circle with any disk bounded by the other one. This correspondence breaks down in $\mathbb{S}^1 \times \mathbb{R}^2 = \partial B^*\tilde{\mathbb{T}}^2$ however: there, the circles $\partial\Delta_{\boldsymbol{q}}$ and $F(\partial\Delta_{\boldsymbol{q}})$ do not bound any disks. We can still define their linking number homologically, and relate it to the Fiber Intersection property, which takes place in the full space $B^*\tilde{\mathbb{T}}^2$. The important point is that the linking of spheres is a purely topological condition which can be read entirely in the boundary.

We first remind the reader of the definition of linking of spheres from algebraic topology. Let $\Delta_{\boldsymbol{q}}$ be a fiber of $B^*\tilde{\mathbb{T}}^n \cong \mathbb{S}^{n-1} \times \mathbb{R}^n$. Then $\partial\Delta_{\boldsymbol{q}} \cong \mathbb{S}^{n-1}$. It make sense to talk about its linking with its image $F(\partial\Delta_{\boldsymbol{q}})$ in $\partial B^*\tilde{\mathbb{T}}^n$: the latter set has dimension $2n-1$ and the dimensions of the spheres add up to $2n-2$. The linking number $F(\partial\Delta_{\boldsymbol{q}})$ with $\partial\Delta_{\boldsymbol{q}}$ is given by the class $[F(\partial\Delta_{\boldsymbol{q}})] \in H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}; \mathbb{Z})$ (from now to the end of this chapter, we only consider homology with integer coefficients). More precisely, we have:

$$
\begin{aligned}
H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}) &\cong H_{n-1}\left(\mathbb{S}^{n-1} \times (\mathbb{R}^n - \{0\})\right) \\
&\overset{\text{Kunneth}}{\cong} H_{n-1}(\mathbb{S}^{n-1}) \oplus H_{n-1}(\mathbb{R}^n - \{0\})
\end{aligned}
\quad . \tag{44.1}
$$

Thus, removing $\partial\Delta_{\boldsymbol{q}}$ from $\partial B^*\tilde{\mathbb{T}}^n$ creates a new generator in the $(n-1)^{\text{st}}$ homology (with integer coefficients) of that set, *i.e.* a generator, call it b, of $H_{n-1}(\mathbb{R}^n - \{0\}) \cong \mathbb{Z}$. As any sphere of dimension $n-1$ in $\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}$, $F(\partial\Delta_{\boldsymbol{q}})$ represents an $n-1$ cohomology class in that set that we can write:

$$[F(\partial\Delta_{\boldsymbol{q}})] = \alpha a \oplus \beta b$$

in the final decomposition in (44.1) . We call the integer β the *linking number* of the spheres $F(\partial\Delta_{\boldsymbol{q}})$ and $\partial\Delta_{\boldsymbol{q}}$. If the linking number is nonzero, we say that the spheres $\partial\Delta_{\boldsymbol{q}}$ and $F(\partial\Delta_{\boldsymbol{q}})$ *link*. Finally, if $\partial\Delta_{\boldsymbol{q}}$ and $F(\partial\Delta_{\boldsymbol{q}})$ link for all $\boldsymbol{q} \in M$, we say that F satisfies the *Linking of Spheres Condition.*

Lemma 44.5 *If F is the lift of a diffeomorphism of $B^*\mathbb{T}^n = \mathbb{T}^n \times B^n$, the Fiber Intersection Property is equivalent to the Linking of Spheres Condition. More precisely, the algebraic intersection number $\#(\Delta_{\boldsymbol{q}} \cap F(\Delta_{\boldsymbol{q}}))$ and the linking number of the spheres $\partial\Delta_{\boldsymbol{q}}$ and $F(\partial\Delta_{\boldsymbol{q}})$ are equal.*

Proof. We complete (44.1) into the following commutative diagram:

$$\begin{array}{ccc} H_{n-1}(\partial B^*\tilde{\mathbb{T}}^n \backslash \partial\Delta_{\boldsymbol{q}}) & \cong & H_{n-1}(\mathbb{R}^n - \{0\}) \oplus H_{n-1}(\mathbb{S}^n) \\ \downarrow i_* & & \downarrow j_* \\ H_{n-1}(B^*\tilde{\mathbb{T}}^n \backslash \Delta_{\boldsymbol{q}}) & \cong & H_{n-1}\left(\{\mathbb{R}^n - \{0\}\} \times B^n\right) \end{array}$$

where i, j are inclusion maps and B^n is the n-ball. It is clear that j_*b generates

$$H_{n-1}\left((\mathbb{R}^n - \{0\}) \times B^n\right) \cong H_{n-1}\left(\{\mathbb{R}^n - \{0\}\} \times \mathbb{R}^n\right).$$

If S is any $n-1$ sphere in $\{\mathbb{R}^n - \{0\}\} \times \mathbb{R}^n$, the class $[S] \in H_{n-1}\left(\{\mathbb{R}^n - \{0\}\} \times \mathbb{R}^n\right)$ ($i.e.$, an integer) measures the (usual) linking number of a sphere with the fiber $\Delta_{\boldsymbol{q}}$ in $B^*\tilde{\mathbb{T}}^n \cong \mathbb{R}^{2n}$. But it is well known that such a number is the intersection number of any ball bounded by the sphere with the fiber $\Delta_{\boldsymbol{q}}$, counted with orientation (see Rolfsen (1976) page 132). $\square$