3

GHOST CIRCLES

In Chapter 2, we saw how traces of the invariant circles of the completely integrable map persist, either as invariant circles, as periodic orbits or as invariant Cantor sets, in any twist map. The main result of this chapter, Theorem 18.1, provides a vertical ordering of these Aubry-Mather sets in the cylinder for each given map. Indeed, we show that each Aubry-Mather set is a subset of a circle in a family of disjoint, homotopically nontrivial circles that are graph over the circle $\{y = 0\}$. The circles in this family are ordered according to the rotation number of the Aubry-Mather sets.

To prove this, we establish important properties of the gradient flow of the action functional in the space of sequences. The central property, given by the Sturmian Lemma, is that the intersection index of two sequences cannot increase under the gradient flow of the action. One consequence is that the flow is *monotone*: it preserves the natural partial order between sequences. This fact yields a new proof of the Aubry-Mather Theorem. It also enables us to define special invariant sets for the gradient flow that we called ghost circles, which we study in some detail here. The family of circles that neatly arranges the Aubry-Mather sets are projections of ghost circles in the cylinder.

The results of this chapter come from three sources: Golé (1992 a), in which properties of ghost circles were systematically investigated; Golé (1992 b), where gradient flow techniques were used to give a proof of the Aubry-Mather theorem. There was a gap in that last paper, pointed out to me by Sinisa Slijepcevic which is fixed here thanks to a lemma from Koch & al. (1994). Finally, the bulk of this chapter comes from Angenent & Golé (1991), in which we gave a proof of the ordering of Aubry-Mather sets via ghost circles. I am deeply indebted to Sigurd Angenent for letting me publish this work here for the first time. The notion of ghost circles originated in my thesis, in which I was looking for regularity properties for ghost tori, their higher dimensional counterparts . In Chapter 5, a link is made between ghost tori and Floer Homology.

14. Gradient Flow of the Action

A. Definition of the Flow

Throughout this chapter, we consider a twist map f of the cylinder and its lift F whose generating function S is C^2. For simplicity, we will also assume that the second derivative of S is bounded. This mild assumption is satisfied for twist maps of the bounded annulus which are extended to maps of the cylinder as in Lemma 8.2, as well as for standard maps. In this section we investigate the property of the "gradient" flow of the action associated with the generating function S of F solution to:

$$\dot{x}_k = -\nabla W(\boldsymbol{x})_k = -[\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k)], \qquad k \in \mathbb{Z} \tag{14.1}$$

Since this is an infinite system of ODEs, we need to set up the proper spaces to talk about such a flow. We endow $\mathbb{R}^{\mathbb{Z}}$ with the norm :

$$\|\boldsymbol{x}\| = \sum_{-\infty}^{+\infty} \frac{|x_k|}{2^{|k|}}$$

We let X be the subspace of $\mathbb{R}^{\mathbb{Z}}$ of elements of bounded norm, which is a Banach space. On bounded subsets of X, the topology given by the above norm is equivalent to the product topology, itself equivalent to the topology of pointwise convergence.

Remember from Chapter 2 that $\mathbb{Z}^2$ acts on $\mathbb{R}^{\mathbb{Z}}$ by:

$$(\tau_{m,n}\boldsymbol{x})_k = x_{k+m} + n$$

The map $\tau_{0,1}$ which we also denote by T has the effect of translating each term of the sequence by 1. The map $\tau_{1,0}$ which we denote also by σ is called the *shift map*, as it shifts the indices of a sequences by 1. We define $X/\mathbb{Z} := X/T$ and we can choose as a representative of a sequence $\boldsymbol{x}$ one such that $x_0 \in [0, 1)$. More generally, in this chapter, the quotient of any subset of $\mathbb{R}^{\mathbb{Z}}$ by $\mathbb{Z}$ will be with respect to the action of the translation $T = \tau_{0,1}$.

Proposition 14.1 *Suppose that the generating function S is C^2 with bounded second derivative. The infinite system of O.D.E's*

$$(14.2) \qquad \dot{x}_k = -\nabla W(\boldsymbol{x})_k = -[\partial_1 S(x_k, x_{k+1}) + \partial_2 S(x_{k-1}, x_k)]$$

defines a C^1 local flow ζ^t on X as well as on $X/\mathbb{Z}$, for the topology of pointwise convergence. The rest points of ζ^t on X correspond to orbits of the map F.

Proof. We prove that the vector field $-\nabla W$ is C^1 by exhibiting its differential. The proposition follows from general theorems on existence and uniqueness of solutions of ODEs in Banach spaces (Lang (1983) , Theorems 3.1 and 4.3). The following map is the derivative of $\boldsymbol{x} \mapsto -\nabla W(\boldsymbol{x})$:

$$L : \{v_k\}_{k\in\mathbb{Z}} \mapsto \{\beta_k v_{k-1} + \alpha_k v_k + \beta_{k+1} v_{k+1}\}_{k\in\mathbb{Z}}$$
$$\alpha_k = -\partial_{22} S(x_{k-1}, x_k) - \partial_{11} S(x_k, x_{k+1}), \quad \beta_k = -\partial_{12} S(x_{k-1}, x_k)$$

Indeed, this map is linear with (uniformly) bounded coefficients, hence a continuous linear operator. Clearly:

$$-\nabla W(\boldsymbol{x} + \boldsymbol{v}) + \nabla W(\boldsymbol{x}) - L(\boldsymbol{v}) = \|\boldsymbol{v}\| \, \psi(\boldsymbol{v})$$

with $\lim_{\boldsymbol{v}\to 0} \psi(\boldsymbol{v}) = 0$. □

B. Order Properties of the Flow

Angenent (1988) was the first author, to my knowledge, to notice the similarity between the ODE (14.1) and the heat flow of parabolic PDEs. Indeed, when we consider the standard map with generating function $S(x, X) = \frac{1}{2}(X - x)^2 + V(x)$, the ODE (14.1) becomes

$$\dot{x}_k = (-\Delta \boldsymbol{x})_k - V'(x_k)$$

where $\Delta(\boldsymbol{x})_k = 2x_k - x_{k-1} - x_{k+1}$ is the discretized Laplacian. It is not too surprising therefore, that the gradient flow solution of (14.1) inherits analogous order properties to those of heat flows (*eg.* , the comparison principle). In a nice reversal of roles, de la Llave (1999) has now proven Aubry-Mather type theorems for certain PDEs, using order properties (see Chapter 9). To explore these properties in twist maps, we come back to the notion of order introduced in Chapter 2. $\mathbb{R}^{\mathbb{Z}}$ is partially ordered by:

$$\boldsymbol{x} \le \boldsymbol{y} \Leftrightarrow \forall k \in \mathbb{Z}, \quad x_k \le y_k.$$

We also define $\boldsymbol{x} < \boldsymbol{y}$ to mean $\boldsymbol{x} \leq \boldsymbol{y}$, but $\boldsymbol{x} \neq \boldsymbol{y}$; and we write $\boldsymbol{x} \prec \boldsymbol{y}$ to denote the condition $x_j < y_j$ for all $j \in \mathbb{Z}$. The *order interval* $[\boldsymbol{x}, \boldsymbol{y}]$ is defined by:

$$[\boldsymbol{x}, \boldsymbol{y}] = \{\boldsymbol{z} \in \mathbb{R}^{\mathbb{Z}} \mid \boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{y}\}$$

The *positive order cone* at $\boldsymbol{x}$

$$V_+(\boldsymbol{x}) = \{\boldsymbol{y} \in X \mid \boldsymbol{x} \leq \boldsymbol{y}\}$$

with a similar definition for $V_-(\boldsymbol{x})$. These cones are closed for the topology of pointwise convergence.

The following statement was observed by Angenent (1988). It is related to the comparison principle for parabolic PDEs (In the case of the standard map.

Theorem 14.2 (Strict Monotonicity of ζ^t) *For $\boldsymbol{x}, \boldsymbol{y} \in X$ with $\boldsymbol{x} < \boldsymbol{y}$ one has $\zeta^t(\boldsymbol{x}) \prec \zeta^t(\boldsymbol{y})$ for all $t > 0$.*

We will give a simple proof of this theorem in Section 22. It is also a consequence of the Sturmian Lemma (see below), which was stated in Angenent (1988), and written in Angenent & Golé (1991). Both proofs were communicated to the author by Sigurd Angenent. In Chapter 2, we defined the notion of crossing of two sequences $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^{\mathbb{Z}}$ in terms of their Aubry diagrams. We remind the reader that such a *crossing* occurs when there is a $k \in \mathbb{Z}$ at which either $x_k - y_k$ and $x_{k+1} - y_{k+1}$ have opposite signs, or $x_k = y_k$ and $x_{k-1} - y_{k-1}$ and $x_{k+1} - y_{k+1}$ have opposite signs. We say that two sequences are *transverse* if they have no *tangency*, *i.e.* there is no $k \in \mathbb{Z}$ at which $x_k = y_k$ and $x_{k-1} - y_{k-1}$ and $x_{k+1} - y_{k+1}$ have same sign. We denote the transversality of $\boldsymbol{x}$ and $\boldsymbol{y}$ by $\boldsymbol{x} \pitchfork \boldsymbol{y}$. We now define the *intersection index* $I(\boldsymbol{x}, \boldsymbol{y})$ to be the number of crossings of transverse sequences.

Lemma 14.3 (Sturmian Lemma) *Let $\boldsymbol{x}, \boldsymbol{y} \in X$ have different rotation numbers. If $\boldsymbol{x}, \boldsymbol{y}$ are not transverse, then for all sufficiently small $\varepsilon > 0$ $\zeta^{\pm\varepsilon}\boldsymbol{x}, \zeta^{\pm\varepsilon}\boldsymbol{y}$ are transverse and:*

$$I\left(\zeta^{-\varepsilon}\boldsymbol{x}, \zeta^{-\varepsilon}\boldsymbol{y}\right) > I\left(\zeta^{\varepsilon}\boldsymbol{x}, \zeta^{\varepsilon}\boldsymbol{y}\right).$$

Otherwise, as long as $\zeta^t\boldsymbol{x}$ and $\zeta^t\boldsymbol{y}$ stay transverse, their intersection index does not change.

Proof. See Section 22.

As in Section 5.C, $X_{p,q}$ is the space of sequences of type p, q and W_{pq} is the periodic action on these sequences.

Corollary 14.4 *The sets* CO, CO_ω, *and* $X_{p,q}$ *are all invariant under the flow* ζ^t, *and so are their quotients by the action of* $T = \tau_{0,1}$.

Proof. The inequalities of the type $\boldsymbol{x} < \tau_{m,n}\boldsymbol{x}$, which define the sets CO and CO_ω are all preserved under ζ^t. The invariance of $X_{p,q}$ comes from the periodicity of the generating function S and its derivatives: when $\boldsymbol{x} \in X_{p,q}$ the infinite dimensional vector field $-\nabla W$ for the ODE (14.1) is a sequence of period n (made of subsequences of length n equal to ∇W_{pq}). □

15. The Gradient Flow and the Aubry-Mather Theorem

In this section, we show how the existence of CO orbits of all rotation numbers can be recovered from the monotonicity of the gradient flow ζ^t. From Lemma 9.2 and Corollary 14.4, we know that the set $\mathrm{CO}_\omega/\mathbb{Z}$ is compact and invariant under the flow ζ^t. Rest points of the flow in this set lift to CO orbits of rotation number ω. It turns out that, even though ζ^t is not the gradient flow of any function, we can still make it gradient like when restricted to the appropriate subsets. Denote by $X^K = \{\boldsymbol{x} \in X \mid \sup_{k\in\mathbb{Z}} |x_k - x_{k-1}| < K\}$.

Theorem 15.1 *Let* $C \subset X^K/\mathbb{Z}$ *be a compact invariant set under* σ *and forward invariant under the flow* ζ^t. *Then* C *must contain a rest point for the flow. In particular* $\mathrm{CO}_\omega/\mathbb{Z}$ *contains a restpoint and thus the map has a* CO *orbit of rotation number* ω.

Proof. Assume, by contradiction, that there are no rest points in C. We show that, for some large enough N, the truncated energy function $W_N = \sum_{-N}^{N} S(x_k, x_{k+1})$ is a strict Lyapunov function for the flow ζ^t on C. More precisely, we find a real $a > 0$ such that $\frac{d}{dt}W_N(\boldsymbol{x}) < -a$ for all $\boldsymbol{x}$ in C. This immediately yields a contradiction since on one hand

W_N decreases to $-\infty$ on any orbit in C, on the other hand, the continuous W_N is bounded on the compact K. To show that W_N is a Lyapunov function for some N, we start with:

Lemma 15.2 *Let C be as in Theorem 15.1. Suppose that there are no rest points in C. Then, there exist a real $\varepsilon_0 > 0$, a positive integer N_0 such that, for all $\boldsymbol{x} \in C$*

$$N \geq N_0 \Rightarrow \forall j \in \mathbb{Z}, \quad \sum_{j}^{j+N} (\nabla W(\boldsymbol{x})_k)^2 > \varepsilon_0.$$

Proof. Suppose by contradiction that there exist sequences j_n, N_n and $\boldsymbol{x}^{(n)}$ with $N_n \to \infty$ such that

$$\sum_{j_n}^{j_n+N_n} \left(\nabla W(\boldsymbol{x}^{(n)})_k\right)^2 \to 0. \tag{15.1}$$

Let $m(n) = -j_n - [N_n/2]$ where $[\cdot]$ is the integer part function, and let $\boldsymbol{x}'^{(n)} = \sigma^{m(n)} \boldsymbol{x}^{(n)}$. This new sequence $\boldsymbol{x}'^{(n)}$ is still in C, and satisfies:

$$\sum_{k=-[N_n/2]}^{N_n-[N_n/2]} \left(\nabla W(\boldsymbol{x}'^{(n)})_k\right)^2 \to 0 \quad \text{as } n \to \infty.$$

By compactness of C, it has a subsequence that converges pointwise to some $\boldsymbol{x}^\infty$ in C. Since S is C^2, $\nabla W(\boldsymbol{x}^\infty)_k = \lim_{n\to\infty} \nabla W(\boldsymbol{x}'^{(n)})_k = 0$ for all k and thus $\boldsymbol{x}^\infty$ is a rest point, a contradiction. □

We now show that W_N is a strict Lyapunov function on C. By chain rule:

$$\begin{aligned} \frac{d}{dt} W_N(\boldsymbol{x}) &= -\sum_{-N}^{N} [\partial_1 S(x_k, x_{k+1}) \nabla W(\boldsymbol{x})_k + \partial_2 S(x_k, x_{k+1}) \nabla W(\boldsymbol{x})_{k+1}] \\ &= -\sum_{-N}^{N} \partial_1 S(x_k, x_{k+1}) \nabla W(\boldsymbol{x})_k - \sum_{-N+1}^{N+1} \partial_2 S(x_{k-1}, x_k) \nabla W(\boldsymbol{x})_k \\ &= -\partial_1 S(x_{-N}, x_{-N+1}) \nabla W(\boldsymbol{x})_{-N} - \partial_2 S(x_N, x_{N+1}) \nabla W(\boldsymbol{x})_{N+1} \\ &\quad - \sum_{-N+1}^{N} \left(\nabla W(\boldsymbol{x})_k\right)^2 \end{aligned} \tag{15.2}$$

For all $\boldsymbol{x}$ in X^K, we have $|x_k - x_{k-1}| < K$ and hence, by periodicity, $S(x_{k-1}, x_k)$, its partial derivatives as wellas ∇W_k are bounded on X^K. In particular, we can find some M depending only on K such that

$$|-\partial_1 S(x_{-N}, x_{-N+1})\nabla W(\boldsymbol{x})_{-N} - \partial_2 S(x_N, x_{N+1})\nabla W(\boldsymbol{x})_{N+1}| < M$$

for all $\boldsymbol{x}$ in X^K and all integer k. Let $p = [M/2\varepsilon_0]$ and $N > (p+1)N_0$, where N_0, ε_0 are as in Lemma 15.2. We claim that for such an N, W_N is a Lyapunov function. Indeed, we can split the sum $\sum_{-N+1}^{N} \left(\nabla W(\boldsymbol{x})_k\right)^2$ into $2p+2$ sums of length greater than N_0. By Lemma 15.2, each of these subsums must be greater than ε_0, and thus the total sum must be greater than $M + 2\varepsilon_0$, making the expression in (15.2) less than $-2\varepsilon_0$. □

Remark 15.3 As in Chapter 2, we can derive from Theorem 15.1 the existence of Aubry-Mather sets of all rotation numbers. This proof does not yield the fact that the orbits found are minimizers. This apparent weakness may be an asset in considering possible generalizations of this theorem to higher dimensions (see Chapter 9). This proof is a variation of the one given in Golé (1992 b). We are very grateful to Sinisa Slijepcevic, who pointed to a gap in Section 3 of that paper. The above is essentially a rewriting of that section. It was inspired by arguments found in Koch & al. (1994), who prove an interesting generalization of the Aubry-Mather Theorem for functions on lattices of any dimensions (see Chapter 9).

16. Ghost Circles

The set of critical sequences corresponding to the orbits of an invariant circle of the twist map f, is itself a circle in $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z}$. Trivially, this circle is invariant under ζ^t, as it is made of rest points of the flow. This circle is one instance of a ghost circle. In general, we think of ghost circles as ζ^t-invariant sets that are the surviving traces *in the sequence space* $\mathbb{R}^{\mathbb{Z}}$ of such critical circles.

Definition 16.1 A subset $\Gamma \subset \mathbb{R}^{\mathbb{Z}}$ is a Ghost Circle, hereafter GC, if it is

1. strictly ordered: $\boldsymbol{x}, \boldsymbol{y} \in \Gamma \Rightarrow \boldsymbol{x} \prec \boldsymbol{y}$ or $\boldsymbol{y} \prec \boldsymbol{x}$.
2. invariant under the $\mathbb{Z}^2$ action (by $\tau_{m,n}$), as well as under the flow ζ^t,

3. closed and connected.

We will see in the Section 17 that GC's can be constructed by bridging the gaps of the Aubry-Mather sets (identified to their corresponding subsets of rest points in $\mathbb{R}^{\mathbb{Z}}$) with connecting orbits of the gradient flow ζ^t.

Any sequence $\boldsymbol{x}$ in a ghost circle Γ is CO: since $\tau_{m,n}\boldsymbol{x}$ must also lie in Γ, which is ordered, we must have $\boldsymbol{x} \prec \tau_{m,n}\boldsymbol{x}$ or $\tau_{m,n}\boldsymbol{x} \prec \boldsymbol{x}$. Moreover, the fact that Γ is ordered implies, by Lemma 13.3, that all sequences in Γ have same rotation number. We will call this number $\rho(\Gamma)$, the *rotation number of the ghost circle*.

Proposition 16.3 *Let Γ be a ghost circle.*
a) The coordinate projection map $\mathbb{R}^{\mathbb{Z}} \mapsto \mathbb{R}$ defined by $\boldsymbol{x} \mapsto x_0$ induces a homeomorphism of Γ to $\mathbb{R}$. The corresponding projection map $\mathbb{R}^{\mathbb{Z}}/\mathbb{Z} \mapsto \mathbb{R}/\mathbb{Z}$ induces a homeomorphism between $\Gamma/\mathbb{Z}$ and the circle.
b) The set of ghost circles is closed in the Hausdorff topology of closed sets of $\mathbb{R}^{\mathbb{Z}}$, and it is compact in $\mathrm{CO}_{[a,b]}/\mathbb{Z}$. The rotation number on GCs is continuous in this topology.

Proposition 20.2 improves on part b) of this proposition by giving a sufficient condition for convergence of sequences of GCs

Proof of Proposition 16.3. We show that, for any $\boldsymbol{x}, \boldsymbol{y}$ in Γ, the projection $\delta : \boldsymbol{x} \mapsto x_0$ defines a homeomorphism from $[\boldsymbol{x}, \boldsymbol{y}] \cap \Gamma$ to the interval $[x_0, y_0]$ in $\mathbb{R}$. As before, we give $\mathbb{R}^{\mathbb{Z}}$ the product topology. The projection map δ is continuous and the set $[\boldsymbol{x}, \boldsymbol{y}]$ is compact, by Tychonov Theorem, as a product of closed intervals. Clearly δ preserves the strict order: $\boldsymbol{x} \prec \boldsymbol{y} \Rightarrow x_0 < y_0$ and hence it is one to one on Γ. Take any two points $\boldsymbol{x} \prec \boldsymbol{y}$ in Γ. As a continuous injection, the map δ defines a homeomorphism between the compact set $\Gamma \cap [\boldsymbol{x}, \boldsymbol{y}]$ and its image. We show that $\delta(\Gamma \cap [\boldsymbol{x}, \boldsymbol{y}]) = [\delta(\boldsymbol{x}), \delta(\boldsymbol{y})]$. For this, it suffices to show that $\Gamma \cap [\boldsymbol{x}, \boldsymbol{y}]$ is connected. Suppose not and $\Gamma \cap [\boldsymbol{x}, \boldsymbol{y}] = A \cup B$ where A and B are closed and disjoint in $\Gamma \cap [\boldsymbol{x}, \boldsymbol{y}]$. There are two possibilities: either both $\boldsymbol{x}$ and $\boldsymbol{y}$ belong to the same set, say A or else $\boldsymbol{x} \in A$, $\boldsymbol{y} \in B$. In the first case, we could write Γ as the union of two disjoint closed sets:

$$\Gamma = [(V_-(\boldsymbol{x}) \cap \Gamma) \cup A \cup (V_+(\boldsymbol{y}) \cap \Gamma)] \bigcup_{\neq} B,$$

a contradiction since Γ is connected. The other case yields the same contradiction. Since Γ is ordered, any bounded open ball for the product topology intersects Γ inside an interval $[\boldsymbol{x}, \boldsymbol{y}]$. Hence what we have shown above implies in particular that δ is a local homeomorphism on Γ. To show that it is a global homeomorphism, it remains to show that it is onto. Since Γ is τ–invariant, if $\boldsymbol{x}$ is a point of Γ, then $\tau_{m,0}\boldsymbol{x}$ is as well, and hence the set $\{x_0 + m \mid m \in \mathbb{Z}\}$ is in $\delta(\Gamma)$. By what we proved above, all the points in between are also in $\delta(\Gamma)$ and hence δ is onto $\mathbb{R}$.

This proves a). To prove b), note that if $\Gamma_k \to \Gamma$ (in the Hausdorff topology) as $k \to \infty$ then any point $\boldsymbol{x} \in \Gamma$ is limit (in the product topology of $\mathbb{R}^{\mathbb{Z}}$) of points $\boldsymbol{x}^{(k)} \in \Gamma_k$. Since $\tau_{m,n}$ and the flow ζ^t are continuous, Γ must be invariant under these maps. “Close” and “connected” are adjectives that also behave well under Hausdorff limits. Finally, to see that Γ is strictly ordered, note that if $\boldsymbol{x} \neq \boldsymbol{y}$ are in Γ, we can find sequences $\boldsymbol{x}^{(k)}, \boldsymbol{y}^{(k)} \in \Gamma_k$ with $\boldsymbol{x} = \lim \boldsymbol{x}^{(k)}, \boldsymbol{y} = \lim \boldsymbol{y}^{(k)}$. If $x_j < y_j$, we can assume $\boldsymbol{x}^{(k)} \prec \boldsymbol{y}^{(k)}$ for all k sufficiently large. Since Γ_k is strictly ordered and ζ^t-invariant, we must have $\zeta^{-t}\boldsymbol{x}^{(k)} \prec \zeta^{-t}\boldsymbol{y}^{(k)}$ and hence $\zeta^{-t}\boldsymbol{x} \leq \zeta^{-t}\boldsymbol{y}$. The strict monotonicity of the flow now implies: $\boldsymbol{x} \prec \boldsymbol{y}$. The continuity of the rotation number is a direct consequences of the continuity of the rotation number on CO sequences, given by Lemma 9.1. □

It follows from this proposition that any GC has a parameterization $\xi \in \mathbb{R} \mapsto \boldsymbol{x}(\xi) \in \Gamma$ of the form

$$\boldsymbol{x}(\xi) = (\cdots, x_{-1}(\xi), \xi, x_1(\xi), x_2(\xi), \cdots)\,. \tag{16.1}$$

where the $x_j(\xi)$ are strictly increasing and continuous functions of ξ. In particular $\xi \mapsto x_1(\xi)$ is a homeomorphism of $\mathbb{R}$. Invariance of Γ under the $\mathbb{Z}^2$ action τ implies that $x_j(\xi + 1) \equiv x_j(\xi) + 1$, so that the x_j define homeomorphisms of the circle as well; τ-invariance also implies that $x_2(\xi) = x_1(x_1(\xi))$, and more generally that the x_n are all iterates of x_1.

Any GC projects naturally to a circle $\pi\Gamma$ in the annulus, where the projection $\pi : \mathbb{R}^{\mathbb{Z}} \to$ **A** is defined by

$$\pi(\boldsymbol{x}) = (x_0, -\partial_1 S(x_0, x_1))$$

Proposition 16.3 *Let Γ be a GC for the twist map f. Then $\pi\Gamma$ and $f(\pi\Gamma)$ are periodic graphs of periodic functions $\varphi(\xi)$ and $\psi(\xi)$ such that there is a constant*

$L < \infty$, depending only on the map, and, where the derivatives are defined,

$$\varphi'(\xi) \geq -L, \qquad \psi'(\xi) \leq L.$$

Proof. If one parameterizes Γ as in (16.1) , then $\pi\Gamma$ is the graph of

$$y = -\partial_1 S(\xi, x_1(\xi)) \stackrel{\text{def}}{=} \varphi(\xi). \tag{16.2}$$

Likewise, the image $f(\pi\Gamma)$ is the graph of $y = \partial_2 S(x_{-1}(\xi), \xi) = \psi(\xi)$. We now give a proof of the Lipschitz estimate. Using the parameterization of the projection of our GC as in (16.2), it is enough to prove that the derivative of φ is bounded below. The same proof would hold for the estimate for the image $f(\pi\Gamma)$ of our circle. Applying the chain rule to (16.2), we find:

$$\varphi' = -\partial_{11} S - \partial_{12} S \cdot \frac{dx_1}{d\xi} \geq -\partial_{11} S.$$

This last term is bounded below by our assumption on the second derivative of S. A similar argument proves the estimate for $\psi'(\xi)$. □

Remark 16.4 As mentioned before (see also Exercise 16.6), the set of critical sequences corresponding to an invariant circle of f is a GC, call it Γ. In this case $\pi\Gamma = f(\pi\Gamma)$, and Proposition 16.3 provides a proof that invariant circles are Lipschitz, a result of Birkhoff (see also Proposition 12.3).

We end this section by giving a condition that insures that GCs do not intersect. We can define a partial ordering on GC's as follows. Let Γ_1, Γ_2 be GCs. We say that $\Gamma_1 \prec \Gamma_2$ if

(i) for all $\boldsymbol{x} \in \Gamma_1, \boldsymbol{x}' \in \Gamma_2$ one has $\boldsymbol{x} \pitchfork \boldsymbol{x}'$ and $I(\boldsymbol{x}, \boldsymbol{x}') = 1$;

(ii) $\rho(\Gamma_1) < \rho(\Gamma_2)$, i.e. $\rho(\boldsymbol{x}) < \rho(\boldsymbol{x}')$.

Lemma 16.5 (Graph Ordering Lemma) *If $\Gamma_1 \prec \Gamma_2$ then the circle $\pi\Gamma_1$ lies below $\pi\Gamma_2$.*

Proof. Let $x_n^{(j)}(\xi)$ be parameterizations of the form (16.1) for Γ_j ($j = 1, 2$). Then $\pi\Gamma_j$ is the graph of $\varphi_j(\xi) = -\partial_1 S(\xi, x_1^{(j)}(\xi))$. We claim that $x_1^{(1)}(\xi) < x_1^{(2)}(\xi)$ for all ξ. Indeed, for a given ξ the sequences $x_n^{(1)}(\xi)$ and $x_n^{(2)}(\xi)$ intersect at site $n = 0$. Since they are transverse, we must have $x_1^{(1)}(\xi) \neq x_1^{(2)}(\xi)$; by comparing rotation numbers we then get

$x_1^{(1)}(\xi) < x_1^{(2)}(\xi)$. By combining this inequality with the twist condition $\partial_{12}S < 0$ we then conclude that $\varphi_1(\xi) < \varphi_2(\xi)$, as claimed. □

Exercise 16.6 Prove that the set of $\boldsymbol{x}$ sequences corresponding to orbits of an nontrivial invariant circle for the map is a GC. [If the map has a transitive invariant circle of rotation number ω, then its associated GC is the only GC with rotation number ω (Golé (1992 a), Lemma 4.22. We conjecture that this remains true when the invariant circle is not transitive (i.e., of Denjoy type).

17. Construction of Ghost Circles

This section will show that GCs are plentiful. In the first subsection we construct GCs whose projection passes through any given Aubry-Mather set. The next subsection will specialize to GCs with rational rotation numbers. For generic twist maps, we construct smooth GCs containing periodic minimizers. In Section 18 we will refine this construction to obtain ordered sets of GCs, whose projections do not intersect.

A*. Ghost Circles Through Any Aubry-Mather Sets

Let M_ω the minimal, recurrent Aubry-Mather set of rotation number ω, as defined in Proposition 12.9. It corresponds bijectively to the set, call it Σ_ω of $\boldsymbol{x}$ sequences of orbits in M_ω. By Aubry's Fundamental Lemma 10.2, Σ_ω is a completely ordered subset of CO_ω. If $\boldsymbol{x}$ is a recurrent minimizer, than so is $\tau_{m,n}\boldsymbol{x}$ for any $m, n \in \mathbb{Z}$, so Σ_ω is invariant under τ. Each point of Σ_ω corresponds to an orbit of F, and thus is a rest point of ζ^t. In Golé (1992 a), we proved the following theorem:

Theorem 17.1 *The set Σ_ω is included in a ghost circle Γ, and hence the Aubry-Mather set M_ω is included in the projection $\pi\Gamma$ of a ghost circle.*

Proof (Sketch). Σ_ω is a Cantor set whose complementary gaps are included in order intervals of the type $]\boldsymbol{x}, \boldsymbol{y}[$ where $\boldsymbol{x}, \boldsymbol{y} \in \Sigma_\omega$. A theorem of Dancer and Hess (1991) on monotone flows implies that, in conditions that are satisfied in the present case, if $\boldsymbol{x} \prec \boldsymbol{y}$ are two rest points for the strictly monotone flow ζ^t and there is no other restpoint in $[\boldsymbol{x}, \boldsymbol{y}]$ then there must be a *monotone orbit* (*i.e.* completely ordered) of ζ^t joining $\boldsymbol{x}$ and $\boldsymbol{y}$. Hence we can bridge all the gaps of Σ_ω with ordered orbits of ζ^t, taking care to do so in an equivariant way with respect to the τ action. The resulting set is a GC. □

B. Smooth, Rational Ghost Circles

We now build rational Ghost Circles by piecing together the unstable manifolds of mountain pass points for W_{pq} in $X_{p,q}$. This construction will be crucial when we build disjoint GCs in Section 18. Let $\omega = p/q$ be given. Beginning here and throughout Sections 18 and 19 , we shall assume the following:

$$\textit{For any } p/q \in \mathbb{Q}, \quad W_{pq} \textit{ is a Morse–function on } X_{pq}. \tag{17.1}$$

This is a generic condition on twist maps, as will be proven in Proposition 29.6. Since a GC consists of CO sequences we may assume that p and q have no common divisor (see the proof of Proposition 10.4). Let $\boldsymbol{x} \in X_{p,q}$ be a critical point of W_{pq}. The second derivative of W_{pq} at $\boldsymbol{x}$ is a *Jacobi matrix*: it is tridiagonal with positive subdiagonal terms and positive "corner" elements as well:

$$-\nabla^2 W_{pq}(\boldsymbol{x}) = \begin{bmatrix} \alpha_1 & \beta_1 & 0 & \cdots & \beta_q \\ \beta_1 & \alpha_2 & \beta_2 & \cdots & 0 \\ 0 & \beta_2 & \alpha_3 & \ddots & \vdots \\ & & \ddots & \ddots & \beta_{q-1} \\ \beta_q & 0 & \cdots & \beta_{q-1} & \alpha_q \end{bmatrix}, \tag{17.2}$$

where $\alpha_j = -\partial_{22}S(x_{j-1}, x_j) - \partial_{11}S(x_j, x_{j+1})$, and $\beta_j = -\partial_{12}S(x_{j-1}, x_j) > 0$. Due to the Perron–Fröbenius theorem, the largest eigenvalue λ_0 of $-\nabla^2 W_{pq}(\boldsymbol{x})$ is simple, and the eigenvector $\boldsymbol{v} = (v_j)$ corresponding to λ_0 can be chosen to be positive: $v_j > 0, j = 1, \ldots, q$. Moreover all other eigenvectors are in different orthants (See Angenent (1988), Proposition 3.2 and Lemma 3.4). If $\boldsymbol{x}$ is a critical point of index 1, there exist two orbits $\alpha_\pm(\boldsymbol{x}; t), t \in \mathbb{R}$ of the gradient flow ζ^t of W_{pq} with $\alpha_\pm(\boldsymbol{x}; t) \to \boldsymbol{x}$ as $t \to -\infty$, and with

$$\alpha_\pm(\boldsymbol{x}; t) = \boldsymbol{x} \pm e^{\lambda_0 t}\xi + o\left(e^{\lambda_0 t}\right).$$

These two orbits, together with $\boldsymbol{x}$ itself, form the unstable manifold of $\boldsymbol{x}$. The orbits $\alpha_\pm(\boldsymbol{x}; t)$ are monotone, α_+ being increasing, and α_- decreasing; since $\tau_{\pm 1,0}\boldsymbol{x} = \boldsymbol{x} \pm 1$ are also critical points, we have $\boldsymbol{x} - 1 \le \alpha_\pm(\boldsymbol{x}; t) \le \boldsymbol{x} + 1$ so that $\alpha_\pm(\boldsymbol{x}; t)$ is bounded. Hence the limits

$$\omega_\pm(\boldsymbol{x}) = \lim_{t\to\infty} \alpha_\pm(\boldsymbol{x};t)$$

exist and they are critical points of W_{pq}. Since ζ^t is monotone, there are no other critical points $\boldsymbol{y}$ with $\omega_-(\boldsymbol{x}) < \boldsymbol{y} < \boldsymbol{x}$ or $\boldsymbol{x} < \boldsymbol{y} < \omega_+(\boldsymbol{x})$. If $\boldsymbol{y} > \boldsymbol{x}$ is another critical point, then $\boldsymbol{y} \geq \omega_+(\boldsymbol{x})$. Moreover, since the Morse index must decrease along the negative gradient flow, the points $\omega_\pm(\boldsymbol{x})$ have index 0, $i.e.$ they are local minima of W_{pq}. We now show that the orbits $\alpha_\pm(\boldsymbol{x};t)$ converge to these points along their "slow stable manifold", tangent to the largest eigenvalue of $-\nabla^2 W_{pq}(\omega_\pm(\boldsymbol{x}))$. Indeed, since $\omega_\pm(\boldsymbol{x})$ are minima, all the eigenvalues are negative, and thus the largest one has the smallest modulus. All orbits in the stable manifold of $\omega_\pm(\boldsymbol{x})$ except for a finite number that are tangent to the eigenspaces of the other eigenvalues, are tangent to this "slow stable manifold". But the other eigenvectors are in different orthants than the positive or negative ones. Hence $\alpha_\pm(\boldsymbol{x};t)$, which are in the positive or negative orthant of $\omega_\pm(\boldsymbol{x})$, must converge to $\omega_\pm(\boldsymbol{x})$ tangentially to the eigenvector of largest eigenvalue.

To construct a GC in W_{pq} we first consider the set of critical points such a GC must contain.

Definition 17.2 A subset $\mathcal{A} \subset X_{p,q}$ is a *skeleton* if the following hold.

S_1 $\mathcal{A}$ consists of critical points of W_{pq} with Morse index ≤ 1,

S_2 $\mathcal{A}$ is invariant under the $\mathbb{Z}^2$ action τ,

S_3 $\mathcal{A}$ is completely ordered.

A skeleton $\mathcal{A}$ is *maximal* if the only skeleton $\mathcal{A}'$ with $\mathcal{A} \subset \mathcal{A}' \subset X_{p,q}$ is $\mathcal{A}$ itself.

Lemma 17.3 *A maximal skeleton $\mathcal{A}$ for W_{pq} exists.*

Proof. Choose r, s with $rp+qs = 1$ and define $\mathcal{T} = \tau_{r,s}$. By Aubry's fundamental lemma the set $\mathcal{A}_0$ of absolute minimisers of W_{pq} is a skeleton. We fix some element $\boldsymbol{x} \in \mathcal{A}_0$. Any skeleton $\mathcal{A} \supset \mathcal{A}_0$ is completely determined by

$$\mathcal{B} = \mathcal{A} \cap [\boldsymbol{x}, \mathcal{T}(\boldsymbol{x})] = \{\boldsymbol{z} \in \mathcal{A} : \boldsymbol{x} < \boldsymbol{z} < \mathcal{T}(\boldsymbol{x})\}.$$

Indeed, given $\mathcal{B}$ we can reconstruct $\mathcal{A}$ as follows:

$$\mathcal{A} = \bigcup_{j=-\infty}^{\infty} \mathcal{T}^j(\mathcal{B}). \tag{17.3}$$

Conversely, any ordered set $\mathcal{B} \subset [x, \mathcal{T}(\boldsymbol{x})]$ of critical points determines a skeleton $\mathcal{A} \supset \mathcal{A}_0$ by (17.3). The closed order interval $[x, \mathcal{T}(\boldsymbol{x})]$ is compact and W_{pq} is a Morse function, so there are only finitely many critical points in $[x, \mathcal{T}(\boldsymbol{x})]$. We can therefore choose a maximal ordered set of critical points $\mathcal{B} \subset [x, \mathcal{T}(\boldsymbol{x})]$ and be sure that the corresponding $\mathcal{A}$ is a maximal skeleton. □

Lemma 17.4 (Mountain Pass Lemma) *If the skeleton $\mathcal{A}$ is maximal, then every other point (according to the order) is a local minimum; the remaining points are minimaxes.*

Proof. If $\boldsymbol{x} < \boldsymbol{y}$ are consecutive elements of a maximal skeleton $\mathcal{A}$ then we must show that exactly one of $\boldsymbol{x}$ and $\boldsymbol{y}$ is a local minimum.

Step 1. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are both local minima then the following standard minimax argument shows that there is a third critical point of index 1 between $\boldsymbol{x}$ and $\boldsymbol{y}$. Define $\mathcal{Q} = [\boldsymbol{x}, \boldsymbol{y}]$ and consider

$$\mathcal{Q}^\gamma = \{\boldsymbol{z} \in \mathcal{Q} : W_{pq}(\boldsymbol{z}) \leq \gamma\}$$

Each $\mathcal{Q}^\gamma$ is compact, and if $\gamma > \max W_{pq}\big|_{\mathcal{Q}}$ then $\mathcal{Q}^\gamma = \mathcal{Q}$ is connected. On the other hand, $\mathcal{Q}^{\gamma_0}$ with $\gamma_0 = \max\left(W_{pq}(\boldsymbol{x}), W_{pq}(\boldsymbol{y})\right)$ is not connected, since $\boldsymbol{x}$ and $\boldsymbol{y}$ are local minima of W_{pq}. Consider

$$\gamma_1 = \inf\{\gamma > \gamma_0 : \boldsymbol{x} \text{ and } \boldsymbol{y} \text{ are in the same connected component of } \mathcal{Q}^\gamma \}.$$

By compactness, $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected in $\mathcal{Q}^{\gamma_1}$, and hence $\gamma_1 > \gamma_0$. Suppose there is no critical point of W_{pq} in $]\boldsymbol{x}, \boldsymbol{y}[$. Note that, by order preservation, $\mathcal{Q} = [\boldsymbol{x}, \boldsymbol{y}]$ is forward invariant under the gradient flow: $\zeta^t(\mathcal{Q}) \subset \mathcal{Q}$ for $t \geq 0$. By compactness of $\mathcal{Q}^{\gamma_1} = \cap_{\gamma > \gamma_1} \mathcal{Q}^\gamma$ there is an $\varepsilon > 0$ such that $\zeta^1(\mathcal{Q}^{\gamma_1}) \subset \mathcal{Q}^{\gamma_1 - \varepsilon}$, which implies that $\boldsymbol{x}$ and $\boldsymbol{y}$ are connected in $\mathcal{Q}^{\gamma_1 - \varepsilon}$, a contradiction. Hence there is at least one critical point $\boldsymbol{z} \in]\boldsymbol{x}, \boldsymbol{y}[$, with $W_{pq}(\boldsymbol{z}) = \gamma_1$. If the Morse index of all such $\boldsymbol{z}$ were 2 or more, then the Morse Lemma 61.1 would show that $\mathcal{Q}^\gamma$ with γ slightly less than γ_1 would still connects $\boldsymbol{x}$ and $\boldsymbol{y}$, so the index of at least one such $\boldsymbol{z}$ is 1. But now we have a contradiction: if $\boldsymbol{x}$ and $\boldsymbol{y}$ are both local minima, then there is a minimax point $\boldsymbol{z} \in]\boldsymbol{x}, \boldsymbol{y}[$ and $\mathcal{A} \cup \{\tau_{m,n}\boldsymbol{z} : m, n \in \mathbb{Z}\}$ is a skeleton; this cannot be since $\mathcal{A}$ is maximal.

Step 2. Next we show that either $\boldsymbol{x}$ or $\boldsymbol{y}$ is a local minimum.If $\boldsymbol{x}$ is not a local minimum, then $\omega_+(\boldsymbol{x}) = \lim_{t\to\infty} \alpha_+(\boldsymbol{x};t)$ is a local minimum. But $\omega_+(\boldsymbol{x}) \leq \boldsymbol{y}$, so $\omega_+(\boldsymbol{x}) = \boldsymbol{y}$, and we find that $\boldsymbol{y}$ must be a local minimum. Likewise, if $\boldsymbol{y}$ is not a local minimum, then $\boldsymbol{x} = \omega_-(\boldsymbol{y})$ must be one. □

We have all the ingredients necessary to show the following, which was proven in a slightly different form in Golé (1992 a), Theorem 3.6.

Theorem 17.5 *Assume W_{pq} is a Morse function. If $\mathcal{A}$ is a maximal skeleton, then*

$$\Gamma_{\mathcal{A}} = \{\alpha_\pm(\boldsymbol{x};t) : t \in \mathbb{R}, \boldsymbol{x} \in \mathcal{A} \textit{ is a minimax}\} \cup \mathcal{A}$$

is a C^1 ghost circle.

Proof. It is simple to check that, by maximality, $\Gamma_{\mathcal{A}}$ is connected, and a ghost circle. As a union of unstable manifolds, $\Gamma_{\mathcal{A}}$ is smooth except perhaps where different unstable manifold meet, at the minima. But we showed above how the orbits $\alpha_\pm(\boldsymbol{x};t)$ must converge tangentially to the one dimensional eigenspace in the positive-negative cone of the minima. Hence the GC constructed is also smooth at the minima. □

Exercise 17.6 Check that $\Gamma_{\mathcal{A}}$ is indeed a GC.

18. Construction of Disjoint Ghost Circles

We now arrive at the main result of this chapter, which provides a vertical ordering of Aubry-Mather sets:

Theorem 18.1 (Ordering of Aubry-Mather Sets) *Given any interval $[a,b]$ in $\mathbb{R}$ there is a family of nontrivial circles $C_\omega, \omega \in [a,b]$ in the cylinder such that:*
(a) Each C_ω is the projection of a GC Γ_ω and hence is a graph over $\{y=0\}$ (as is $f(C_\omega)$).
(b) The C_ω are mutually disjoint and if $\omega > \omega'$, C_ω is above $C_{\omega'}$.
(c) Each C_ω contains the Aubry-Mather set M_ω of recurrent minimizer of rotation number ω.

This section and the next two are devoted to the proof of this theorem. We will first construct, in this section and the next one, finite families of rational ghost circles. In Section 20, we will take limits of such families and conclude the proof of the theorem.

Let $\omega_1, \ldots, \omega_k$ be distinct rational numbers. The construction of the preceding section provides us with maximal skeletons $\mathcal{A}_1, \ldots, \mathcal{A}_k$ and corresponding GC's $\Gamma_{\mathcal{A}_1}, \ldots, \Gamma_{\mathcal{A}_k}$. It is not immediatly clear from this construction that the projections $C_j = \pi \Gamma_{\mathcal{A}_j}$ are disjoint. In this section we show that the skeletons can be chosen so that the C_j are indeed disjoint.

Definition 18.2 A family of skeletons $\mathcal{A}_j \subset X_{p_j q_j}$ is *minimally linked* if any pair $\boldsymbol{x} \in \mathcal{A}_i, \boldsymbol{y} \in \mathcal{A}_j$ with $i \neq j$ is transverse with $I(\boldsymbol{x}, \boldsymbol{y}) = 1$.

Theorem 18.3 (Disjointness Theorem) *If $\mathcal{A}_j \subset X_{p_j q_j}$ is a minimally linked family of maximal skeletons, then the projected ghost circles $C_j = \pi \Gamma_{\mathcal{A}_j}$ are disjoint.*

Proof. Order the $\mathcal{A}_j$ so that their rotation numbers $\rho_j = p_j / q_j$ are increasing. Then we claim that

$$\Gamma_{\mathcal{A}_1} \prec \Gamma_{\mathcal{A}_2} \prec \Gamma_{\mathcal{A}_3} \prec \cdots \prec \Gamma_{\mathcal{A}_k}. \tag{18.1}$$

Disjointness of the projected GCs then follows directly from the Graph Ordering Lemma 16.5. To see why (18.1) holds, we consider any pair $\boldsymbol{x}^{(i)} \in \Gamma_{\mathcal{A}_i}$, $\boldsymbol{x}^{(j)} \in \Gamma_{\mathcal{A}_j}$ and assume that they are not transverse. Since $\rho(\zeta^t \boldsymbol{x}^{(i)}) \neq \rho(\zeta^t \boldsymbol{x}^{(j)})$ we must always have $I\left(\zeta^t \boldsymbol{x}^{(i)}, \zeta^t \boldsymbol{x}^{(j)}\right) \geq 1$ when defined. By the Sturmian Lemma 14.3,

$$I\left(\zeta^t \boldsymbol{x}^{(i)}, \zeta^t \boldsymbol{x}^{(j)}\right) > 1 \tag{18.2}$$

for all those $t < 0$ at which $\zeta^t \boldsymbol{x}^{(i)} \pitchfork \zeta^t \boldsymbol{x}^{(j)}$. But $\lim_{t \to -\infty} \zeta^t \boldsymbol{x}^{(l)} = \boldsymbol{y}^{(l)}$ for some $\boldsymbol{y}^{(l)} \in \mathcal{A}_l$ ($l = i, j$). Since the $\mathcal{A}_l$ are minimally linked we must have $I(\boldsymbol{y}^{(i)}, \boldsymbol{y}^{(j)}) = 1$, which contradicts (18.2) . $\square$

Theorem 18.4 *For any k-tuple $\omega_1, \ldots, \omega_k$ of rational numbers there exists a minimally linked family of skeletons $\mathcal{A}_1, \ldots, \mathcal{A}_k$ such that each $\mathcal{A}_j$ is a maximal skeleton.*

This theorem, combined with the Disjointness Theorem, provides us with a disjoint family of circles $C_j = \pi \Gamma_{\mathcal{A}_j}$ in the annulus. The construction of the $\mathcal{A}_j$'s will be such that they automatically contain the absolute minimizers of $W_{p_i q_i}$, which by Proposition 10.4 are the minimal energy orbits of Aubry–Mather. In our proof of Theorem 18.4 we begin with constructing a maximal k-tuple of skeletons, and then show that each skeleton in this k-tuple is maximal.

Proof of Theorem 18.4. Let $\mathcal{M}_j$ be the set of absolute minimizers of $W_{p_j q_j}$ on $X_{p_j q_j}$. Aubry's fundamental lemma implies that $\mathcal{M}_1, \ldots, \mathcal{M}_k$ is a minimally linked family of skeletons. As in the proof of Lemma 17.3 one easily finds a maximal k-tuple of *minimally linked* skeletons $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with $\mathcal{M}_j \subset \mathcal{A}_j$, by observing that there are only finitely many such extensions. We shall now show that each $\mathcal{A}_j$ is a maximal skeleton.

Assume that one of the $\mathcal{A}_j$, say $\mathcal{A}_1$ is not maximal. Then there is a critical point $z \in W_{p_1 q_1}$ with index 0 or 1, such that $\mathcal{A}_1 \cup \{z\}$ is completely ordered. In particular, there must exist a couple of adjacent critical points $\boldsymbol{x} < \boldsymbol{y}$ in $\mathcal{A}_1$ with $\boldsymbol{z} \in]\boldsymbol{x}, \boldsymbol{y}[$. We must deal with two different cases:

A. Both $\boldsymbol{x}$ and $\boldsymbol{y}$ are local minima of $W_{p_1 q_1}$.

B. At least one of the critical points $\boldsymbol{x}$ or $\boldsymbol{y}$ has index 1.

Case A. By a minimax argument we will show that there is a critical point between $\boldsymbol{x}$ and $\boldsymbol{y}$ which allows us to extend $\mathcal{A}_1$ to a larger skeleton $\mathcal{A}_1'$ for which $(\mathcal{A}_1', \ldots, \mathcal{A}_k)$ is still minimally linked. This would then contradict maximality of $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$, and thereby show that Case A cannot occur. To carry out the minimax argument we consider

$$\Omega = \{\boldsymbol{w} \in W_{p_1 q_1} : \boldsymbol{x} < \boldsymbol{w} < \boldsymbol{y}, \forall j \geq 2, \forall \boldsymbol{v} \in \mathcal{A}_j, \boldsymbol{v} \pitchfork \boldsymbol{w} \text{ and } I(\boldsymbol{v}, \boldsymbol{w}) = 1\}.$$

and its closure $\bar{\Omega}$. The Sturmian Lemma implies that Ω, and hence $\bar{\Omega}$ are forward invariant under the gradient flow ζ^t. Also, as in Mountain Pass Lemma 17.4, $\bar{\Omega}$ is compact, as are the sublevel sets $\bar{\Omega}^\gamma = \{\boldsymbol{w} \in \bar{\Omega} : W_{p_1 q_1}(\boldsymbol{w}) \leq \gamma\}$. To obtain a critical point other than $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\bar{\Omega}$ we must show that not all the $\bar{\Omega}^\gamma$'s have the same topology. If $\gamma_0 = \max(W_{p_1 q_1}(\boldsymbol{x}), W_{p_1 q_1}(\boldsymbol{y}))$, then $\bar{\Omega}^{\gamma_0}$ is again not connected, since $\boldsymbol{x}$ and $\boldsymbol{y}$ are local minima. On the other hand we have

Lemma 18.5 *$\bar{\Omega}$ is connected.*

Postponing the proof of this statement to the next section, we can now easily complete the minimax argument. Indeed, as in the Mountain Pass Lemma,

$$\gamma_1 = \inf\left(\gamma > \gamma_0 : \bar{\Omega}^\gamma \text{ connected}\right)$$

is a critical value of $W_{p_1 q_1}$, so there must be a third critical point $\boldsymbol{w} \in \bar{\Omega}$. By the Sturmian Lemma $\boldsymbol{w}$ must lie in Ω, and it follows from the Morse lemma that $\boldsymbol{w}$ has index 1. Put

$$\mathcal{A}_1' = \mathcal{A}_1 \cup \{\tau_{m,n}\boldsymbol{w} : m, n \in \mathbb{Z}\}; \tag{18.3}$$

then $(\mathcal{A}_1', \ldots, \mathcal{A}_k)$ is a minimally linked family of skeletons extending $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$, and we have our contradiction.

Case B. Assume that $\boldsymbol{x}$ has Morse index 1, and put $\boldsymbol{w} = \omega_+(\boldsymbol{x})$. Then $\boldsymbol{w}$ is a critical point of $W_{p_1 q_1}$ and is therefore transverse to any $\boldsymbol{v} \in \mathcal{A}_j$ with $j \geq 2$, by the Sturmian Lemma. We claim that $I(\boldsymbol{w}, \boldsymbol{v}) = 1$. Indeed, for $t \to -\infty$ we have $\alpha_+(\boldsymbol{x}; t) \to \boldsymbol{x}$. Since $(\mathcal{A}_1, \ldots, \mathcal{A}_k)$ is minimally linked, we find that for all t sufficiently large negative $\alpha_+(\boldsymbol{x}; t)$ and $\boldsymbol{v}$ are transverse with $I(\alpha_+(\boldsymbol{x}; t), \boldsymbol{v}) = 1$. By the Sturmian Lemma $I(\alpha_+(\boldsymbol{x}; t), \boldsymbol{v})$ cannot increase, and since $\alpha_+(\boldsymbol{x}; t)$ and $\boldsymbol{v}$ have different rotation numbers $I(\alpha_+(\boldsymbol{x}; t), \boldsymbol{v}) \geq 1$ for all t: hence $I(\alpha_+(\boldsymbol{x}; t), \boldsymbol{v}) = 1$ for all t. Letting $t \to +\infty$ we get $I(\boldsymbol{w}, \boldsymbol{v}) = 1$, as claimed. Defining $\mathcal{A}_1'$ as in ((18.3)) we again get a larger minimally linked family of skeletons, a contradiction. If $\boldsymbol{x}$ is a local minimum then $\boldsymbol{y}$ cannot be one by Case A, and considering $\omega_-(\boldsymbol{y})$ leads to a similar contradiction. $\square$

19. Proof of Lemma 18.5

We must show that $\bar{\Omega}$ is connected. We shall do this by showing that any $\boldsymbol{w} \in \Omega$ can be connected to $\boldsymbol{x}$ via a path $\gamma : [0,1] \to \Omega \cup \{\boldsymbol{x}\}$.

For any $j \in \mathbb{Z}$ and any $\boldsymbol{x} < \boldsymbol{w} \in X_{p_1 q_1}$ we put

$$A_j(\boldsymbol{x} : \boldsymbol{w}) = \{v_j : \boldsymbol{v} \in \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k\} \cap [x_j, w_j).$$

For simplicity we shall write $\boldsymbol{x} \pitchfork \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ when we mean that $\boldsymbol{x} \pitchfork \boldsymbol{v}$ for every $\boldsymbol{v} \in \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$.

Proposition 19.1 *Given* $\boldsymbol{x} < \boldsymbol{w}$ *in* $X_{p_1q_1}$,
(i) $A_j(\boldsymbol{x} : \boldsymbol{w})$ *is finite, for each* $j \in \mathbb{Z}$.
(ii) $A_{j+q_1}(\boldsymbol{x} : \boldsymbol{w}) = A_j(\boldsymbol{x} : \boldsymbol{w}) + p_1$.
(iii) If $\boldsymbol{z} \in X_{p_1q_1}$ *and* $\boldsymbol{x} \leq \boldsymbol{z} \leq \boldsymbol{w}$, *then* $\boldsymbol{z} \pitchfork \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$, *if and only if they are tranverse in the index range* $0 \leq j \leq q_1$.

Proof. (i) : $W_{p_jq_j}$ is a Morse function. (ii) holds because $\boldsymbol{x}, \boldsymbol{w} \in X_{p_1q_1}$ and the $\mathcal{A}_l$ are invariant under the action of $\tau_{m,n}$, $m, n \in \mathbb{Z}$. (iii) is a consequence of (ii). □

We define the *height* of $\boldsymbol{w}$ over $\boldsymbol{x}$ by

$$h(\boldsymbol{x} : \boldsymbol{w}) = \sum_{j=0}^{q_1-1} \# \left(A_j(\boldsymbol{x} : \boldsymbol{w}) \right).$$

If the height $h(\boldsymbol{x} : \boldsymbol{w})$ vanishes then all the $A_j(\boldsymbol{x} : \boldsymbol{w})$ are empty and we can define $\gamma(t) = t\boldsymbol{w} + (1-t)\boldsymbol{x}$. Since $x_j \leq \gamma_j(t) \leq w_j$ for all j and $0 \leq t \leq 1$, it follows from part (iii) of our last proposition that $\gamma(t) \pitchfork \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ for $0 \leq t \leq 1$, so that $\gamma(t)$ stays within $\bar{\Omega}$. Call this a move of type 0.

We now assume that $h(\boldsymbol{x} : \boldsymbol{w}) > 0$, and we show how to decrease it to zero. Suppose that for some l one has $w_l = v_l > x_l$ for some $\boldsymbol{v} \in \mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$. Then there is an ε such that $0 < \varepsilon < w_l - x_l$ and $(w_l - \varepsilon, w_l) \cap A_l(\boldsymbol{x} : \boldsymbol{w})$ is empty and we can define

$$w'_j = \begin{cases} w_j - \varepsilon & \text{if } j = l \bmod q_1, \\ w_j & \text{otherwise.} \end{cases}$$

As before one can connect $\boldsymbol{w}$ and $\boldsymbol{w}'$ by $\gamma(t) = t\boldsymbol{w} + (1-t)\boldsymbol{w}'$ without leaving $\bar{\Omega}$. Call this a move of type 1.

Assuming now that $w_i \neq v_i$ for all i, we will move the sequence $\boldsymbol{w}$ down by interpolating it linearly to:

$$z_i^{(l)} = \begin{cases} \max A_i(\boldsymbol{x} : \boldsymbol{w}) & \text{if } i = l \bmod q_1, \\ w_i & \text{otherwise} \end{cases}$$

for some judiciously chosen l. Call this a move of type 2. Clearly $\boldsymbol{z}^{(l)} \in X_{p_1q_1}$ and $\boldsymbol{x} \leq \boldsymbol{z}^{(l)} \leq \boldsymbol{w}$, $\boldsymbol{z}^{(l)} = \boldsymbol{z}^{(l+q)}$ and $h(\boldsymbol{x} : \boldsymbol{z}^{(l)}) = h(\boldsymbol{x} : \boldsymbol{w}) - 1$. We need to show that for at least one $l \in \mathbb{Z}$, this move does not change the intersection index of $\boldsymbol{w}$ with the elements of $\mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$. Consider the set of elements in $\mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$ that are immediately below $\boldsymbol{w}$:

$$a_i^{(s_i)} \stackrel{\text{def}}{=} \max A_i(\boldsymbol{x} : \boldsymbol{w}).$$

Assume that, among the sequences $\boldsymbol{a}^{(s_i)}$ at least one has rotation number greater than that of $\boldsymbol{x}$ and pick the one, say $\boldsymbol{a}^{(s_j)}$ which has the largest rotation number (If all $\boldsymbol{a}^{(s_i)}$ have lower rotation number than $\boldsymbol{x}$, pick the one that has the lowest and proceed similarly). In the following, we only worry about the possible changes of intersection index in the range $0 \leq j \leq q_1$. The periodicity condition (ii) of Proposition 19.1 insures that if there are changes of index, they must occur periodically. There are three cases (see Figure 19.1) to consider:

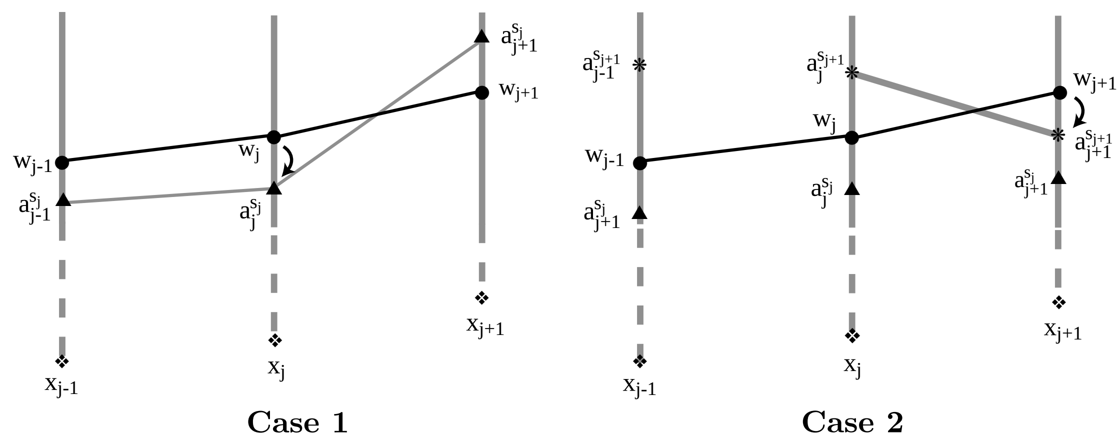

Fig. 19.1. The two possible moves of type 2.

Case 1: $a_{j+1}^{(s_j)} > w_{j+1}$

Choose $l = j$ and move $\boldsymbol{w}$ to $\boldsymbol{z}^{(l)}$ as defined above. This could only change the intersection index of $\boldsymbol{w}$ with $\boldsymbol{a}^{(s_j)}$. But in this case this intersection index remains the same: since $\rho(\boldsymbol{a}^{(s_j)}) > \rho(\boldsymbol{w}) = \rho(\boldsymbol{x})$, and $I(\boldsymbol{a}^{(s_j)}, \boldsymbol{w}) = 1$, we must have $a_{j-1}^{(s_j)} \leq a_{j-1}^{(s_{j-1})} < w_{j-1}$. Hence the one crossing of $\boldsymbol{w}$ and $\boldsymbol{a}^{(s_j)}$, which occured between j and $j+1$ is now moved to a crossing that occurs at j, with no other crossing introduced with this or any other sequence of $\mathcal{A}_2 \cup \cdots \cup \mathcal{A}_k$.

Case 2: $a_{j+1}^{(s_j)} < a_{j+1}^{(s_{j+1})}$

Since by assumption $\rho(\boldsymbol{a}^{(s_{j+1})}) \leq \rho(\boldsymbol{a}^{(s_j)})$, we must have $a_j^{(s_{j+1})} > a_j^{(s_j)}$ and thus $a_j^{(s_{j+1})} > w_j$, by maximality of $a_j^{(s_j)}$. Now choose $l = j+1$ and move $\boldsymbol{w}$ to $\boldsymbol{z}^{(l)}$: the one crossing of $\boldsymbol{w}$ and $\boldsymbol{a}^{(s_{j+1})}$, which occured between j and $j+1$ is now moved to a crossing that occurs at $j+1$.

Case 3: $a_{j+1}^{(s_j)} = a_{j+1}^{(s_{j+1})}$
The equality $a_i^{(s_j)} = a_i^{(s_i)}$ cannot be true for all $i > j$ since otherwise $\boldsymbol{w}$ and $\boldsymbol{a}^{(s_j)}$ would have same rotation number. Hence for some $i > j$, Case 1 or 2 must occur. Apply the procedure for these cases there.

Concatenating moves alternating between type 1 and 2, we get a curve in $\bar{\Omega}$ between $\boldsymbol{w}$ and and a sequence which has zero height. Concatenate this with a move of type 0 to get a curve in $\bar{\Omega}$ between $\boldsymbol{w}$ and $\boldsymbol{x}$. □

20. Proof of Theorem 18.1

Let $\omega_1, \omega_2, \cdots$ be an enumeration of the rational numbers in the interval (a, b).

Proposition 20.1 *There is a family of GCs* $\{\Gamma_1^{(n)}, \dots, \Gamma_n^{(n)}\}$*, where* $\Gamma_j^{(n)}$ *has rotation number* ω_j*, and where* $\Gamma_i^{(n)} \prec \Gamma_j^{(n)}$ *if* $\omega_i < \omega_j$*. Each* $\Gamma_i^{(n)}$ *contains at least one minimizing periodic orbit of rotation number* ω_i*, and generically all of them.*

Proof. If one assumes that the map f is such that the Morse property 17.1 holds, then, according to Theorem 18.4, one can find a minimally linked family of maximal skeletons $\{\mathcal{A}_1^{(n)}, \dots, \mathcal{A}_n^{(n)}\}$ such that $\mathcal{A}_j^{(n)}$ has rotation number ω_j and contains all the absolute minimizers of that rotation number. The corresponding GCs $\Gamma_i^{(n)} = \Gamma_{\mathcal{A}_i^{(n)}}$ then satisfy the required conditions.

In general, when the Morse property 17.1 is not satisfied, one can approximate f by smooth twist maps f_ε which do satisfy 17.1 (since this condition is generic); One thus obtains ghost circles $\Gamma_{j,\varepsilon}^{(n)}$, and by the compactness of the set of GCs with a fixed rotation number (Proposition 16.3) one can extract a convergent subsequence whose limit will then be a family $\{\Gamma_1^{(n)}, \dots, \Gamma_n^{(n)}\}$ of GCs. But we need to make sure that limits of strictly ordered rational GCs stay strictly ordered. To see this, notice that the set $\Gamma_{i,\varepsilon}^{(n)} \times \Gamma_{j,\varepsilon}^{(n)}$ is, when $i \neq j$, included in:

$$\Omega_{ij} = \left\{(\boldsymbol{v}, \boldsymbol{w}) \in \mathrm{PCO}_{\omega_i} \times \mathrm{PCO}_{\omega_j} \;:\; \boldsymbol{v} \pitchfork \boldsymbol{w} \quad \text{and} \quad I(\boldsymbol{v}, \boldsymbol{w}) = 1\right\}$$

where PCO_ω is the set of periodic CO sequences of rotation number ω:

$$\mathrm{PCO}_{p/q} = CO_{p/q} \cap X_{p,q}.$$

The set Ω_{ij} is, by the Sturmian lemma, positively invariant under the product gradient flow $\zeta^t \times \zeta^t$ corresponding to any twist map. In fact: $(\zeta^t \times \zeta^t)(Clos\ \Omega_{ij}) \subset (Int\ \Omega_{ij})$, as can easily be checked (i.e. $Clos\ \Omega_{ij}$ is an "attractor block" in the sense of Conley). As Hausdorff limit of compact sets in Ω_{ij}, the set $\Gamma_i^{(n)} \times \Gamma_j^{(n)}$ is in $Clos\ \Omega_{ij}$. But, since it is both positively and negatively invariant under $\zeta^t \times \zeta^t$, $\Gamma_i^{(n)} \times \Gamma_j^{(n)}$ must in fact be in $Int\ \Omega_{ij}$ where the intersection number is well defined and always equal to 1. In other words, we have shown that, whenever $\omega_i < \omega_j$ one must have $\Gamma_i^{(n)} \prec \Gamma_j^{(n)}$. Finally, the set $\Gamma_i^{(n)}$ contains at least a minimizing periodic orbit, since the sets $\Gamma_{i,\varepsilon}^{(n)}$ contain by construction all the minimizing periodic orbits of period ω_i for f_ε, and limits of minimizers are minimizers. □

A. Rational C_ω's

We now construct the C_ω's of Theorem 18.1, starting with all the rational $\omega \in [a, b]$. Again, we use the compactness of the set of GCs: For each n, Proposition 20.1 provides us with GCs $\Gamma_1^{(n)}, \ldots, \Gamma_n^{(n)}$ with rotation numbers $\omega_1, \ldots, \omega_n$. By compactness we can extract a subsequence $\{n_j\}$ for which the $\Gamma_1^{(n_j)}$ converge as $j \to \infty$ to a GC of rotation number ω_1. Using compactness again, we can extract a further subsequence n'_j for which $\Gamma_1^{(n'_j)}$ and $\Gamma_2^{(n'_j)}$ both converge; repetition of this argument and application of the diagonal trick then finally gives a subsequence n''_j for which all $\Gamma_k^{(n''_j)}$ converge to some limiting GC $\Gamma_k^{(\infty)}$ (of rotation number ω_k) as $j \to \infty$. By the same argument as in the previous proposition, the limits $\Gamma_k^{(\infty)}$ satisfy $\Gamma_i^{(\infty)} \prec \Gamma_j^{(\infty)}$ whenever $\omega_i < \omega_j$. We then define $C_{\omega_k} = \pi\Gamma_k^{(\infty)}$ and by the Graph Ordering Lemma 16.5, the C_{ω_k}'s are disjoint. In the generic case, each $\Gamma_i^{(n)}$ contains all the periodic minimizers of rotation number ω_i, and hence so must the limit $\Gamma_i^{(\infty)}$. In the non generic case, $\Gamma_i^{(\infty)}$ must contain at least one periodic minimizer of the energy.

B. Irrational C_ω's

To complete our family of rational GCs with irrational ones, we once again take a limit. We could proceed in a way similar to what we did in order to get all rational GCs, but we would have to appeal to the axiom of choice (no diagonal tricks on uncountable sets!). To avoid this, we first prove a proposition of monotone convergence of GCs. We shall write $\Gamma_1 \preceq \Gamma_2$ if either $\Gamma_1 \prec \Gamma_2$ or $\rho(\Gamma_1) = \rho(\Gamma_2)$ and $\pi\Gamma_1$ is (not necessarily strictly) below $\pi\Gamma_2$.This last condition is equivalent to $x_1^{(1)}(\xi) \le x_1^{(2)}(\xi)$ in the notation of the proof of the Graph Ordering Lemma 16.5.

Proposition 20.2 (Monotone Convergence for Ghost Circles) *Let $\Gamma^{(j)}$ be an increasing sequence of GCs, i.e. assume that*

$$\Gamma^{(1)} \preceq \Gamma^{(2)} \preceq \Gamma^{(3)} \preceq \cdots.$$

Assume also that the rotation numbers $\rho_j = \rho(\Gamma^{(j)})$ are bounded from above. Then there is a unique GC $\Gamma^{(\infty)}$ such that $\Gamma^{(j)} \to \Gamma^{(\infty)}$ as $j \to \infty$. Moreover, if $x^{(j)}(\xi)$ is the parametrization of $\Gamma^{(j)}$ with $x_0^{(j)}(\xi) \equiv \xi$, then the $x_k^{(j)}(\xi)$ converge monotonically and uniformly to $x_k^{(\infty)}(\xi)$, where $x^{(\infty)}(\xi)$ is the parametrization of $\Gamma^{(\infty)}$ with $x_0^{(\infty)}(\xi) \equiv \xi$.

Of course, the corresponding theorem for decreasing sequences of GCs also holds. We postpone the proof of this proposition till the end of this section.

Assume now that we have constructed the rational GCs $\Gamma_k^{(\infty)}$ as above. For any number $\omega \in (a, b)$, rational or otherwise, we can then define two GCs $\Gamma_\omega^\pm$ as follows. Choose a sequence of rational numbers ω_{n_j} which increases monotonically to ω. The Monotone Convergence Theorem tells us that the limit of the corresponding GCs $\Gamma_{n_j}^{(\infty)}$ must exist. We denote this limit by Γ_ω^-. This procedure might produce an ambiguous definition of Γ_ω^-, since the result could depend on the choice of the sequence n_j: If one has two such sequences, n_j and n'_j, then the $\Gamma_{n_j}^{(\infty)}$ and $\Gamma_{n'_j}^{(\infty)}$ might have two different limits Γ and Γ'. However, one can take the union of the two sequences to obtain a third sequence n''_j, i.e. $\{n''_j\} = \{n_j\} \cup \{n'_j\}$. The $\omega_{n''_j}$ then also increase to ω, so that the $\Gamma_{n''_j}^{(\infty)}$ also must converge to some GC Γ''. Since n_j and n'_j are subsequences of n''_j, both sequences n_j and n'_j must produce the same limiting GC: hence $\Gamma = \Gamma' = \Gamma''$, and the definition of Γ_ω^- is independent of the choice of the n_j. We choose to define $C_\omega = \pi\Gamma_\omega^-$ (or $\pi\Gamma_\omega^+$, but with the same choice of $+$ or $-$ for all ω in order to avoid using the axiom of choice...).

We now check that, for ω irrational, the unique Aubry-Mather set M_ω of recurrent minimizers (see Proposition 12.9) is included in C_ω. We can take a sequence of periodic Aubry minimizing sequences $\boldsymbol{x}^k \in \Gamma_k^{(\infty)}$ where $\omega_k \nearrow \omega$ ($\searrow$ if one chose $C_\omega = \pi\Gamma^+$). Then $\boldsymbol{x}^k \to \boldsymbol{x}$, an Aubry minimizing sequence in Γ_ω^-. The orbit that $\boldsymbol{x}$ corresponds to is recurrent and minimizing, as limit of recurrent and minimizing orbits. Its closure, which is also included in C_ω, must be the Aubry-Mather set M_ω . From our definition of $\Gamma_\omega^\pm$, it is clear that:

$$\omega_i < \omega < \omega_j \Rightarrow \Gamma_i^{(\infty)} \prec \Gamma_\omega^- \preceq \Gamma_\omega^+ \prec \Gamma_j^{(\infty)},$$

for rational ω_i, ω_j and irrational ω. Hence the set formed by the rational GCs $\Gamma_k^{(\infty)}$ and the irrational ones Γ_ω is completely ordered according to their rotation numbers. By the Graph Ordering Lemma 16.5, the C_ω's (irrational and rational) that we have constructed are mutually disjoint. □

Remark 20.2 If ω is a rational number, Γ_ω^- is no longer necessarily in PCO_ω but is certainly in CO_ω. It may contain the sequences corresponding to homo(hetero)clinic orbits joining hyperbolic periodic orbits of rotation number ω. Hence we may (and, probably, generically do) have three distinct Ghost Circles $\Gamma_\omega^- \preceq \Gamma_\omega \preceq \Gamma_\omega^+$ for each rational ω where Γ_ω is $\Gamma_k^{(\infty)}$ for some k. We will call their projections C_ω^-, C_ω and C_ω^+ respectively. Instead of the set $\{C_\omega\}_{\omega\in[a,b]}$ of strictly non mutually intersecting curves that we have found in Theorem 18.1, one might prefer to consider the bigger set $\{C_\omega \cup C_\omega^+ \cup C_\omega^-\}_{\omega\in[a,b]}$. It is not hard to check that this is a closed set of GCs.

Proof of Proposition 20.2. It follows from the Graph Ordering Lemma 16.5 that the $x_k^{(j)}(\xi)$ are monotonic in j. We have assumed that the rotation numbers of the $\Gamma^{(j)}$ are bounded, say by some integer M. Since $\boldsymbol{x}^{(j)}$ is CO, this bound implies for $l > 0$ that $x_l^{(j)}(\xi) \le \xi + l(M+1)$, and in view of the monotonicity of the $x_l^{(j)}(\xi)$ they converge to some $x_l^{(\infty)}(\xi)$. For negative l one finds that $x_l^{(j)}(\xi) \ge \xi + l(M+1)$, so that the $x_l^{(j)}(\xi)$ decrease to some $x_l^{(\infty)}(\xi)$. Clearly $x_1^{(\infty)}(\xi)$ is a nondecreasing function of ξ. We shall show that it is strictly increasing, and continuous.

$x_1^{(\infty)}(\xi)$ *is strictly increasing.* Let $\xi < \eta$ be given. Then $t \mapsto \zeta^t(\boldsymbol{x}^{(j)}(\xi))$ and $t \mapsto \zeta^t(\boldsymbol{x}^{(j)}(\eta))$ both are on the GC $\Gamma^{(j)}$, so that they must be ordered in the same way for all $t \in \mathbb{R}$. At $t = 0$ we have

$$\xi = \zeta^t(\boldsymbol{x}^{(j)}(\xi))_0 < \zeta^t(\boldsymbol{x}^{(j)}(\eta))_0 = \eta$$

so this ordering must hold for all t. Upon taking the limit $j \to \infty$ we find that $\zeta^t(\boldsymbol{x}^{(\infty)}(\xi)) \le \zeta^t(\boldsymbol{x}^{(\infty)}(\eta))$ holds for all t. By the strict monotonicity of ζ^t, we must have strict inequality for all t, unless we have equality for all t. Equality cannot happen of course, since $x_0^{(\infty)}(\xi) = \xi < \eta = x_0^{(\infty)}(\eta)$. Hence we have $\boldsymbol{x}^{(\infty)}(\xi) < \boldsymbol{x}^{(\infty)}(\eta)$; in particular $x_1^{(\infty)}(\xi) < x_1^{(\infty)}(\eta)$.

$x_1^{(\infty)}(\xi)$ *is continuous.* Since the $x_1^{(j)}(\xi)$ are monotonically increasing in both j and ξ, their limit is increasing and lower semicontinuous in ξ. Thus we only have to show that $x_1^{(\infty)}(\xi) = x_1^{(\infty)}(\xi+0)$. Assume that $x_1^{(\infty)}(\xi) < x_1^{(\infty)}(\xi+0)$ and define $a = \{x_1^{(\infty)}(\xi) + x_1^{(\infty)}(\xi+0)\}/2$. Then there is a sequence $\delta_j \downarrow 0$ such that $x_1^{(j)}(\xi+\delta_j) = a$. As before we consider $\zeta^t\left(\boldsymbol{x}^{(j)}(\xi+\delta_j)\right)$ and $\zeta^t\left(\boldsymbol{x}^{(j)}(\xi)\right)$, and take the limit $j \to \infty$. Then, after passing to a subsequence if necessary, $\zeta^t\left(\boldsymbol{x}^{(j)}(\xi+\delta_j)\right) \to \zeta^t\left(\boldsymbol{x}^*\right)$ for some $\boldsymbol{x}^*$ with $x_0^* = \xi$ and $x_1^* = a$, while $\zeta^t\left(\boldsymbol{x}^{(j)}(\xi)\right) \to \zeta^t\left(\boldsymbol{x}^{(\infty)}(\xi)\right)$. Moreover we will have $\zeta^t\left(\boldsymbol{x}^*\right) \geq \zeta^t\left(\boldsymbol{x}^{(\infty)}(\xi)\right)$ for all t, again with either strict inequality for all t or equality for all t. But this contradicts the fact that at $t=0$ we have $x_0^* = \xi = x_0^{(\infty)}(\xi)$ and $x_1^* = a > x_1^{(\infty)}(\xi)$. Thus $x_1^{(\infty)}(\xi)$ is indeed continuous. Since the $x_1^{(j)}(\xi)$ increase monotonically to $x_1^{(\infty)}(\xi)$, and since $x_1^{(\infty)}(\xi)$ is continuous, the convergence must be uniform (Dini's theorem). Therefore the $x_l^{(j)}(\xi)$, being iterates of $x_1^{(j)}(\xi)$ (see (16.1) and below) also converge uniformly.

One now easily verifies that $\Gamma^{(\infty)} = \{\boldsymbol{x}^{(\infty)}(\xi) : \xi \in \mathbb{R}\}$ is a GC, and that it is the limit in the Hausdorff metric of the $\Gamma^{(j)}$s. □

Exercise 20.3 Complete the sketch of the following alternate conclusion to the proof of Theorem 18.1, which does not use Proposition 20.3, but uses the axiom of choice. For each $\rho = (\omega_1, \ldots, \omega_k)$ in $\mathbb{Q}^k$, and k arbitrary, consider the set, given by Theorems 18.3 and 18.4, $\mathcal{G}_\rho = \bigcup_{\omega_i \in \rho} \Gamma_{\omega_i}$, union of GC's whose projections do not intersect. Let

$$J_{[a,b]} = \text{closure}\{(\boldsymbol{x},\boldsymbol{y}) \in \left(\text{CO}_{[a,b]}\right)^2 \bigm| I(\tau_{m,n}\boldsymbol{x},\boldsymbol{y}) \leq 1, \quad \forall (m,n) \in \mathbb{Z}^2\} .$$

This is a compact attractor block for the flow $\zeta^t \times \zeta^t$ on the cartesian product $\left(\text{CO}_{[a,b]}\right)^2$. Let $K \subset J_{[a,b]}$ be the maximum invariant set in $J_{[a,b]}$. Then K and its projection K' on the first component are both compact. Take an increasing (for the inclusion) sequence of finite subsets $\mathcal{R}$ of $\mathbb{Q}$, say $\{\mathcal{R}^j\}_{j\in\mathbb{N}}$ such that $\bigcup_{j\in\mathbb{N}} \mathcal{R}^j = \mathbb{Q} \cap [a,b]$. Since K' is compact, assume that the sequence of compact sets $\{\mathcal{G}_{\mathcal{R}^i}\}_{i\in\mathbb{N}}$ converges (in the Hausdorff topology) to a set $\mathcal{L}$ in K'. Now show that for all $\omega \in [a,b]$, $\mathcal{L} \cap \text{CO}_\omega$ contains at least one GC. Show that two GCs $\Gamma_\omega, \Gamma_{\omega'}$ of different rotation numbers in $\mathcal{L}$ must satisfy $\Gamma_{\omega'} \cap \Gamma_{\omega'} = \emptyset$. To construct a partition, i.e. a family of non intersecting circles , pick (using the axiom of choice!) one GC of $\mathcal{L}$ for each ω in $[a,b]$.

21.* Remarks and Applications

A*. Remarks

1) The techniques introduced in this chapter have a scope that goes beyond proving the vertical ordering of Aubry-Mather sets. Angenent (1988) introduced the flow ζ^t and its monotonicity. He used it to prove, for instance, the existence of periodic orbits that, in the generic case, would come from “elliptic islands around elliptic islands”, as well as homoclinic and heteroclinic orbits between hyperbolic points. The remarkable fact is that his results do not make any generic assumption. This is a definite advantage of the variational techniques over the hyperbolic techniques with which removing generic assumptions about transversality of unstable manifolds is often a major hurdle. As an example, it was this kind of technical hurdle that barred Tangerman & Veerman (1990a) to obtain a complete proof that the Aubry-Mather sets are vertically ordered, a fact that they conjecture in that paper. In Chapter 9, we review work by Angenent (1990), Koch & al. (1994) and Candel & de la Llave (1997) which use the monotone properties of variational problem in higher dimensional and PDE contexts.

2) Ghost circles first appeared in Golé (1992 a). They stemmed from an effort I was making in understanding the ghost tori of my thesis (ζ^t-invariant sets for symplectic twist maps, see Chapter 5). In the realm of twist maps, I had constructed ζ^t invariant circles within the ghost tori. My advisor G. Hall as well as R. MacKay and J. Meiss asked me if their projections were graphs. I proved that in Golé (1992 a), where I also recover a result similar to that of Mather (1986) on the existence of invariant circles. MacKay and Muldoon showed numerical evidence that well chosen ghost circles were disjoint, which led to the work of Angenent & Golé (1991) which makes the bulk of this chapter.

In his work on toral and annulus homeomorphisms, LeCalvez (1997) proposes another way to construct ghost circles: take an ordered circle in $\mathrm{CO}_\omega/\mathbb{Z}$ which is $\mathbb{Z}^2$ invariant, but not necessarily ζ^t invariant. A simple choice is the “straight” circle with $\boldsymbol{x}_k(\xi) = k\omega + \xi$. Apply the flow ζ^t to this whole circle, and take a limit as the time $t \to \infty$. Le Calvez suggested to us that letting the flow act on non–intersecting collections of rational GCs may be a way to prove Theorem 18.4. In a way that is reminiscent to Le Calvez’ construction of GCs, Fathi (1997) has obtained, in the context of convex Lagrangian systems, the generalized Aubry-Mather sets of Mather (see Chapter 9) by applying a flow in a functional analytic space of Lagrangian graphs. Finally Katznelson & Ornstein (1997) find Aubry-Mather sets on a collection of pseudo–graphs that are (not strictly) ordered vertically. They do this by iterating the map on circles in the annulus, trimming the image of the circles at each step so that they remain pseudo–graphs (see Chapter 6). It would be interesting to investigate the parallel between these different methods.

B. Approximate Action-Angle Variables for an Arbitrary Twist Map

If in some well chosen coordinate system (say (x, y)) of $\mathbb{R}^2$ a twist map is completely integrable, these coordinates are called *Action-Angle variables* (x is the angle, y the action).

Dewar & Meiss (1992) attempt the construction of approximate action-angle variables using almost–invariant circles defined through a mean square flux variational principle. We refer the reader to their paper as to the physical relevance of such coordinates. We show here that similar approximate action variables can easily be defined from our GC's. Given any finite number of minimal Aubry-Mather sets, we will produce a continuous foliation of the annulus by circles such that each of the Aubry-Mather set of our chosen collection is contained in a different circle of the foliation. Moreover, such a construction will also produce a completely integrable, albeit not necessarily differentiable map of the annulus that coincides with the original map on the collection of Aubry-Mather sets and leaves the foliation invariant. We sketch here the simple construction.

Let $M_{\omega_1}, \dots, M_{\omega_n}$ be an arbitrary collection of minimal Aubry-Mather sets. From Theorem 18.1, we know that we can produce a corresponding collection $\Gamma_1, \dots, \Gamma_n$ of GC's whose disjoint projections contain the chosen Aubry-Mather sets. Parameterize these GC's by their coordinate x_0 as in (16.1) and order them by increasing rotation number. Between two succesive GC's , say Γ_k and Γ_{k+1}, construct the continuous family:

$$\begin{aligned} & \Gamma_t(\xi) = \left(\cdots, x^t_{-1}(\xi), \xi, x^t_1(\xi), \cdots\right) \\ \text{with} \quad & x^t_1(\xi) = (1-t)x_1^{(k)}(\xi) + t x_1^{(k+1)}(\xi) \\ & x^t_j(\xi) = (x^t_1)^j(\xi) \end{aligned}$$

where, since both $x_1^{(k)}$ and $x_1^{(k+1)}$ are lifts of homeomorphisms of the circle, x_1^t also is (it must be periodic and monotone); $(x_1^t)^j$ represents the jth iterate of this homeomorphism. It is not hard to see that, for $t \neq 0$ or 1, Γ_t has all the properties of a GC except for that of being invariant under the flow. In particular it is a circle in $\mathrm{CO}_{\omega_t}/\tau_{0,1}$ on which the shift $\tau_{1,0}$ acts as a circle homeomorphism with rotation number $\omega_t = (1-t)\omega_k + t\omega_{k+1}$. Its projection $\pi\Gamma_t$ is a graph in the annulus. The circles $\pi\Gamma_t$ do not intersect for different t's since in the (x_0, x_1) coordinates, they are the linear interpolation along the x_1 axis of the non intersecting graphs of $x_1^{(k)}$ and $x_1^{(k+1)}$. Repeating this process between each pair of adjacent Γ_k's in our finite collection gives the continous foliation $\pi\Gamma_t$ advertised. The completely integrable map is given by $\tau_{1,0}$ acting on the family Γ_t of Ghost Circles, or alternatively by $\pi \circ \tau_{1,0} \circ \pi^{-1}$ acting on the annulus, which is the topologically embedded image (by π) of the family Γ_t.

Since for generic maps the rational GC's can be made C^1, the above construction yields, when starting with a generic map and rational Aubry-Mather sets, a C^1 foliation (after smoothing the glueing of our interpolations with suitable time reparameterizations). All the minimizing periodic orbits of the chosen rotation numbers are then embedded in the construction. One can also take a limit of this process, by adding more and more Aubry-Mather sets. One obtains an ordered continuum of circles in $\mathbb{R}^{\mathbb{Z}}$ which contains our set $\mathcal{L}$ of the proof of Theorem 18.1. Alternatively, we could have started with the set $\mathcal{L}$ of GCs and filled its gaps as above, all at once (gaps will occur between the Γ_ω^- and the Γ_ω^- of a given rotation number).

Further study of this object might be interesting in order to draw a parallel between twist maps and families of circle maps, *eg.* in the theory of renormalization (see MacKay (1993)).

C*. Partition for Transport

In the theory of transport of MacKay, Meiss & Percival (1984) and (1986), it is sought to use almost invariant circles in order to form disjoint boxes containing the "resonance zones" around the elliptic islands (or hyperbolic points with reflexion) of the periodic minimax orbits of different rational rotation numbers. It is not hard to see that the pairs $C_{p/q\pm}$ of projections of the $p/q\pm$ GC's each enclose the circle $C_{p/q}$ of Theorem 18.1: they are defined as limits of circles that are respectively strictly above or strictly below $C_{p/q}$. Moreover, as in the almost invariant circles (or partial separatrices) of MacKay, Meiss & Percival (1986), $C_{p/q}$ and the $C_{p/q\pm}$ all meet at the minimum p/q orbits, at least when there are finitely many of these minima (*i.e.* generically). $C_{p/q+}$ (resp. $C_{p/q-}$) contains the advance (resp. retrograde) homoclinic orbits (min and minimax), by an argument of Hasselblat & Katok (1995) , in their Proposition 13.2.11. We therefore hope that the boxes defined by the pairs

$C_{p/q\pm}$ of GC's may be used as intended for the partial separatrices in MacKay, Meiss & Percival (1986). The advantage of our boxes over those formed by partial separatrices is that their boundaries are graphs and that they are disjoint from one another (statements unproven to our knowledge for partial separatrices in the general case. See Tangerman & Veerman (1990a) for partial results). Hence the calculation of the flux through them does not rely on the hypothesis that the turnstiles of MacKay et al. always have the simple shape of a figure 8. One of the advantages of their partial barriers is that they can canalise the flux through "cheminees", i.e., points exit a resonance zone through one turnstile (as opposed to infinitely many in our case).

D*. An extension of Aubry's Fundamental Lemma

As a consequence of Theorem 18.4, we get that any pairs of points in two unlinked maximal skeletons of distinct rotation numbers have intersection index 1. By Aubry's Fundamental Lemma, we knew this to be the case for minimizers, but our results shows that it is also true for the minimaxes and local minima in the skeletons. The relevance of this appears clearer in the light of LeCalvez (1991), where he shows that this intersection number is geometrically a linking number for the corresponding orbits of the suspension flow of the map. Extending an idea of Hall (1984), he shows that this linking is an obstruction to continue periodic orbits simultaneously, through paths of periodic orbits in an isotopy of the map to some completely integrable twist map. In our terminology, his result implies that the periodic orbits corresponding to critical sequences in a set of minimally linked skeletons can "continue" simultaneously through curves of periodic orbits of an isotopy of our map to a well chosen completely integrable map. In particular, LeCalvez already noted that, because of Aubry's Fundamental Lemma, any collection of minimum periodic orbits can be continued simultaneously to orbits of a completely integrable map. A consequence of Theorem 18.4, where we construct minimally linked sets that contain minimum and minimax orbits, we get, using LeCalvez' result, periodic local minimizers as well as orbits of minimax type continuing simultaneously to orbits of a completely integrable map f_0, through paths of periodic orbits of a curve of maps joining f to f_0.

22. Proofs of Monotonicity and of the Sturmian Lemma

In this section, we give the proofs of Theorem 14.2 and Lemma 14.3. Eventhough it is a consequence of the latter, we start with a simpler, direct proof of the former. Both proofs are by S. Angenent.

A. Proof of Strict Monotonicity

We let the reader show that if the operator solution of the linearised equation:

$$\dot{\boldsymbol{u}}(t) = L\boldsymbol{u}(t) \tag{22.1}$$

with

$$L : \{v_k\}_{k\in\mathbb{Z}} \mapsto \{\beta_k v_{k-1} + \alpha_k v_k + \beta_{k+1} v_{k+1}\}_{k\in\mathbb{Z}}$$
$$\alpha_k = -\partial_{22} S(x_{k-1}, x_k) - \partial_{11} S(x_k, x_{k+1}), \quad \beta_k = -\partial_{12} S(x_{k-1}, x_k)$$

is strictly positive, then the flow ζ^t is strictly monotone. $L(\boldsymbol{x}(t))$ is an infinite tridiagonal matrix with positive off diagonal terms $-\partial_{12} S(x_k, x_{k+1})$ (see Formula (17.1) for a finite dimensional version of this matrix). The diagonal terms $\partial_{11} S(x_k, x_{k+1}) + \partial_{22}\partial_2 S(x_{k-1}, x_k)$ are uniformaly bounded by assumption on S. Hence, for any $T > 0$ for which $\boldsymbol{x}(t) = \zeta^t(\boldsymbol{x})$ is defined when $0 \le t \le T$, we can find a positive λ such that:

$$B(t) = L(\boldsymbol{x}(t)) + \lambda Id$$

is a strictly positive matrix. If $\boldsymbol{u}(t)$ is solution of the equation (22.1) then $e^{\lambda t}\boldsymbol{u}(t)$ is solution of :

$$\dot{\boldsymbol{v}}(t) = B(t)\boldsymbol{v}(t), \tag{22.2}$$

hence the strict positivity of the solution operator for (22.1) is equivalent to that of (22.2) . Looking at the integral equation:

$$\boldsymbol{v}(t) = \boldsymbol{v}(0) + \int_0^t B(s)\boldsymbol{v}(s)ds,$$

one sees that Picard's iteration will give positive solutions for a positive vector $\boldsymbol{v}(0)$. This will imply, assuming that $v_k(0) > 0, v_l(0) \ge 0$, for $l \ne k$:

$$v_{k+1}(t) \geq v_{k+1}(0) + \int_0^t B_{k,k+1}(s) v_k(s) ds > 0$$

The same holding for v_{k-1}. By induction, $v_k(t) > 0, \forall k \in \mathbb{Z}$ and the operator solution is strictly positive. This finishes the proof of Theorem 14.2. □

B. Proof of the Sturmian Lemma

Lemma 22.1 (Sturmian Lemma) *Let $\boldsymbol{x}(\cdot), \boldsymbol{y}(\cdot) \in CO$ be different solutions of*

$$\frac{dx_k}{dt} = -\partial_2 S(x_{k-1}, x_k) - \partial_1 S(x_k, x_{k+1}) \ ;$$

then $I(\boldsymbol{x}(t), \boldsymbol{y}(t))$ does not increase, and decreases whenever $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ are not transverse.

To prove this lemma, we will examine a more general situation.

Let $x_i(t)$ $(i_0 \leq i \leq i_1, -T \leq t \leq T)$ be a solution of

$$\frac{dx_i}{dt} = a_i(t)x_{i-1} + b_i(t)x_i(t) + c_i(t)x_{i+1}(t) \quad (i_0 < i < i_1) \tag{22.3}$$

where we assume that the coefficients $a_i(t), b_i(t), c_i(t)$ are continuous and satisfy

$$a_i(t), c_i(t) \geq \delta; \qquad a_i, b_i, c_i \leq M \tag{22.4}$$

for all $-T \leq t \leq T$, $i_0 < i < i_1$, and for some constants δ, $M > 0$.

Lemma 22.2 *Assume*

$$x_i(0) \begin{cases} = 0 & \text{for } i_0 < i < i_1 \\ \neq 0 & \text{if } i = i_0 \text{ or } i = i_1 . \end{cases}$$

Then the sequence $\{x_{i_0}(t), \ldots, x_{i_1}(t)\}$ has less sign changes when $t > 0$ than when $t < 0$.

We will now see how Lemma 22.2. gives us a proof of the Sturmian Lemma 22.1.

Proof of Lemma 22.1. By the mean value theorem the difference $\boldsymbol{z}(t) = \boldsymbol{x}(t) - \boldsymbol{y}(t)$ satisfies a linear equation of the form (22.3). If $\boldsymbol{x}(t_0) \pitchfork \boldsymbol{y}(t_0)$, then $I(\boldsymbol{x}(t), \boldsymbol{y}(t))$ is constant for t near t_0.

If $\boldsymbol{x}(t_0)$ and $\boldsymbol{y}(t_0)$ are not transverse, then since $\boldsymbol{x}(t_0) \neq \boldsymbol{y}(t_0)$ one can choose $i_0 < i_1$ such that $z^{i_0}(t_0) \neq 0$, $z^{i_1}(t_0) \neq 0$, while $z^i(t_0) = 0$ for $i_0 \leq i \leq i_1$. Lemma 22.2 then implies the Sturmian Lemma. □

Proof of Lemma 22.2. First a few reductions. Consider

$$y_i(t) = B_i(t)x_i(t)$$

with $B_i(t) = \exp\{-\int_0^t b_i(\tau)d\tau\}$; then

$$\frac{dy_i}{dt} = A_i(t)y_{i-1} + C_i(t)y_{i+1},$$

where

$$A_i(t) \stackrel{\text{def}}{=} \frac{B_{i-1}(t)}{B_i(t)}a_i(t), \qquad C_i(t) \stackrel{\text{def}}{=} \frac{B_{i+1}(t)}{B_i(t)}c_i(t).$$

In other words, we may assume that the $b_i(t)$ vanish. Note that $\{x_i(t)\}$ and $\{y_i(t)\}$ have the same sign changes.

The coefficients A_i, C_i satisfy

$$\delta e^{-MT} \leq A_i(t), C_i(t) \leq Me^{+MT} \tag{22.5}$$

By integrating the differential equation for $y_i(t)$ we find that for $i_0 < i < i_1$ one has

$$y_i(t) = \int_0^t \{A_i(\tau)y_{i-1}(\tau) + C_i(\tau)y_{i+1}(\tau)\}d\tau \tag{22.6}$$

Proposition 22.3 *For $i_0 < i < i_1$ one has*

$$y_i(t) = M_i t^{i-i_0} + N_i t^{i_1-i} + o\left(|t|^{i-i_0} + |t|^{i_1-i}\right) \qquad (t \to 0) \tag{22.7}$$

where the constants M_i and N_i are given by

$$M_i = A_i(0)A_{i-1}(0)\cdots A_{i_0+1}(0)\frac{y_{i_0}(0)}{(i-i_0)!},$$

$$N_i = C_i(0)C_{i+1}(0)\cdots C_{i_1-1}(0)\frac{y_{i_1}(0)}{(i_1-i)!}.$$

We shall prove this by induction. The relevant property of the coefficients M_i, N_i is that the M_i have the same sign as $y_{i_0}(0)$, and the N_i have the same sign as $y_{i_1}(0)$. Furthermore, one of the two terms in (22.7) always dominates the other, unless $i - i_0 = i_1 - i$, i.e. unless $i = \frac{i_0+i_1}{2}$; if $i < \frac{i_0+i_1}{2}$ then $y_i(t) = M_i t^{i-i_0} + o\left(t^{i-i_0}\right)$, if $i > \frac{i_0+i_1}{2}$ then $y_i(t) = N_i t^{i_1-i} + o\left(t^{i_1-i}\right)$.

Proof. We may assume $i_1 - i_0 \geq 2$. The $y_i(t)$ are continuous, and hence bounded as $t \to 0$. Therefore it follows from (22.6) that $|y_i(t)| \leq C\,|t|$ for $|t| \leq T$.

If $i_1 - i_0 = 2$, then the only i with $i_0 < i < i_1$ is $i = i_0 + 1 = i_1 - 1$, and we have

$$\begin{aligned} y_{i_0+1}(t) &= \int_0^t \{A_{i_0+1}(0)y_{i_0}(0) + C_{i_1-1}(0)y_{i_1}(0) + o(1)\}d\tau \\ &= M_{i_0+1}t + N_{i_0-1}t + o(t), \end{aligned}$$

as claimed.

If $i_1 - i_0 > 2$, then $y_{i_0+2}(t) = o(1)$, and (22.6) implies

$$\begin{aligned} y_{i_0+1}(t) &= \int_0^t \{A_{i_0+1}(0)y_{i_0}(0) + o(1)\}d\tau \\ &= M_{i_0+1}y_{i_0}(0)t + o(t). \end{aligned}$$

Likewise (22.6) implies $y_{i_1-1}(t) = N_{i_0-1}y_{i_1}(0)t + o(t)$. If $i_1 - i_0 = 3$ this proves the claim; if $i_1 - i_0 > 3$, then for all $i_0 + 1 < i < i_1 - 1$ one deduces from (22.6) and the estimate $|y_{i\pm1}(t)| \leq C\,|t|$ that $|y_i(t)| \leq Ct^2$.

The general induction step in the derivation of (22.7) is as follows. Assume that it has been shown that (22.7) holds for all i with $i_0 < i < i_0 + k$, or $i_1 - k < i < i_1$; moreover assume it has been shown that $|y_i(t)| \leq C\,|t|^k$ for $i_0 + k \leq i \leq i_1 - k$. If $i_0 + k = i_1 - k$, then (22.7) implies

$$\begin{aligned} y_{i_0+k}(t) &= \int_0^t \{A_{i_0+k}(0)M_{i_0+k-1}\tau^{k-1} + C_{i_1-k}(0)N_{i_1-k+1}\tau^{k-1} + o\left(\tau^{k-1}\right)\}d\tau \\ &= M_{i_0+k}t^k + N_{i_1-k}t^k + o\left(t^k\right), \end{aligned}$$

with

$$\begin{aligned} M_{i_0+k} &= A_{i_0+k}(0)\frac{1}{k}M_{i_0+k-1}, \\ N_{i_1-k} &= C_{i_1-k}(0)\frac{1}{k}N_{i_1-k+1}. \end{aligned}$$

In this case the claim is proved. Otherwise $i_0 + k < i_1 - k$, and a similar computation shows that (22.7) holds when $i = i_0 + k$ and $i = i_1 - k$. Finally, using (22.6) again, one finds that for $i_0+k<i<i_1-k$ the estimate $\left|y^{i\pm1}(t)\right|\le C\,|t|^k$ implies $|y_i(t)|\le C\,|t|^{k+1}$, which completes the induction step. □

Lemma 22.2 follows directly from the proposition. If $y_{i_0}(0)$ and $y_{i_1}(0)$ have the same sign, say they are positive, then the expansion (22.7) implies that all $y_i(t)$ are positive for $t>0$; For small negative t the sequence $y_{i_0}(t)$, $y_{i_0+1}(t)$, . . ., $y_{i_1}(t)$ alternates signs, except in the middle, i.e. if i_1-i_0 is odd then $y_{i_0+k}(t)$ and $y_{i_0+k+1}(t)$ (with $k=\left[\frac{i_1-i_0}{2}\right]$) will have the same sign. Indeed, (22.7) says the sequence $\{y_{i_0}(t),\dots,y_{i_1}(t)\}$ has the signs as the sequence

$$(c_0, c_1t, c_2t^2,\dots,c_{k-1}t^k, c_kt^k, c_{k+1}t^{k-1},\dots,c_{2k-1}t, c_{2k})$$

if $i_1-i_0=2k$ is even, and $\{y_{i_0}(t),\dots,y_{i_1}(t)\}$ will have the same signs as the sequence

$$(c_0, c_1t, c_2t^2,\dots,c_kt^{k+1}, c_{k+1}t^k,\dots,c_{2k}t, c_{2k+1})$$

if $i_1-i_0=2k+1$ is odd; here the c_j's are positive constants, with the possible exception of the coefficient c_k of t^{k+1} in the second sequence. If $y_{i_0}(0)$ and $y_{i_1}(0)$ have opposite signs, then one can again use the expansion (22.7) to derive that the sequence $\{y_i(t)\}$ has exactly one sign change for $t>0$, and i_1-i_0-1 sign changes for $t<0$. If $i_1-i_0=2$, then $\{y_{i_0}(t), y_{i_0+1}(t), y_{i_0+2}(t)\}$ is "transverse" to the zero sequence for all small t, whatever the sign of $y_{i_0+1}(t)$ is. Thus, if $\{y_{i_0}(t),\dots,y_{i_1}(t)\}$ is *not* transverse to the zero sequence at $t=0$, then either $i_1>i_0+2$, or $i_1=i_0+2$, and $y_{i_0}(0)$ and $y_{i_1}(0)$ have the same sign. In either case we have shown that the number of sign changes of $\{y_{i_0}(t),\dots,y_{i_1}(t)\}$ drops at $t=0$. □

Lemma 22.2 implies the following:

Lemma 22.4 *If $\{x_{i_0}(t),\dots,x_{i_1}(t)\}$ is a C^1 solution of (22.3), with $x_{i_0}(t), x_{i_1}(t)\neq 0$ for all $t_0<t<t_1$, then*
(a) the number of sign changes of $\{x_{i_0}(t),\dots,x_{i_1}(t)\}$ does not increase;
(b) this number drops whenever $\{x_{i_0}(t),\dots,x_{i_1}(t)\}$ is not transverse to the zero sequence.