4

SYMPLECTIC TWIST MAPS

In this chapter, we generalize the definition of twist maps of the annulus to that of symplectic twist maps in higher dimensions. We will do that first in the natural higher dimensional analog of the cylinder, $\mathbb{T}^n \times \mathbb{R}^n = \mathcal{C}^n$, and then treat the case of general cotangent bundles. There are several reasons to favor the cotangent bundle of the torus $\mathbb{T}^n \times \mathbb{R}^n$. This space arises naturally in many classical mechanical settings whose configuration spaces can be described by n angles. It also arises as local polar coordinate systems near elliptic fixed points of symplectic maps. Another reason to start with maps of $\mathbb{T}^n \times \mathbb{R}^n$ is that they are the most accessible: although these notions are at least implicitly present, little knowledge of manifolds, fiber bundles and differential forms is needed in the study of this case. Finally, these maps are prone to numerical studies, and for this reason have given rise to many studies by physicists and astronomers.

Nonetheless, cotangent bundles of other manifolds than the torus do occur in mechanics (*eg.* the configuration space of the solid rigid body is SO(3)) and there too it is possible to define and make use of symplectic twist maps. We will see in later chapters that they offer a convenient handle on Hamiltonian systems on cotangent bundles. For the part of the chapter treating general symplectic twist maps , the reader should be familiar with the notion of cotangent bundle and differential forms which are reviewed in Section 58 of Appendix 1.

Higher dimensional Symplectic Twist Maps have appeared in many forms, under many names in the literature. I owe the name to MacKay & al. (1989), who use a more restrictive twist condition. The work of Herman (1990), as well as my desire to find a geometric twist condition suitable for general cotangent bundles inspired the definition given here, which first appeared (to my knowledge) in Banyaga & Golé (1993) and Golé (1994).

23. Symplectic Twist Maps of $\mathbf{T}^n \times \mathbf{R}^n$

A. Definition

Let $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ be the n–dimensional torus. An analog to the annulus in higher dimensions which is most natural in mechanics is the space $\mathbb{T}^n \times \mathbb{R}^n$, which can be seen as the cartesian product of n annuli. We give $\mathbb{T}^n \times \mathbb{R}^n$ the coordinate $(\boldsymbol{q}, \boldsymbol{p}) = (q_1, \ldots, q_n, p_1, \ldots, p_n)$. In mechanics, $q_1, \ldots, q_n$ would be n angular configuration variables of the system, whereas $p_1, \ldots, p_n$ would be their conjugate momentum, and $\mathbb{T}^n \times \mathbb{R}^n$ is the cotangent bundle $T^*\mathbb{T}^n$ of the torus $\mathbb{T}^n$. The following is a generalization of the definition of twist maps of the cylinder:

Definition 23.1 Let F be a diffeomorphism of $\mathbb{R}^{2n}$ and write $(\boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p}), \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p})) = F(\boldsymbol{q}, \boldsymbol{p})$. Let F satisfies:

1) $F(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}) = F(\boldsymbol{q}, \boldsymbol{p}) + (\boldsymbol{m}, 0), \qquad \forall \boldsymbol{m} \in \mathbb{Z}^n$
2) *Twist Condition:* the map $\psi_F : (\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q}, \boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p}))$ is a diffeomorphism of $\mathbb{R}^{2n}$.
3) *Exact Symplectic:* in the coordinates $(\boldsymbol{q}, \boldsymbol{Q})$,

$$\boldsymbol{P} d\boldsymbol{Q} - \boldsymbol{p} d\boldsymbol{q} = dS(\boldsymbol{q}, \boldsymbol{Q}) \tag{23.1}$$

where S is a real valued function on $\mathbb{R}^{2n}$ satisfying:

$$S(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S(\boldsymbol{q}, \boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n. \tag{23.2}$$

Then the map f that F induces on $\mathbb{T}^n \times \mathbb{R}^n$ is called a *Symplectic Twist Map.*

As for maps of the annulus, $S(\boldsymbol{q}, \boldsymbol{Q})$ is called a *generating function* of the map F: Equation (23.1) is equivalent to

$$\begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q}) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}), \end{cases}$$

and thus F is implicitly given by S since

$$\begin{gathered} F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q} \circ \psi_F(\boldsymbol{q}, \boldsymbol{p}), \partial_2 S \circ \psi_F(\boldsymbol{q}, \boldsymbol{p})) \\ \text{with} \quad \psi_F^{-1}(\boldsymbol{q}, \boldsymbol{Q}) = (\boldsymbol{q}, -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})) \end{gathered} \tag{23.3}$$

Note that the prescription of F through its generating function S is often more theoretical than computational: it involves the inversion of the diffeomorphism ψ_F^{-1}.

B. Comments on the Definition

•The periodicity condition $F(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = F(\boldsymbol{q},\boldsymbol{p})+(\boldsymbol{m},0)$ insures that F induces a map f on $\mathbb{T}^n \times \mathbb{R}^n$. It also implies that (in fact is equivalent to) f is homotopic to Id (see the Exercise 23.3, b)). The periodicity condition is also a consequence of (23.2) (Exercise 23.3 , a). We chose to include it for its importance.

•The twist condition (2) of definition 23.0 implies the local twist condition often used in the literature:

$$\text{Local Twist Condition (2')} \quad \det\ \partial\boldsymbol{Q}/\partial\boldsymbol{p} \neq 0.$$

We will explore in Section 25 extra assumptions under which this local twist condition implies the global twist of Condition (2).

•In terms of differential forms, $\boldsymbol{P}d\boldsymbol{Q}-\boldsymbol{p}d\boldsymbol{q} = F^*\boldsymbol{p}d\boldsymbol{q}-\boldsymbol{p}d\boldsymbol{q}$. The periodicity of S given by $S(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{Q}+\boldsymbol{m}) = S(\boldsymbol{q},\boldsymbol{Q})$ in the $(\boldsymbol{q},\boldsymbol{Q})$ coordinates becomes $S(\boldsymbol{q}+m,\boldsymbol{p}) = S(\boldsymbol{q},\boldsymbol{p})$ in the $(\boldsymbol{q},\boldsymbol{p})$ coordinates ($i.e.$ applying Ψ_F^{-1}). In particular S induces a function s on $\mathbb{T}^n \times \mathbb{R}^n$ such that $f^*\boldsymbol{p}d\boldsymbol{q}-\boldsymbol{p}d\boldsymbol{q} = ds$ ($\boldsymbol{q}$ is seen as coordinate on $\mathbb{T}^n$ here). This last equality expresses the fact that f is *exact symplectic*. As we have seen in Chapter 1(see also Appendix 1), if f is exact symplectic it is also *symplectic*:

$$f^*\, pd\boldsymbol{q}-\boldsymbol{p}d\boldsymbol{q} = ds \Rightarrow d(f^*\boldsymbol{p}d\boldsymbol{q}-\boldsymbol{pdq}) = 0 \Rightarrow f^*(d\boldsymbol{q}\wedge d\boldsymbol{p}) = d\boldsymbol{q}\wedge d\boldsymbol{p}.$$

Any symplectic map of $\mathbb{R}^{2n}$ is exact symplectic, but it is not true of maps of $\mathbb{T}^n \times \mathbb{R}^n$: the map $f(\boldsymbol{q},\boldsymbol{p}) \mapsto (\boldsymbol{q},\boldsymbol{p}+\boldsymbol{m}), \boldsymbol{m} \neq 0$ is symplectic but not exact symplectic. As in the case of maps of the annulus, exact symplecticity can be interpreted as a zero flux condition, but the flux is now an n dimensional quantity.

C. The Variational Setting

As in the case of monotone twist maps of the annulus, the generating function of a symplectic twist map induces a variational approach to finding orbits of the map.

Proposition 23.2 (Critical Action Principle) *Let $f_1,\dots,f_N$ be symplectic twist maps of $T^*\mathbb{T}^n$, and let F_k be a lift of F_k, with generating function S_k.There is a one to one correspondence between orbits $\{(\boldsymbol{q}_{k+1},\boldsymbol{p}_{k+1}) = F_k(\boldsymbol{q}_k,\boldsymbol{p}_k)\}$ under the successive F_k's and the sequences $\{\boldsymbol{q}_k\}_{k\in\mathbb{Z}}$ in $(\mathbb{R}^n)^{\mathbb{Z}}$ satisfying:*

$$\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}) + \partial_2 S_{k-1}(\boldsymbol{q}_{k-1}, \boldsymbol{q}_k) = 0 \tag{23.4}$$

The correspondence is given by: $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$.

Proof. It is identical to the case $n = 1$, see Lemma 5.4. □

Remark 23.3 As in the case $n = 1$, Equation (23.4) can be interpreted as:

$$\nabla W = 0 \quad \text{with} \quad W(\boldsymbol{q}_0, \ldots, \boldsymbol{q}_N) = \sum_0^{N-1} S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}).$$

And, as in the proof of Corollary 5.5, (23.4) can also be written:

$$\boldsymbol{P}_{k-1} - \boldsymbol{p}_k = 0$$

where we use the notation $\boldsymbol{p}_k = -\partial_1 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1}),\ \boldsymbol{P}_k = \partial_2 S_k(\boldsymbol{q}_k, \boldsymbol{q}_{k+1})$.

Exercise 23.4 a) Prove that Condition 1) in Definition 23.3 is a consequence of Conditions 2) and 3).
b) You will now prove that Condition 1) implies that the map f induced by F is homotopic to Id. It is known that each homeomorphism of the torus $\mathbb{T}^n$ is homotopic to a unique torus map induced by a linear map A of $Gl(n, \mathbb{Z})$ (the group of invertible integer $n \times n$ matrices, see Stillwell (1980)). Likewise, each homotopy classes of homeomorphisms of $\mathbb{T}^n \times \mathbb{R}^n$ has exactly one represent of the form $A \times Id$ where $A \in Gl(n, \mathbb{Z})$. Show that any lift F of a map homotopic to $A \times Id$ satisfies:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Rightarrow F(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}) = (\boldsymbol{Q} + A\boldsymbol{m}, \boldsymbol{P}).$$

Exercise 23.5 Show that if F and F' are two lifts of the same symplectic twist map F, their corresponding generating functions S and S' satisfy:

$$S(\boldsymbol{q}, \boldsymbol{Q}) = S'(\boldsymbol{q}, \boldsymbol{Q} + \boldsymbol{m}),$$

where $\boldsymbol{m} \in \mathbb{Z}^n$ is such that $F'(\cdot) = F(\cdot) + (\boldsymbol{m}, 0)$.

24. Examples

A. The Generalized Standard Map

The *generalized standard map* or *standard family* is the family of symplectic twist map whose lift is generated by the following type of functions:

$$S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\|\boldsymbol{Q} - \boldsymbol{q}\|^2 + V_\lambda(\boldsymbol{q}).$$

where V_λ is a family of C^2 functions that are $\mathbb{Z}^n$–periodic, with λ a (possibly multidimensional) parameter and $V_0 \equiv 0$. In the following, we assume that the norm is the usual Euclidean one, but that could be changed. It is trivial to see that S satisfies the periodicity condition $S_\lambda(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S_\lambda(\boldsymbol{q}, \boldsymbol{Q})$. To find the corresponding map, we compute:

$$\begin{aligned} \boldsymbol{p} &= -\partial_1 S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{q} - \nabla V_\lambda(\boldsymbol{q}) \\ \boldsymbol{P} &= \partial_2 S_\lambda(\boldsymbol{q}, \boldsymbol{Q}) = \boldsymbol{Q} - \boldsymbol{q} \end{aligned}$$

from which we immediately get:

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{q} + \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q}) \\ \boldsymbol{P} &= \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q}) \end{aligned}$$

In other words, the standard map is given by:

$$F_\lambda(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q}), \boldsymbol{p} + \nabla V_\lambda(\boldsymbol{q})). \tag{24.1}$$

In the case $n = 2$, the following is the most widely studied potential. It is due to Froeschlé (1972) (see also Kook & Meiss (1989), Froeschlé, Laskar & Celletti (1992)):

$$V_\lambda(q_1, q_2) = \frac{1}{(2\pi)^2}\{K_1\cos(2\pi q_1) + K_2\cos(2\pi q_2) + h\cos(2\pi(q_1 + q_2))\}.$$

In this case $\lambda = (K_1, K_2, h) \in \mathbb{R}^3$, and the standard family attached to this potential is a three parameter family of symplectic maps of $\mathbb{T}^2 \times \mathbb{R}^2$. The picture on the book cover, gracefully provided by Eduardo Tabacman, represents the stable and unstable manifolds of a periodic orbit for this map, with parameter $k_0 = k_1 = 2, h = 1/2$. The difference in colour is to suggest the fourth dimension.

When $\lambda = 0$, the map F_λ of (24.1) becomes:

$$F_0(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + \boldsymbol{p}, \boldsymbol{p}).$$

This is an instance of a *completely integrable* symplectic twist map: such maps preserve a foliation of $\mathbb{T}^n \times \mathbb{R}^n$ by tori homotopic to $\mathbb{T}^n \times \{0\}$. On the covering space of each of these tori, the lift of the map is conjugate to a rigid translation. The term "completely integrable" comes from the corresponding notion in Hamiltonian systems (see Example 24.2.)

The reason why the standard map has attracted so much research is that it is a $computable$ example of a higher dimensional conservative system. Because of the relative tractability of this system, one may understand questions about persistence of invariant tori as the parameter λ varies away from 0, study the various properties of its periodic orbits as well as problems of diffusion.

B. Hamiltonian systems

As we will see, the relationship between symplectic twist maps and Hamiltonian systems runs wide and deep. Chapter 7 is devoted to this relationship. Let us say here that there are two ways to generate a symplectic twist map from a Hamiltonian system. They occur either as Poincaré return maps around elliptic orbits in Hamiltonian systems (we develop this idea in the next subsection and Section 40) or as time ϵ maps of a Hamiltonian system, when restricted to an appropriate domain. As a basic example of the latter, the Hamiltonian flow generated by:

$$H(\boldsymbol{q}, \boldsymbol{p}) = \frac{1}{2}\langle A\boldsymbol{p}, \boldsymbol{p}\rangle \quad \text{with} \quad A^t = A, \; \det A \neq 0$$

is completely integrable, in that it preserves each torus $\{\boldsymbol{p} = \boldsymbol{p}_0\}$ and its time t map:

$$g^t(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{q} + t(A\boldsymbol{p}), \boldsymbol{p})$$

is a completely integrable symplectic twist map. If A is positive definite, g^t restricted to $\{H = 1\}$ is just the geodesic flow for the flat metric $\frac{1}{2}\langle A^{-1}\boldsymbol{v}, \boldsymbol{v}\rangle$ on $\mathbb{T}^n$ (See Section 38).

More generally, if $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$ is the lift of the time ε of some Hamiltonian flow generated by the function H, then:

$$\begin{aligned}
\boldsymbol{Q} &= \boldsymbol{q}(\epsilon) = \boldsymbol{q}(0) + \epsilon.H_{\boldsymbol{p}} + o(\epsilon^2) \\
\boldsymbol{P} &= \boldsymbol{p}(\epsilon) = \boldsymbol{p}(0) - \epsilon.H_{\boldsymbol{q}} + o(\epsilon^2),
\end{aligned}$$

and F satisfies the local twist condition $\det \frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}}(\boldsymbol{z}(0)) \neq 0$ whenever $H_{\boldsymbol{pp}}$ is non degenerate. This remark was made by Moser (1986a) in the dimension 2 case. From this local argument we will derive in Chapter 7 conditions under which the time ε of a Hamiltonian is a symplectic twist map . We will also see that, even if the time ϵ map of a Hamiltonian system is not twist, its time 1 map can, for large classes of Hamiltonian systems, still be decomposed into the product of twist maps (see Chapter 7).

C. Elliptic Fixed Points

As we will see in Section 40, the study of Hamiltonian dynamics around a periodic orbit of a time independent Hamiltonian reduces to that of a symplectic map:

$$\mathcal{R} : \mathbb{R}^{2n} \to \mathbb{R}^{2n}, \quad \text{such that} \quad \mathcal{R}(0) = 0,$$

called the Poincaré return map. The case where 0 is an elliptic fixed point ($i.e.$ the differential $D\mathcal{R}$ has all its eigenvalues on the unit circle, see Sections 33and 56) is of particular interest, as the dynamics around it offers a microcosm of all the possible symplectic dynamics. Elliptic fixed points have also been a focus of attention in the discussion on stability with the KAM theory, Nekhoroshev estimates and Arnold diffusion (see Chapter 6). We now follow Moser (1977) in proving that, generically, one can find good symplectic coordinates around the elliptic fixed point which makes the map $\mathcal{R}$ a symplectic twist map . A normal form theorem for elliptic fixed points says that the map $\mathcal{R}$ is, around 0 given by:

$$\begin{aligned} Q_k &= q_k cos\Phi_k(\boldsymbol{q},\boldsymbol{p}) - p_k sin\Phi_k(\boldsymbol{q},\boldsymbol{p}) + f_k(\boldsymbol{q},\boldsymbol{p}) \\ P_k &= q_k sin\Phi_k(\boldsymbol{q},\boldsymbol{p}) + p_k cos\Phi_k(\boldsymbol{q},\boldsymbol{p}) + g_k(\boldsymbol{q},\boldsymbol{p}) \\ \Phi_k(\boldsymbol{q},\boldsymbol{p}) &= \alpha_k + \sum_{l=1}^{n} \beta_{kl}(q_l^2 + p_l^2). \end{aligned}$$

where the error terms f_k, g_k are C^3.[8] We now show that this map is, in "polar coordinates", a symplectic twist map of $T^*\mathbb{T}^n$, *whenever the matrix* $\boldsymbol{B} = \{\beta_{kl}\}$ *is non singular.* Let V be a punctured neighborhood of 0 defined by: $0 < \sum_k (q_k^2 + p_k^2) < \epsilon$. We introduce on V new coordinates (r_k, θ_k) by:

$$q_k = \sqrt{2\epsilon r_k} cos 2\pi\theta_k, \quad p_k = \sqrt{2\epsilon r_k} sin 2\pi\theta_k$$

where θ_k is determined modulo 1. One can check that V is transformed into the "annular" set:

[8] actually, one only need them to have vanishing derivatives up to order 3 at the origin and be C^1 otherwise.

$$U = \left\{ (\theta_k, r_k) \in \mathbb{T}^n \times \mathbb{R}^n \quad \Big| \quad r_k > 0 \quad \text{and} \quad \sum_k r_k < \frac{1}{2} \right\}$$

Since the symplectic form $d\boldsymbol{q} \wedge d\boldsymbol{p}$ is transformed into $2\pi\epsilon d\boldsymbol{r} \wedge d\boldsymbol{\theta}$, $\mathcal{R}$ remains symplectic in these new coordinates, with the symplectic form $d\boldsymbol{r} \wedge d\boldsymbol{\theta}$. In fact, $\mathcal{R}$ is exact symplectic. To check this, it is enough to show that (Exercise 58.6), for any simple closed curve γ:

$$\int_{\mathcal{R}\gamma} \boldsymbol{r} d\boldsymbol{\theta} = \int_{\gamma} \boldsymbol{r} d\boldsymbol{\theta}.$$

It is easy to see that $4\pi\epsilon r_k d\theta_k = p_k dq_k - q_k dp_k$, so by Stokes' theorem:

$$4\pi\epsilon \int_{\gamma} \boldsymbol{r} d\boldsymbol{\theta} = \int_{\partial D} \boldsymbol{p} d\boldsymbol{q} - \boldsymbol{q} d\boldsymbol{p} = -2 \int_{D} d\boldsymbol{q} \wedge d\boldsymbol{p}$$

where D is a 2 manifold in V with boundary $\partial D = \gamma$. Since $\mathcal{R}$ preserves $d\boldsymbol{q} \wedge d\boldsymbol{p}$ in V, it must preserve the last integral, and hence the first.

To see that $\mathcal{R}$ satisfies the two other conditions for being a symplectic twist map, we write $\mathcal{R}(\boldsymbol{\theta}, \boldsymbol{r}) = (\boldsymbol{\Theta}, \boldsymbol{R})$ in the new coordinates:

$$\begin{aligned} \Theta_k =& \theta_k + \psi_{F_k}(\boldsymbol{r}) + o_1(\epsilon) \\ R_k =& r_k + o_1(\epsilon) \\ \text{with } \psi_{F_k} =& \alpha_k + \epsilon \sum_{l=1}^{n} 2\beta_{kl} r_l. \end{aligned}$$

where $\epsilon^{-1} o_1(\epsilon, \boldsymbol{\theta}, \boldsymbol{r})$ and its first derivatives in $\boldsymbol{r}, \boldsymbol{\theta}$ tend to 0 uniformly as $\epsilon \to 0$. We can rewrite this as:

$$\mathcal{R}(\boldsymbol{\theta}, \boldsymbol{r}) = \left(\boldsymbol{\theta} + \epsilon \boldsymbol{B} \boldsymbol{r} + \boldsymbol{\alpha} + o_1(\epsilon), \boldsymbol{r} + o_1(\epsilon) \right).$$

So for small ϵ, the condition $\det\ \partial\boldsymbol{\Theta}/\partial\boldsymbol{r} \neq 0$ is given by the nondegeneracy of $\boldsymbol{B} = \{\beta_{kl}\}$, one uses the fact that $\mathcal{R}$ is C^1 close to a completely integrable symplectic twist map to show that $\mathcal{R}$ is twist in U (the twist condition is open). The fact that it is homotopic to Id derives from Exercise 23.4. Note that the set V and therefore U are not necessarily invariant under $\mathcal{R}$. Note also that the symmetric matrix $\boldsymbol{B}$, even though it is generically nondegenerate, is not necessarily positive definite. Herman (1992 b) has examples of Hamiltonian systems and symplectic maps arbitrarily close to completely integrable which have elliptic fixed point with $\boldsymbol{B}$ not positive definite.

Exercise 24.3 Compute the expression of the lift of a symplectic twist map generated by:

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2} \langle A(\boldsymbol{Q} - \boldsymbol{q}), (\boldsymbol{Q} - \boldsymbol{q}) \rangle + \boldsymbol{c}.(\boldsymbol{Q} - \boldsymbol{q}) + V(\boldsymbol{q}),$$

where A is a nondegenerate $n \times n$ symmetric matrix (This is yet a further generalization of the standard map).

25. More on Generating Functions

A. Homeomorphism Between Twist Maps and Generating Functions

The following proposition justifies the name "generating function".

Proposition 25.1 *There is a homeomorphism*[9] *between the set of lifts F of C^1 symplectic twist maps of $T^*\mathbb{T}^n$ and the set of C^2 real valued functions S on $\mathbb{R}^{2n}$ satisfying the following:*

(a) $S(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S(\boldsymbol{q}, \boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n,$

(b) The maps: $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$ are diffeomorphisms of $\mathbb{R}^n$ for any $\boldsymbol{Q}_0$ and $\boldsymbol{q}_0$ respectively,

(c) $S(0, 0) = 0.$

This homeomorphism is implicitely given by:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Leftrightarrow \begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q}, \boldsymbol{Q})). \end{cases} \tag{25.1}$$

Proof. Let F be a lift of a symplectic twist map and $S(\boldsymbol{q}, \boldsymbol{Q})$ be its generating function. For such F and S, we have already derived (25.1) from $\boldsymbol{P} d\boldsymbol{Q} - \boldsymbol{p} d\boldsymbol{q} = dS$, and (a) is part of our definition of symplectic twist maps. To show that S satisfies (b), first note that, by (25.1), $\boldsymbol{Q} \to -\partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$ is just the inverse of the map $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q}_0, \boldsymbol{p})$, which is a diffeomorphism since $\psi_F : (\boldsymbol{q}, \boldsymbol{p}) \mapsto (\boldsymbol{q}, \boldsymbol{Q})$ is a diffeomorphism by the twist condition. We also have the composition of diffeomorphisms:

$$(\boldsymbol{q}, \boldsymbol{Q}) \stackrel{\psi_F^{-1}}{\to} (\boldsymbol{q}, \boldsymbol{p}) \stackrel{F}{\to} (\boldsymbol{Q}, \boldsymbol{P})$$

[9] In the compact open topologies of the corresponding sets

which implies that the map $\boldsymbol{q} \to \boldsymbol{P}(\boldsymbol{q}, \boldsymbol{p}_0) = \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ is a diffeomorphism, which finishes to prove that S satisfies (b). Since two generating functions of the same F only differ by a constant there is exactly one such $S(0,0) = 0$.

Conversely, given an S satisfying (b), we can define a C^1 map F of $\mathbb{R}^{2n}$ by:

$$\begin{aligned} F(\boldsymbol{q}, \boldsymbol{p}) &= (\boldsymbol{Q} \circ \psi_F(\boldsymbol{q}, \boldsymbol{p}), \partial_2 S \circ \psi_F(\boldsymbol{q}, \boldsymbol{p})) \\ \text{where} \quad \psi_F^{-1}(\boldsymbol{q}, \boldsymbol{Q}) &= (\boldsymbol{q}, -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})). \end{aligned} \tag{25.2}$$

F is a diffeomorphism, as it is the composition of the two diffeomorphisms:

$$(\boldsymbol{q}, \boldsymbol{p}) \stackrel{\psi_F}{\to} (\boldsymbol{q}, \boldsymbol{Q}) \to (\boldsymbol{Q}, \partial_2 S(\boldsymbol{q}, \boldsymbol{Q})).$$

It is easy to check that such a pair F, S satisfies (25.1). Since S satisfies (a), F is a lift of a diffeomorphism of $T^*\mathbb{T}^n$: (a) also holds for $\partial_1 S$ and $\partial_2 S$, which implies (as the reader should check) that $F(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{p}) = (\boldsymbol{Q} + \boldsymbol{m}, \boldsymbol{P})$ whenever $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$. Exercise 23.4 shows that F must be homotopic to the Identity. Because of (b), F satisfies the twist condition. Hence the map F (uniquely) defined from (25.1) is a symplectic twist map and it is not hard to see that the correspondence we built between the maps F and the functions S is continuous in the C^1 and C^2 compact open topologies respectively. □

B. Local vs. Global Twist

It is useful to have "local" computational criteria to determine when a function S satisfies the hypotheses of Proposition 25.1 in order to be the generating function of some map. This is the purpose of the following proposition:

Proposition 25.2 *Let $S: \mathbb{R}^{2n} \to \mathbb{R}$ be a C^2 function satisfying:*

$$\begin{aligned} &(i)\ S(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \boldsymbol{m}) = S(\boldsymbol{q}, \boldsymbol{Q}), \quad \forall \boldsymbol{m} \in \mathbb{Z}^n \\ &(ii)\ \det \partial_{12} S \neq 0 \\ &(iii)\ \sup_{(\boldsymbol{q}, \boldsymbol{Q}) \in \mathbb{R}^{2n}} \left\| (\partial_{12} S(\boldsymbol{q}, \boldsymbol{Q}))^{-1} \right\| = K < \infty. \end{aligned} \tag{25.3}$$

Then S is the generating function for the lift of a symplectic twist map .

The next proposition gives a way to insure the "global" twist condition from a local condition on the map:

Proposition 25.3 *Let* $F(\boldsymbol{q},\boldsymbol{p}) = (\boldsymbol{Q},\boldsymbol{P})$ *be a symplectic map of* $\mathbb{R}^{2n}$ *with* $F(\boldsymbol{q}+\boldsymbol{m},\boldsymbol{p}) = (\boldsymbol{Q}+\boldsymbol{m},\boldsymbol{P})$*. Suppose that*

$$\sup_{(\boldsymbol{q},\boldsymbol{p})\in\mathbb{R}^{2n}} \left\|(\partial \boldsymbol{Q}(\boldsymbol{q},\boldsymbol{p})/\partial \boldsymbol{p})^{-1}\right\| < \infty. \tag{25.4}$$

Then F *is the lift of a symplectic twist map .*

The proof of both these propositions are direct consequences of the following:

Lemma 25.4 *Let* $f:\mathbb{R}^N \to \mathbb{R}^N$ *be a local diffeomorphism at each point, such that:*

$$\sup_{x\in\mathbb{R}^N} \left\|(Df_x)^{-1}\right\| = K < \infty.$$

Then f *is a global diffeomorphism.*

Proof. We first prove that f is onto. Let $y_0 = f(0)$ and take any $y \in \mathbb{R}^N$. Let $y(t) = (1-t)y_0 + ty$. By the inverse function theorem, f^{-1} is defined and differentiable on an interval $y([0,\epsilon))$. Let a be the supremum of all such ϵ in $[0,1]$. If we prove that f^{-1} is also defined and differentiable at a, then $a=1$, otherwise, by the inverse function theorem, we get the contradiction that f^{-1} is defined on $[0,a+\alpha)$, for some $\alpha>0$. For any $t_0,t_1\in[0,a)$, we have:

$$\begin{aligned}\left\|f^{-1}(y(t_1)) - f^{-1}(y(t_0))\right\| &\le \sup_{t\in[0,a)} \left\|Df^{-1}(y(t))\right\| \|y-y_0\| \, |t_1-t_0| \\ &\le K\|y-y_0\|\,|t_1-t_0|\,.\end{aligned}$$

So that, for any sequence $t_k \to a$, the sequence $f^{-1}(y(t_k))$ is Cauchy. This proves the existence of $f^{-1}(y(a))$. By the Inverse Function Theorem, f^{-1} is differentiable at $y(a)$. This finishes the proof that f is onto. Since f is onto and open, it is a covering map from $\mathbb{R}^N$ to $\mathbb{R}^N$. Such a covering has to be one sheeted, *i.e.* a diffeomorphism, since $\mathbb{R}^N$ is connected and simply connected (see Appendix 2). This finishes the proof. □

Proof of Proposition 25.2. In order to apply Proposition 25.1, we need to show that S satisfies condition (b) in that proposition. But this is an immediate consequence of Lemma 25.4 applied to the two maps $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q},\boldsymbol{Q}_0)$ and $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0,\boldsymbol{Q})$ (note that $\left\|(\partial_{21}S)^{-1}\right\| = \left\|(\partial_{12}S)^{-1}\right\|$). □

Proof of Proposition 25.3. By Lemma 25.4, for each fixed $\boldsymbol{q}$, the map $\boldsymbol{p} \to \boldsymbol{Q}(\boldsymbol{q}, \boldsymbol{p})$ is a global diffeomorphism of $\mathbb{R}^n$. This implies that $\psi_F : (\boldsymbol{q}, \boldsymbol{p}) \to (\boldsymbol{q}, \boldsymbol{Q})$ is a global diffeomorphism of $\mathbb{R}^{2n}$. □

C. Differential of the Map vs. Generating Function

Proposition 25.5 *The following formula relates the differential of a symplectic twist map F to the second derivatives of its generating function S:*

$$DF_{(\boldsymbol{q},\boldsymbol{p})} = \begin{pmatrix} -\partial_{11}S.(\partial_{12}S)^{-1} & -(\partial_{12}S)^{-1} \\ \partial_{21}S - \partial_{22}S.\partial_{11}S.(\partial_{12}S)^{-1} & -\partial_{22}S.(\partial_{12}S)^{-1} \end{pmatrix}.$$

where all the partial derivatives are taken at the point $(\boldsymbol{q}, \boldsymbol{Q}) = \psi_F(\boldsymbol{q}, \boldsymbol{p})$.

Proof. We will show that $\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}} = -(\partial_{12}S)^{-1}(\boldsymbol{q}, \boldsymbol{Q})$, where, as usual, we have set $F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P})$. Differentiating the equality: $\boldsymbol{p} = -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})$ with respect to $\boldsymbol{p}$, viewing $\boldsymbol{Q}$ as a function of $\boldsymbol{q}, \boldsymbol{p}$, one gets:

$$Id = -\partial_{12}S(\boldsymbol{q}, \boldsymbol{Q}) \left(\frac{\partial \boldsymbol{Q}}{\partial \boldsymbol{p}} \right).$$

The computations for the other terms are similar. □

Exercise 25.6 a) Show that if instead of Condition (1) in the definition of symplectic twist maps we ask F to be homotopic to $A \times Id$, where a lift $\tilde{A}$ of A is in $Gl^+(n, \mathbb{Z})$, then Proposition 25.1 remains true, replacing (a) by:

$$S\left(\boldsymbol{q} + \boldsymbol{m}, \boldsymbol{Q} + \tilde{A}(\boldsymbol{m})\right) = S(\boldsymbol{q}, \boldsymbol{Q}).$$

b) Find the map generated by

$$S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}(\boldsymbol{q} - \tilde{A}^{-1}\boldsymbol{Q})^2 + V(\boldsymbol{q})$$

Note that this exercise shows, in particular, that there are plenty of examples of exact symplectic maps of $T^*\mathbb{T}^n$ that are not homotopic to Id and hence cannot be Hamiltonian maps.

Exercise 25.7 Let $\mathbb{B}^n$ denote a *compact* ball in $\mathbb{R}^n$. Show that if $f : \mathbb{B}^n \to \mathbb{R}^n$ is a differentiable map satisfying :

$$\inf_{x\in \mathbb{B}^n} \langle df_x \boldsymbol{v}, \boldsymbol{v}\rangle \geq a\langle \boldsymbol{v}, \boldsymbol{v}\rangle, \quad \forall \boldsymbol{v} \in \mathbb{R}^n$$

then f is an *embedding* (diffeomorphism on its image) of $\mathbb{B}^n$ in $\mathbb{R}^n$.

26. Symplectic Twist Maps on Cotangent Bundles of Compact Manifolds

A. Definition

Our definition of symplectic twist maps of $\mathbb{T}^n \times \mathbb{R}^n$ is geometric enough to allow a generalization to cotangent bundles of general compact manifolds. The main difference between our general definition and the one in the case of $\mathbb{T}^n \times \mathbb{R}^n \cong T^*\mathbb{T}^n$ is that we do not work with the universal covering space of our manifold any more, to the cost of a less global definition.[10] In this book, the main examples of symplectic twist maps on general cotangent bundles will arise in the context of Hamiltonian systems (see Chapter 7). We also present, in the next section, a generalization of the standard map in cotangent of hyperbolic manifolds. We refer the reader to Appendix 1 for a review of the concepts of cotangent bundles and their symplectic structure.

In the following, U will denote an open subset of T^*M such that:

$$\pi^{-1}(\boldsymbol{q}) \cap U \simeq interior(\mathbb{B}^n) \tag{26.1}$$

where $\pi : T^*M \to M$ is the canonical projection, and $\mathbb{B}^n \subset \mathbb{R}^n$ denotes the n-ball. Hence U is a relatively compact ball bundle over M, diffeomorphic to T^*M. As in Appendix 1, we denote by λ the canonical 1-form on a cotangent bundle.

Definition 26.1 A *symplectic twist map* F is a diffeomorphism of an open ball bundle $U \subset T^*M$ (as in (26.1)) onto itself satisfying the following:

(1) F is homotopic to Id.

(2) F is *exact symplectic*: $F^*\lambda - \lambda = \underline{S}$ for some real valued function $\underline{S}$ on U.

(3) *Twist condition*: the map $\psi_F : U \to M \times M$ given by $\psi_F(\boldsymbol{z}) = (\pi(\boldsymbol{z}), \pi \circ F(\boldsymbol{z}))$ is an embedding.

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[10] If the manifold M is not covered (topologically) by $\mathbb{R}^n$, problems occur when we want to make the definition of symplectic twist maps of T^*M as global as in $T^*\mathbb{T}^n$: there cannot be a global diffeomorphism from a fiber of T^*M to the universal cover $\tilde{M}$.

The function $S = \underline{S} \circ \psi_F^{-1}$ on $\psi_F(U)$ is called the *generating function* for F.

Often, the kind of neighborhood we have in mind is of the form:

$$U = \{(\boldsymbol{q}, \boldsymbol{p}) \in T^*\tilde{M} \mid H(\boldsymbol{q}, \boldsymbol{p}) < K\}$$

for some function H convex in $\boldsymbol{p}$. One could use a less restrictive class of neighborhoods, to the cost of possible domain complications.

If $\tilde{M} \cong \mathbb{R}^n$, one can take $U = T^*M$ and modify the above definition slightly to make it more global by changing (2) into: (2') If $\tilde{F} : T^*\tilde{M} \to T^*\tilde{M}$ is a lift of F, the map $\psi_{\tilde{F}} : \tilde{U} \to \tilde{M} \times \tilde{M}$ given by $\psi_{\tilde{F}}(\boldsymbol{z}) = (\pi(\boldsymbol{z}), \pi \circ F(\boldsymbol{z}))$ is a diffeomorphism (of $\mathbb{R}^{2n}$). We leave the reader check that when $M = \mathbb{T}^n$, Definition 26.1 with 2') replacing 2) is an obvious, coordinate free generalization of the definition of symplectic twist map of $T^*\mathbb{T}^n$, with the appropriate restrictions of domains.

Finally, one could further relax the above definition by asking that F be only an *embedding* of U into T^*M, letting go of the invariance of U.

B. Maps vs. Functions, Revisited

It is not hard to adapt the proof of Proposition 25.1 to the more general:

Proposition 26.2 *Given a point $\boldsymbol{q}^*$ in the compact manifold M, there is a homeomorphism between the set of pairs (F, U) where F is a C^1 symplectic twist map of an open ball bundle $U \subset T^*M$ and the pairs (S, V) , where S is in the set of C^2 real valued functions S on an open set V (diffeomorphic to U) of $M \times M$ satisfying the following:*

(i) The map $\boldsymbol{q} \to \partial_2 S(\boldsymbol{q}, \boldsymbol{Q}_0)$ (resp. $\boldsymbol{Q} \to \partial_1 S(\boldsymbol{q}_0, \boldsymbol{Q})$) is a diffeomorphism of the open set $\{(\boldsymbol{q}, \boldsymbol{Q}_0)\} \cap V$ (resp. $\{(\boldsymbol{q}_0, \boldsymbol{Q})\} \cap V$) of M into $\left(T^*_{\boldsymbol{Q}_0} M\right) \cap U$ (resp. $\left(T^*_{\boldsymbol{q}_0} M\right) \cap U$) for each $\boldsymbol{Q}_0$ (resp. $\boldsymbol{q}_0$).

(ii) $S(\boldsymbol{q}^*, \boldsymbol{q}^*) = 0$.

This correspondence is given by:

$$F(\boldsymbol{q}, \boldsymbol{p}) = (\boldsymbol{Q}, \boldsymbol{P}) \Leftrightarrow \begin{cases} \boldsymbol{p} = & -\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})) \\ \boldsymbol{P} = & \partial_2 S(\boldsymbol{q}, \boldsymbol{Q})). \end{cases} \tag{26.2}$$

Remark 26.3 1) As noted before, if $\tilde{M} \cong \mathbb{R}^n$, we can choose $\tilde{U} = \tilde{M} \times \mathbb{R}^n \cong \mathbb{R}^{2n}$ in the above proposition. In this case Corollaries 25.2 and 25.3 also remain valid.
2) Even though we have written Proposition 26.2 in coordinates, the relationship between symplectic twist maps and their generating function is independent of the *canonical* system of coordinates chosen, see Exercise 26.4. In particular, the prescription of Formula (26.2) will yield the same covectors $\boldsymbol{p}$ and $-\boldsymbol{P}$ regardless of the local coordinates $\boldsymbol{q}$ and $\boldsymbol{Q}$ are expressed in.

C. Examples

Time t of Hamiltonian Systems. The examples of symplectic twist maps in general cotangent bundles will mainly come from Chapter 7, as time t of Hamiltonian systems satisfying the Legendre condition. We will also show how to decompose any Hamiltonian map of a large class into symplectic twist maps .

Standard Map on Hyperbolic Manifolds. In Proposition 38.7 of Chapter 7, we will prove that the function $S(\boldsymbol{q}, \boldsymbol{Q}) = \frac{1}{2}\mathrm{Dis}^2(\boldsymbol{q}, \boldsymbol{Q}) + V(\boldsymbol{q})$, where Dis is the distance function on the hyperbolic half space $\mathbb{H}^n$ and $V : \mathbb{H}^n \to \mathbb{R}$ is some C^2 function, generates a *global* symplectic twist map that we call generalized standard map on hyperbolic space. By using potential functions V that are equivariant under discrete groups of isometries, this type of map provides many examples of symplectic twist maps on compact hyperbolic manifolds that are covered by $\mathbb{H}^n$.

Exercise 26.4 Check that, with the appropriate restriction on the set of sequences, Proposition 23.2 and Remark 23.3 remain valid for symplectic twist maps on general cotangent bundles.

Exercise 26.5 a) Prove Proposition 26.2. Verify that, although we have written things in local coordinates, everything in Proposition 26.2 has intrinsic meaning (e.g. $\partial_1 S(\boldsymbol{q}, \boldsymbol{Q})$ *is* an element of $T^*_{\boldsymbol{q}}M$, which is independent of the *canonical* systems of coordinate chosen above either $\boldsymbol{q}$ or $\boldsymbol{Q}$).
b) Prove that if M in Proposition 26.2 is the covering space of a manifold N with fundamental group Γ, and if S satisfy $S(\gamma \boldsymbol{q}, \gamma \boldsymbol{Q}) = S(\boldsymbol{q}, \boldsymbol{Q}), \forall \gamma \in \Gamma$, as well as (i) and (ii), then the symplectic twist map that S generates is a lift of a symplectic twist map on N.

Exercise 26.6 Show that the set of C^1 twist maps on a relatively compact, open neighborhood in the cotangent bundle of a manifold is open in the set of C^1 symplectic maps on that neighborhood (*Hint*: prove first that the twist condition is an open condition).