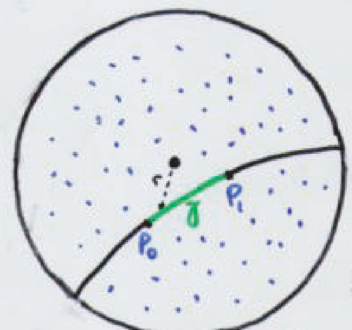
In Euclidean Plane:

Area $A=\pi r^2$

In Hyperbolic Plane:

What is area of a disk with hyperbolic radius r_H?

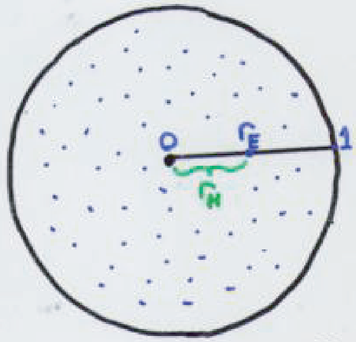
$\cancel{A=\pi r_H^2}$?



Poincaré Disk Model

Defn. The distance between P_0 and P_1 is

$$d_H(P_0,P_1)=\int_\gamma \frac{2\,ds}{1-r^2}$$



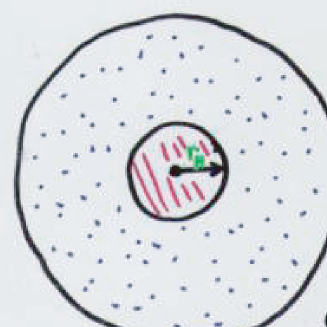
$0\le r_E<1$

$$r_H=d_H(0,r_E)=\int_0^{r_E}\frac{2\,dx}{1-x^2}=\int_0^{r_E}\left(\frac{1}{1-x}+\frac{1}{1+x}\right)dx$$

$$=-\ln(1-x)+\ln(1+x)\Big]_0^{r_E}=\ln\left(\frac{1+x}{1-x}\right)\Big]_0^{r_E}$$

$$=\ln\left(\frac{1+r_E}{1-r_E}\right)$$

$$\boxed{r_H=\ln\left(\frac{1+r_E}{1-r_E}\right)} \quad \text{OR} \quad \boxed{r_E=\frac{e^{r_H}-1}{e^{r_H}+1}}$$



Hyperbolic Area of disk of hyperbolic radius r_H (and Euclidean radius r_E)?

$$A=\int_0^{2\pi}\int_0^{r_E}\left(\frac{2}{1-r^2}\right)^2 r\,dr\,d\theta=\frac{4\pi}{1-r^2}\Big]_0^{r_E}$$

$$=4\pi\left(\frac{r_E^2}{1-r_E^2}\right)=\frac{4\pi(e^{r_H}-1)^2}{(e^{r_H}+1)^2-(e^{r_H}-1)^2}$$

$$=4\pi\left(\frac{e^{\frac{r_H}{2}}-e^{-\frac{r_H}{2}}}{2}\right)^2$$

$$\boxed{A(r_H)=4\pi\sinh^2\left(\frac{r_H}{2}\right)} \quad \sim\pi e^{r_H}$$