Curves, surfaces and their tangent lines, planes. We have seen 5 different types of functions in the last few weeks: Functions from $\mathbb{R}$ to $\mathbb{R}^2$ (parametric curves in the plane), from $\mathbb{R} \to \mathbb{R}^3$ (parametric curves in the space: Chapter 10), from $\mathbb{R}^2 \to \mathbb{R}^3$ (parametric surfaces in the space Section 10.5), from $\mathbb{R}^2 \to \mathbb{R}$ (whose graph form surfaces and whose level sets are curves in the plane (Ch. 11), and $\mathbb{R}^3 \to \mathbb{R}$ (whose level sets are surfaces, and whose graphs we don't even try to draw). To each one of these types of functions, we associated "tangent objects": velocity vectors and tangent lines for parametric curves (Section 10.2 and 10.4), tangent planes for parametric surfaces (10.5), graphs of functions $\mathbb{R}^2 \to \mathbb{R}$ (11.4) or level surfaces of functions $\mathbb{R}^3 \to \mathbb{R}$ (11.6)

Problem 1

Describe what kind of geometric object (using the terminology of the above paragraph) each of the following equations represents, and give its corresponding "tangent object" at the point indicated.

$$\text{(I)} \quad z = x\sin(\pi xy), \quad (x_0, y_0) = (1, \frac{1}{2}).$$

$$\text{(II)} \quad \begin{cases} x &= -1+6t \\ y &= 3+\sqrt{2}t^2 \\ z &= -5\cos(\pi t) \end{cases}, \qquad t_0 = 2.$$

$$\text{(III)} \quad \mathbf{r}(s,t) = (-s+3u)\mathbf{i} + (s^2-3t)\mathbf{j} + (-3st)\mathbf{k}, \quad (s_0, t_0) = (1,1).$$

$$\text{(VI)} \quad x^2 - 3y^3 + z^3 = 2, \quad (x_0, y_0, z_0) = (1, -1, 0).$$

Linear approximation. In most cases, the "tangent object" provides a way to make simple approximations for the value of a function near a point where the value is known. For instance, if $f : \mathbb{R}^2 \to \mathbb{R}$ and we know the value of the function at a point (x_0, y_0), we can approximate the value of $f(x,y)$ for any point (x,y) close to (x_0, y_0) by plugging (x,y) in the equation of the tangent plane:

$$(1) \qquad f(x,y) \approx f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

(compare with the equation of the tangent plane: $z = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$). This is called the linear approximation. Sometimes, the following notation is also used in equation (1): $\Delta f \approx f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y$ where $\Delta f, \Delta x, \Delta y$ mean $f(x,y) - f(x_0,y_0)$, $x - x_0$, $y - y_0$ respectively. This approximation formula easily generalizes to functions of three variables (or more): just add a term $f_z(x_0,y_0,y_0)(z-z_0)$ to the right hand side.

Problem 2

Use the linear approximation at well chosen points with the function indicated to approximate the following values:

a) $\sqrt{1.02}e^{.06}$, $f(x,y) = \sqrt{x}e^y$

b) $(\sin(.1) + .2)(\cos(.02))/.2$, $\quad f(x,y,z) = (\sin(x)+z)(\cos y)/z$

c) If you have a calculator, estimate the error made with these approximations.

In many "real life" situations, the function f is not given by a formula, but a table of data. One can still use formula (1) by first computing an approximation of $f_x \approx \Delta f/\Delta x$ and $f_y = \Delta f/\Delta y$ from the table.

Problem 3

Do Problem 4, page 776.

Partial derivatives, Directional derivatives and Gradient. If $f : \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^n \to \mathbb{R}$), the partial derivative $f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,y_0)-f(x_0,y_0)}{h}$ is the slope of the graph of f when one travels at unit speed in the positive x direction (i.e. in the direction of $< 1, 0 >$). Likewise for f_y. The directional derivative along the unit vector $\mathbf{u} =< a, b >$, $D_{\mathbf{u}}f(x_0,y_0) = f_x(x_0,y_0)a + f_y(x_0,y_0)b$ generalizes this notion: it gives the slope of the graph when one travels in the direction of $\mathbf{u}$ at unit speed. The gradient vector $\nabla f =< f_x, f_y >$ gives the direction where this slope is greatest (and positive): taking $\mathbf{u} = \nabla f/\|f\|$ gives $D_{\mathbf{u}}f(x_0,y_0) = \|\nabla f(x_0,y_0)\|$ which is the greatest slope the tangent plane at $(x_0, y_0, f(x_0,y_0))$ has. It is important to note that the gradient lives in domain space of the function ($x - y$ plane for a function $f(x,y)$, xyz-space for a function $f(x,y,z)$). [One way to relate the gradient vector to the tangent plane is that ∇f is the projection on the $x - y$ plane of the vector $< f_x, f_y, -1 >$ normal to the tangent plane. Try to visualize the fact that this normal vector points to the direction of "greatest tilt" of the tangent plane.] One of the most useful properties of the gradient is that it is perpendicular to the level set of f at the point at which it is based. This enables one to compute equations of tangent lines or planes to level sets. It is also the basis to the Lagrange multipliers method.

Problem 4

Let $f(x, y) = xe^{xy^2/10}$

a) Compute the gradient of f.

b) What is the slope of the tangent plane at $x_0, y_0) = (1, 0)$ in the direction of $< 1/\sqrt{2}, 1/\sqrt{2} >$? What is the direction and slope of steepest increase at that point?

c) Find the equation of the tangent line to the level set of f at $(1, 0)$

d) Find the equation of the tangent plane to $xy - yz = 3$ at the point $(2, 1, 1)$.

Chain rule Chain rule is a general formula that enables one to compute derivatives of compositions of functions. One important example we have seen is the following. If $\mathbf{r}(t) =< x(t), y(t) >$ is a curve in the $x - y$ plane and $f : \mathbb{R}^2 \to \mathbb{R}$, then

$$\frac{df}{dt}(x(t), y(t)) = f_x(x(t), y(t))\frac{dx}{dt} + f_y(x(t), y(t))\frac{dy}{dt},$$

which can also be written: $\dfrac{df(\mathbf{r}(t))}{dt} = \nabla f(\mathbf{r}(t)) \cdot \mathbf{r}'(t)$. One geometric interpretation of this quantity is as z-component of the tangent vector of the curve $(x(t), y(t), f(x(t), y(t))$ above $\mathbf{r}(t)$ on the graph of f.

Problem 5

A Monarch butterfly is travelling in a region where the temperature is estimated by the function $T(x, y) = x^2y - .1y^3$, where x, y represent geographical coordinates.

a) If the Monarch travels along the curve $\mathbf{r}(t) =< 3\sin(\pi t), 2\cos(\pi t) >$, at what rate will it feel the temperature increase when $t = 1/2$?

b) In which direction should the Monarch fly at that point for a maximum increase in temperature?

Critical points, min, max and saddles, Lagrange multipliers A critical point (x_0, y_0) for a function $f : \mathbb{R}^2 \to \mathbb{R}$ is one above which the tangent plane to the graph is horizontal. This is equivalent to $\nabla f(x_0, y_0) = \mathbf{0}$, that is $f_x(x_0, y_0) = 0 = f_y(x_0, y_0)$. Most critical points of most functions come in three flavors: local minima, local maxima or saddles. To decide which is which, one uses the second derivative test.

- To find absolute minima or maxima of a function f on a bounded region, one looks for the critical points inside the region, records the values of f at these points and then proceed to look for the min and max of f on the boundary of the region. One finally compares the values of f at all of these points. The second step in the previous process is often of interest on its own. We learned of two ways to do this: either by parameterizing the boundary or by the Lagrange multiplier method.
- The Lagrange multiplier method is for finding the min and max of a function $f(x, y)$ under a constraint given by an equation of the form $g(x, y) = c$ (e.g. $x^2 + y^2 = 4$). The min and max of f on this "constraint set" occur at points where the level set of f is tangent to that of g, i.e. where

$$\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0) \tag{2}$$

(why?). To find the min and max, one must solve Equation (2) for (x_0, y_0) by trying to eliminate λ.

Problem 6

Problem 51 (add: find the min of f with constraint $xy = 2$) and Problem 63.

Integration

Problem 7

a) $\iint_R xy - x\cos(yx^2)dA$ where $R = [0, 1] \times [0, 2\pi]$

b) $\iint_D \frac{dA}{9-x^2-y^2}$ where D is the disk of radius 2 centered at the origin (Use polar coordinates).