The mixed Riemann curvature tensor

The key to eliminating the nontensorial terms in the transformed version of R^i_{jkl} is Lemma 1, which, in effect, "inverts" the expression

$$\frac{\partial^2 x^i}{\partial \xi^f \partial \xi^a}\frac{\partial \xi^f}{\partial x^k} = \frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^a}\right).$$

Lemma 1 $\dfrac{\partial}{\partial x^k}\left(\dfrac{\partial x^i}{\partial \xi^a}\right) = -\dfrac{\partial^2 \xi^g}{\partial x^k \partial x^p}\dfrac{\partial x^p}{\partial \xi^a}\dfrac{\partial x^i}{\partial \xi^g}.$

PROOF: Using the basic fact

$$\delta^i_p = \frac{\partial x^i}{\partial \xi^a}\frac{\partial \xi^a}{\partial x^p},$$

we find

$$0 = \frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^a}\frac{\partial \xi^a}{\partial x^p}\right) = \frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^a}\right)\frac{\partial \xi^a}{\partial x^p} + \frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^a}{\partial x^k \partial x^p}.$$

Replace the dummy index $a \mapsto g$ in the second term on the right, and then solve for the first:

$$\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^a}\right) = -\frac{\partial x^i}{\partial \xi^g}\frac{\partial^2 \xi^g}{\partial x^k \partial x^p}\cdot\frac{\partial x^p}{\partial \xi^a}. \qquad \square$$

Theorem 1 $R^i_{jkl} = \dfrac{\partial \Gamma^i_{jl}}{\partial x^k} - \dfrac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl}$ *is a* $(1,3)$ *tensor.*

PROOF: We assemble transformed versions of each term on the right, and show that all nontensorial terms cancel. The basic transformation is

$$\Gamma^i_{jl} = \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial x^i}{\partial \xi^f}.$$

Step 1

$$\begin{aligned}
\frac{\partial \Gamma^i_{jl}}{\partial x^k} &= \frac{\partial \Gamma^a_{bd}}{\partial \xi^c}\frac{\partial \xi^c}{\partial x^k}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bd}\frac{\partial^2 \xi^b}{\partial x^k \partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial^2 \xi^d}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \xi^a}\\
&\quad + \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\cdot\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^a}\right) + \frac{\partial^3 \xi^f}{\partial x^k \partial x^j \partial x^l}\frac{\partial x^i}{\partial \xi^f} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\cdot\frac{\partial}{\partial x^k}\left(\frac{\partial x^i}{\partial \xi^f}\right)\\
&= \frac{\partial \Gamma^a_{bd}}{\partial \xi^c}\frac{\partial \xi^c}{\partial x^k}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bd}\frac{\partial^2 \xi^b}{\partial x^k \partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial^2 \xi^d}{\partial x^k \partial x^l}\frac{\partial x^i}{\partial \xi^a}\\
&\quad - \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial^2 \xi^g}{\partial x^k \partial x^p}\frac{\partial x^p}{\partial \xi^a}\frac{\partial x^i}{\partial \xi^g} + \frac{\partial^3 \xi^f}{\partial x^k \partial x^j \partial x^l}\frac{\partial x^i}{\partial \xi^f}\\
&\quad - \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial^2 \xi^g}{\partial x^k \partial x^p}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.
\end{aligned}$$

To get the next term from the preceding one, make the substitutions $k \leftrightarrow l$, $c \leftrightarrow d$. Step 2

$$\frac{\partial \Gamma^i_{jk}}{\partial x^l} = \frac{\partial \Gamma^a_{bc}}{\partial \xi^d}\frac{\partial \xi^d}{\partial x^l}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bc}\frac{\partial^2 \xi^b}{\partial x^l \partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a} + \Gamma^a_{bc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial^2 \xi^c}{\partial x^l \partial x^k}\frac{\partial x^i}{\partial \xi^a}$$
$$- \Gamma^a_{bc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial^2 \xi^g}{\partial x^l \partial x^p}\frac{\partial x^p}{\partial \xi^a}\frac{\partial x^i}{\partial \xi^g} + \frac{\partial^3 \xi^f}{\partial x^l \partial x^j \partial x^k}\frac{\partial x^i}{\partial \xi^f}$$
$$- \frac{\partial^2 \xi^f}{\partial x^j \partial x^k}\frac{\partial^2 \xi^g}{\partial x^l \partial x^p}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.$$

In the third and fifth terms in steps 1 and 2, the variables x^k and x^l appear symmetrically; therefore, those terms vanish when the difference is taken (and dummy indices are adjusted): Step 3

$$\frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} = \left(\frac{\partial \Gamma^a_{bd}}{\partial \xi^c} - \frac{\partial \Gamma^a_{bc}}{\partial \xi^d}\right)\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a}$$
$$+ \Gamma^a_{bd}\frac{\partial^2 \xi^b}{\partial x^k \partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} - \Gamma^a_{bc}\frac{\partial^2 \xi^b}{\partial x^l \partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a}$$
$$- \Gamma^a_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial^2 \xi^g}{\partial x^k \partial x^p}\frac{\partial x^p}{\partial \xi^a}\frac{\partial x^i}{\partial \xi^g} + \Gamma^a_{bc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial^2 \xi^g}{\partial x^l \partial x^p}\frac{\partial x^p}{\partial \xi^a}\frac{\partial x^i}{\partial \xi^g}$$
$$- \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial^2 \xi^g}{\partial x^k \partial x^p}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^k}\frac{\partial^2 \xi^g}{\partial x^l \partial x^p}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.$$

Only the first term is tensorial; the other six must cancel with terms in the remaining two expressions, which we now compute.

The product of Step 4

$$\Gamma^p_{jl} = \Gamma^e_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^p}{\partial \xi^e} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial x^p}{\partial \xi^f}$$

and

$$\Gamma^i_{pk} = \Gamma^a_{hc}\frac{\partial \xi^h}{\partial x^p}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a} + \frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^i}{\partial \xi^g}$$

contributes four terms:

$$\Gamma^p_{jl}\Gamma^i_{pk} = \Gamma^e_{bd}\Gamma^a_{hc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^p}{\partial \xi^e}\frac{\partial \xi^h}{\partial x^p}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a} + \Gamma^e_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^p}{\partial \xi^e}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^i}{\partial \xi^g}$$
$$+ \Gamma^a_{hc}\frac{\partial \xi^h}{\partial x^p}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial x^p}{\partial \xi^f} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.$$

The first and third terms simplify, owing to the factors

$$\frac{\partial x^p}{\partial \xi^e}\frac{\partial \xi^h}{\partial x^p} = \delta^h_e \qquad \text{and} \qquad \frac{\partial \xi^h}{\partial x^p}\frac{\partial x^p}{\partial \xi^f} = \delta^h_f;$$

the result is

$$\begin{aligned}\Gamma^p_{jl}\Gamma^i_{pk} = \Gamma^e_{bd}\Gamma^a_{ec}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a} + \Gamma^e_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^p}{\partial \xi^e}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^i}{\partial \xi^g} \\ + \Gamma^a_{fc}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^f}{\partial x^j \partial x^l} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.\end{aligned}$$

The second product is Step 5

$$\begin{aligned}\Gamma^p_{jl}\Gamma^i_{pk} = \Gamma^e_{bc}\Gamma^a_{ed}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} + \Gamma^e_{bc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^p}{\partial \xi^e}\frac{\partial^2 \xi^g}{\partial x^p \partial x^l}\frac{\partial x^i}{\partial \xi^g} \\ + \Gamma^a_{ed}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^e}{\partial x^j \partial x^k} + \frac{\partial^2 \xi^f}{\partial x^j \partial x^k}\frac{\partial^2 \xi^g}{\partial x^p \partial x^l}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.\end{aligned}$$

The difference between these products yields one tensorial term and six nontensorial ones: Step 6

$$\begin{aligned}\Gamma^p_{jl}\Gamma^i_{pk} - \Gamma^p_{jk}\Gamma^i_{pl} = (\Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed})\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a} \\ + \Gamma^e_{bd}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^p}{\partial \xi^e}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^i}{\partial \xi^g} - \Gamma^e_{bc}\frac{\partial \xi^b}{\partial x^j}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^p}{\partial \xi^e}\frac{\partial^2 \xi^g}{\partial x^p \partial x^l}\frac{\partial x^i}{\partial \xi^g} \\ + \Gamma^a_{ec}\frac{\partial \xi^c}{\partial x^k}\frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^e}{\partial x^j \partial x^l} - \Gamma^a_{ed}\frac{\partial \xi^d}{\partial x^l}\frac{\partial x^i}{\partial \xi^a}\frac{\partial^2 \xi^e}{\partial x^j \partial x^k} \\ + \frac{\partial^2 \xi^f}{\partial x^j \partial x^l}\frac{\partial^2 \xi^g}{\partial x^p \partial x^k}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g} - \frac{\partial^2 \xi^f}{\partial x^j \partial x^k}\frac{\partial^2 \xi^g}{\partial x^p \partial x^l}\frac{\partial x^p}{\partial \xi^f}\frac{\partial x^i}{\partial \xi^g}.\end{aligned}$$

The final step in the proof is to note that the twelve nontensorial terms in the sum of the two differences (steps 3 and 6) cancel in pairs when dummy indices are adjusted. □ Step 7

I thank Egbert K. Buning for raising a question about Proposition 6.14 that led me to note the role of Lemma 1, above, in its proof.