MATH 325, Complex Analysis **Midterm** **Prof. Golé, October 26 1999**

The personal version of this midterm is due in class on Tuesday November 2. At that point, you will hand me a photocopy of your work and keep the original. You can then work on a second version with your group. Put the names of your group members on your second version, and hand it in in class on Thursday November 4. It can consist of annotation of the original written in a different colour. The grade for the midterm is an average of these two versions. You are allowed to use any material you want, but *you must justify your answers* to get full credit. Do not stop working on a problem because you can't do one question. Often the questions are independant, or you can assume (without penalty) the result of previous questions to solve subsequent ones. Good luck!

Problem 1
In this problem, we study the map:

$$f(z) = \frac{1}{z}$$

a) Show that f maps $\mathbb{C}\backslash\{0\}$ to $\mathbb{C}\backslash\{0\}$ one to one.
b) Define an *inversion* to be a map $z \mapsto w$ such that w lies on the same radius as z and $|z||w| = 1$. Express $f(z) = 1/z$ in polar coordinates ($z = re^{i\theta}$) and show that f is the composition of an inversion $z \mapsto w$ and the conjugation $w \mapsto \overline{w}$.
c) Express f as a map $F : \mathbb{R}^2 \mapsto \mathbb{R}^2$ of the variable (x, y).
d) Show that F maps any circle or line in the plane into some circle or line. (*Hint.* Any circle *or* line can be given an equation of the form $a(x^2 + y^2) + bx + cy + d = 0$ for the appropriate reals a, b, c, d.) Which circles transform into lines?
e) Find the image of the right half plane $Re(z) > c$ under f . Find the image of the strip $0 < Im(z) < \frac{1}{2c}$ under f .
f) Show that f satisfies the Cauchy-Riemann equations at all $z \neq 0$ and find $f'(z)$.
g) (Xtra credit) Show how $f'(z)$ can be obtained geometrically.

Problem 2
In this problem , we study the Möbius transformations, which are complex maps of the type:

$$M(z) = \frac{az + b}{cz + d}, \quad a, b, c, d, \in \mathbb{C}.$$

a) Under what conditions on a, b, c, d does M have an inverse for all but one point? (Remember that an inverse M^{-1} of M is defined by: $M^{-1}(M(z)) = z = M(M^{-1}(z))$
b) Show that M can be written as the composition of the translation $z \mapsto z + d/c$, the map $z \mapsto 1/z$, the amplitwist $z \mapsto \frac{ad-bc}{c^2}z$ and the translation $z \mapsto z + a/c$.
c) Deduce from b) and Problem 1 that M transforms circles or lines into circles or lines, and that M is analytic whenever $z \neq d/c$.

e) If we draw the correspondence $M(z) = \frac{az+b}{cz+d} \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $M^*(z) = \frac{a^*z+b^*}{c^*z+d^*} \leftrightarrow \begin{pmatrix} a^* & b^* \\ c^* & d^* \end{pmatrix}$, what is the matrix corresponding to $M \circ M^*(z) = M(M^*(z))$? How could you answer question a) in the light of this?
f) (Xtra credit) Show that M maps the upper plane $\{Im(z) \geq 0\}$ onto itself if and only if $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has real entries and positive determinant.
g) (Xtra credit) Show that the invertible Möbius transformations form a group. Can you find a subgroup?

Problem 3
a) Find all the solutions to the equation $f(z) = z^5 - 3i = 0$ geometrically and algebraically.
b) Give a formula for the inverse f^{-1} of f and find its branch points and their order.
c) What would be the radius of convergence of a power series centered at π for f^{-1}?
d) (Xtra credit) Describe a Riemann surface on which f^{-1} is a single valued function.

Problem 4
a) Use the ratio test to find the disk of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{(z-\pi)^n}{n}$.
b) Find a known function whose power series is the one in a) and explain your result in a).
c) (unrelated to a) and b)) Show that $e^z = e \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$

Problem 5
Do Problem 19, page 214 of the textbook.