Math 212 Practice Final Examination with Solutions Prof. Golé, May 2000

For the final, you may use a (two-sided) sheet of notes and a (graphing) calculator. The final will not have the computational glitches this one has (eg, Pb1 a)

Problem 1

Calculate the following integrals:

a) $\iint_R y^2 e^{xy^3} dA$ where R is the rectangular domain within the points $(1,3), (-1,3)$, $(1,-1)$ and $(-1,-1)$.

You run into weird functions when you try to integrate this integral....(the Exponential integral function $\int_1^\infty e^{-xt}/t dt$. Sorry! Try the following instead. $\iint_R y^2 e^{y^3/x} dA$ where R is the region bounded by the lines $y = x, y = 2x, x = -1, x = 1$. You should get 0.

b) $\iint_D \sin(1 - x^2 - y^2) dA$ where D is the region in the second quadrant bounded by the x-axis, the line $y = \sqrt{3}x$, the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Use Polar coordinates. You should get: $\int_{\pi/2}^{\pi} \int_2^3 r \sin(1 - r^2) dr d\theta$, which yields: $\frac{\pi}{4}\left(\cos(-8) - \cos(-3)\right)$. Note that the line $y = \sqrt{3}x$ does not cross the second quadrant...

Problem 2

a) Give the equation of the tangent plane to the surface of equation $z = xy$ at the point $(1, 2, 2)$.

$z - 2 = 2(x - 1) + (y - 2)$

b) Show that $(1, 0, 0)$ is one of the points of intersection of the unit circle parameterized by $\langle \cos t, \sin t, 0 \rangle$ and the graph of f above.

Set $x = \cos t, y = \sin t, z = 0$ in $z = xy$. Then $0 = \cos t \sin t$, which is satisfied for $t = 0$, ie at the point $(1, 0, 0)$

c) Find the angle the circle makes with $z = xy$ at $(1, 0, 0)$.

This angle is $\pi/2 - \theta$ where θ is the angle between the tangent vector to thecircle at $(1, 0, 0)$ and the normal vector to the tangent plane, $\mathbf{n} = (2, 1, -1)$. To get θ, use the dot product. I got $\pi/2 - \theta = .4205...$(radiants).

Problem 3

Establish whether the following are true or false. Justify your answers!

a) $f_y(a, b) = \lim_{y \to b} \frac{f(y) - f(b)}{b - y}$

False: $f_y(a, b) = \lim_{y \to b} \frac{f(a,y) - f(a,b)}{y - b}$.

b) $\int_{-1}^1 \int_0^1 e^{x^2 + y^2} y^3 dx dy = 0$ (No computation necessary!).

True. The inner integral yields a function $g(y)$ which is such that $g(-y) = -g(y)$. By symmetry, such a function integrate to 0 on $[-1, 1]$.

c) The line integral $\int_C 2x \sin y dx + x^2 \cos y dy = 9$ where C is any curve joining the points $(33333, 0)$ and $(3, \pi/2)$

True. I would not ask you such a question if the vector field was not conservative. Here, check that the "potential" function $f(x, y) = x^2 \sin y$ has $\nabla f(x, y) = \langle 2x \sin y, x^2 \cos y \rangle$. The fundamental theorem for line integrals yields $\int_C 2x \sin y dx + x^2 \cos y dy = f(3, \pi/2) - f(33333, 0) = 9 \sin(\pi/2) - 0 = 9$

Problem 4

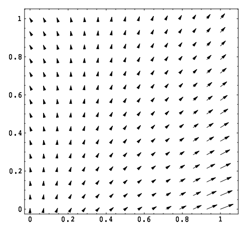
a) Calculate the iterated integral : $\int_0^2 \int_1^{y^2+1} \int_0^{x+y} (x - 2y)\ dz\ dx\ dy$

$-1327/105 = -12.64$

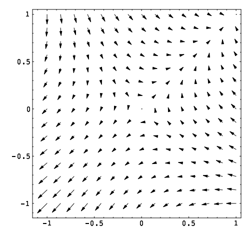
b) Describe the region of integration in the previous integral.

The "floor" of the region of integration is a wedge in the xy-plane between the parabola $x = y^2 + 1$, and the straight lines $x = 1$ and $y = 2$. The "roof" is the plane $z = x + y$.

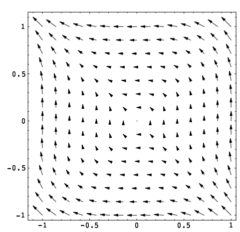
Problem 5



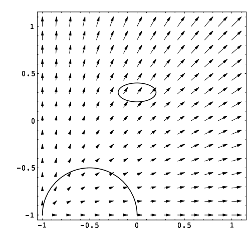
(1)



(2)



(3)



(4)

a) Draw a few flowlines for the vector fields (1) and (2).

Just follow the arrows, starting at a few, well chosen points...

b) Match the vector fields (1)-(4) with the following (justify your answers!):

$$\mathbf{F}(x, y) = \langle x + 1, y + 1 \rangle \qquad \mathbf{G}(x, y) = \left\langle -x^2 + y, -y^2 + x \right\rangle$$
$$\mathbf{H}(x, y) = \left\langle -y^2, x^2 \right\rangle \qquad \mathbf{J}(x, y) = \left\langle e^{3x-y} - 1, e^{2x} \right\rangle$$

(1)-$\mathbf{J}$: The only one that's vertical at $(0, 0)$, (2)-$\mathbf{G}$: it is 0 at $(0, 0)$, and is parallele to (x, y) when $x = y$, (3)-$\mathbf{H}$: it is 0 at $(0, 0)$ and parallele to $(-1, 1)$ when $y = \pm x$, (4)-$\mathbf{F}$: it is horizontal along the line $y = -1$, vertical along $x = -1$. You could get this by other arguments as well...

c) *(You will need to do (b) to solve this question)* Show that the line integral of the vector field (4) along the ellipse shown is 0. Compute the line integral of (4) along the upper half of circle shown, traversed clockwise.

(4)-$\mathbf{F}$ is conservative: $\mathbf{F} = \nabla f$ where $f(x, y) = (x^2/2 + y^2/2 + x + y)$, so the integral of $\mathbf{F}$ along the ellipse is 0, whereas the integral along the semi circle is $f(-1, -1) - f(0, -1)$.

d) Compute the line integral of $\mathbf{H}$ along the line segment between $(-1, 0)$ and $(0, -1)$.

The line segment is parameterized by $\langle -1 + t, -t \rangle$, yielding: $\int_C \mathbf{F} d\mathbf{r} = \int_0^1 t^2 - (-1 + t)^2 dt$

e) Compute the line integral of $\mathbf{J}$ along the rectangle with vertices $(3, 1), (-1, 1)$, $(3, -1)$ and $(-1, -1)$ traversed once counterclockwise.

As I pointed out in class, this example is not so interesting as $\mathbf{J}$ is conservative: the answer is 0. Use $\mathbf{H}$ instead, and Green's theorem: $\int_C \mathbf{H} d\mathbf{r} = \iint_D Q_x - P_y\, dA = \int_{-1}^1 \int_1^3 2x + 2y\, dx dy$