Calculus in Context

The Five College Calculus Project

James Callahan
Kenneth Hoffman

David Cox
Donal O'Shea
Harriet Pollatsek
Lester Senechal

DVI file created at 14:14, 31 January 2008

Advisory Committee of the Five College Calculus Project

Peter Lax, *Courant Institute, New York University*, Chairman
Solomon Garfunkel, *COMAP, Inc.*
John Neuberger, *The University of North Texas*
Barry Simon, *California Institute of Technology*
Gilbert Strang, *Massachusetts Institute of Technology*
John Truxal, *State University of New York, Stony Brook*

Preface: 2008 edition

We are publishing this edition of *Calculus in Context* online to make it freely available to all users. It is essentially unchanged from the 1994 edition.

The continuing support of Five Colleges, Inc., and especially of the Five College Coordinator, Lorna Peterson, has been crucial in paving the way for this new edition. We also wish to thank the many colleagues who have shared with us their experiences in using the book over the last twenty years—and have provided us with corrections to the text.

Preface: 1994 edition

Our point of view We believe that calculus can be for our students what it was for Euler and the Bernoullis: A language and a tool for exploring the whole fabric of science. We also believe that much of the mathematical depth and vitality of calculus lies in these connections to the other sciences. The mathematical questions that arise are compelling in part because the answers matter to other disciplines as well.

The calculus curriculum that this book represents started with a "clean slate;" we made no presumptive commitment to any aspect of the traditional course. In developing the curriculum, we found it helpful to spell out our **starting points**, our **curricular goals**, our **functional goals**, and our view of the **impact of technology**. Our starting points are a summary of what calculus is really about. Our curricular goals are what we aim to convey about the subject in the course. Our functional goals describe the attitudes and behaviors we hope our students will adopt in using calculus to approach scientific and mathematical questions. We emphasize that what is missing from these lists is as significant as what appears. In particular, we did *not* not begin by asking what parts of the traditional course to include or discard.

Starting Points

- Calculus is fundamentally a way of dealing with functional relationships that occur in scientific and mathematical contexts. The techniques of calculus must be subordinate to an overall view of the underlying questions.
- Technology radically enlarges the range of questions we can explore and the ways we can answer them. Computers and graphing calculators are much more than tools for teaching the traditional calculus.

Starting Points—continued

- The concept of a dynamical system is central to science Therefore, differential equations belong at the center of calculus, and technology makes this possible *at the introductory level*.
- The process of successive approximation is a key tool of calculus, even when the outcome of the process—the limit—cannot be explicitly given in closed form.

Curricular Goals

- Develop calculus in the context of scientific and mathematical questions.
- Treat systems of differential equations as fundamental objects of study.
- Construct and analyze mathematical models.
- Use the method of successive approximations to define and solve problems.
- Develop geometric visualization with hand-drawn and computer graphics.
- Give numerical methods a more central role.

Functional Goals

- Encourage collaborative work.
- Empower students to use calculus as a language and a tool.
- Make students comfortable tackling large, messy, ill-defined problems.
- Foster an experimental attitude towards mathematics.
- Help students appreciate the value of approximate solutions.
- Develop the sense that understanding concepts arises out of working on problems, not simply from reading the text and imitating its techniques.

Impact of Technology

- Differential equations can now be solved numerically, so they can take their rightful place in the introductory calculus course.
- The ability to handle data and perform many computations allows us to explore examples containing more of the messiness of real problems.
- As a consequence, we can now deal with credible models, and the role of modelling becomes much more central to our subject.

Impact of Technology—continued

- In particular, introductory calculus (and linear algebra) now have something more substantial to offer to life and social scientists, as well as to physical scientists, engineers and mathematicians.
- The distinction between pure and applied mathematics becomes even less clear (or useful) than it may have been.

By studying the text you can see, quite explicitly, how we have pursued the curricular goals. In particular, every one of those goals is addressed within the very first chapter. It begins with questions about describing and analyzing the spread of a contagious disease. A model is built, and the model is a system of coupled non-linear differential equations. We then begin a numerical assault on those equations, and the door is opened to a solution by successive approximations.

Our implementation of the functional goals is less obvious, but it is still evident. For instance, the text has many more words than the traditional calculus book—it is a book to be read. Also, the exercises make unusual demands on students. Most exercises are not just variants of examples that have been worked in the text. In fact, the text has rather few simple "template" examples.

Shifts in Emphasis It will also become apparent to you that the text reflects substantial shifts in emphasis in comparison to the traditional course. Here are some of the most striking:

How the emphasis shifts:	
INCREASE	DECREASE
concepts	techniques
geometry	algebra
graphs	formulas
brute force	elegance
numerical solutions	closed-form solutions

Euler's method is a good example of what we mean by "brute force." It is a general method of wide applicability. Of course when we use it to solve a differential equation like $y'(t) = t$, we are using a sledgehammer to crack a peanut. But at least the sledgehammer *does* work. Moreover, it works with coconuts (like $y' = y(1-y/10)$), and it will just as happily knock down a house (like $y' = \cos^2(t)$). Of course, students also see the elegant special methods that can be invoked to solve $y' = t$ and $y' = y(1-y/10)$ (separation of variables and partial fractions are discussed in chapter 11), but they understand that they are fortunate indeed when a real problem will succumb to these special methods.

Audience Our curriculum is not aimed at a special clientele. On the contrary, we think that calculus is one of the great bonds that unifies science, and all students should have an opportunity to see how the language and tools of calculus help forge that bond. We emphasize, though, that this is not a "service" course or calculus "with applications," but rather a course rich in mathematical ideas that will serve all students well, including mathematics majors. The student population in the first semester course is especially diverse. In fact, since many students take only one semester, we have aimed to make the first six chapters stand alone as a reasonably complete course. In particular, we have tried to present contexts that would be more or less broadly accessible. The emphasis on the physical sciences is clearly greater in the later chapters; this is deliberate. By the second semester, our students have gained skill and insight that allows them to tackle this added complexity.

Handbook for Instructors Working toward our curricular and functional goals has stretched us as well as our students. Teaching in this style is substantially different from the calculus courses most of us have learned from and taught in the past. Therefore we have prepared a handbook based on our experiences and those of colleagues at other schools. We urge prospective instructors to consult it.

Origins The Five College Calculus Project has a singular history. It begins almost thirty years ago, when the Five Colleges were only Four: Amherst, Mount Holyoke, Smith, and the large Amherst campus of the University of Massachusetts. These four resolved to create a new institution which would be a site for educational innovation at the undergraduate level; by 1970, Hampshire College was enrolling students and enlisting faculty.

Early in their academic careers, Hampshire students grapple with primary sources in all fields—in economics and ecology, as well as in history and literature. And journal articles don't shelter their readers from home truths: if a mathematical argument is needed, it is used. In this way, students in the life and social sciences found, sometimes to their surprise and dismay, that they needed to know calculus if they were to master their chosen fields. However, the calculus they needed was not, by and large, the calculus that was actually being taught. The journal articles dealt directly with the relation between quantities and their rates of change—in other words, with differential equations.

Confronted with a clear need, those students asked for help. By the mid-1970s, Michael Sutherland and Kenneth Hoffman were teaching a course for those students. The core of the course was calculus, but calculus as it is *used* in contemporary science. Mathematical ideas and techniques grew out of scientific questions. Given a process, students had to recast it as a model; most often, the model was a set of differential equations. To solve the differential equations, they used numerical methods implemented on a computer.

The course evolved and prospered quietly at Hampshire. More than a decade passed before several of us at the other four institutions paid some attention to it. We liked its fundamental premise, that differential equations belong at the center of calculus. What astounded us, though, was the revelation that differential equations could really *be* at the center—thanks to the use of computers.

This book is the result of our efforts to translate the Hampshire course for a wider audience. The typical student in calculus has not been driven to study calculus in order to come to grips with his or her own scientific questions—as those pioneering students had. If calculus is to emerge organically in the minds of the larger student population, a way must be found to involve that population in a spectrum of scientific and mathematical questions. Hence, calculus *in context*. Moreover, those contexts must be understandable to students with no special scientific training, and the mathematical issues they raise must lead to the central ideas of the calculus—to differential equations, in fact.

Coincidentally, the country turned its attention to the undergraduate science curriculum, and it focused on the calculus course. The National Science Foundation created a program to support calculus curriculum development. To carry out our plans we requested funds for a five-year project; we were fortunate to receive the only multi-year curriculum development grant awarded in the first year of the NSF program. This text is the outcome of our effort.

Acknowledgements

Certainly this book would have been possible without the support of the National Science Foundation and of Five Colleges, Inc. We particularly want to thank Louise Raphael who, as the first director of the calculus program at the National Science Foundation, had faith in us and recognized the value of what had already been accomplished at Hampshire College when we began our work. Five College Coordinators Conn Nugent and Lorna Peterson supported and encouraged our efforts, and Five College treasurer and business manager Jean Stabell has assisted us in countless ways throughout the project.

We are very grateful to the members of our Advisory Board: to Peter Lax, for his faith in us and his early help in organizing and chairing the Board; to Solomon Garfunkel, for his advice on politics and publishing; to Barry Simon, for using our text and giving us his thoughtful and imaginative suggestions for improving it; to Gilbert Strang, for his support of a radical venture; to John Truxal, for his detailed commentaries and insights into the world of engineering.

Among our colleagues, James Henle of Smith College deserves special thanks. Besides his many contributions to our discussions of curriculum and pedagogy, he developed the computer programs that have been so valuable for our teaching: Graph, Slinky, Superslinky, and Tint. Jeff Gelbard and Fred Henle ably extended Jim's programs to the MacIntosh and to DOS Windows and X Windows. All of this software is available on anonymous ftp at emmy.smith.edu. Mark Peterson, Robert Weaver, and David Cox also developed software that has been used by our students.

Several of our colleagues made substantial contributions to our frequent editorial conferences and helped with the writing of early drafts. We offer thanks to David Cohen, Robert Currier and James Henle at Smith; David Kelly at Hampshire; and Frank Wattenberg at the University of Massachusetts. Mary Beck, who is now at the University of Virginia, gave heaps of encouragement and good advice as a co-teacher of the earliest version of the course at Smith. Anne Kaufmann, an Ada Comstock Scholar at Smith, assisted us with extensive editorial reviews from the student perspective.

Two of the most significant new contributions to this edition are the appendix for graphing calculators and a complete set of solutions to all the exercises. From the time he first became aware of our project, Benjamin Levy has been telling us how easy and natural it would be to adapt our

Basic programs for graphing calculators. He has always used them when he taught *Calculus in Context*, and he created the appendix which contains translations of our programs for most of the graphing calculators in common use today. Lisa Hodsdon, Diane Jamrog, and Marcia Lazo have worked long hours over an entire summer to solve all the exercises and to prepare the results as LaTeX documents for inclusion in the Handbook for Instructors. We think both these contributions do much to make the course more useful to a wider audience.

We appreciate the contributions of our colleagues who participated in numerous debriefing sessions at semester's end and gave us comments on the evolving text. We thank George Cobb, Giuliana Davidoff, Alan Durfee, Janice Gifford, Mark Peterson, Margaret Robinson, and Robert Weaver at Mount Holyoke; Michael Albertson, Ruth Haas, Mary Murphy, Marjorie Senechal, Patricia Sipe, and Gerard Vinel at Smith. We learned, too, from the reactions of our colleagues in other disciplines who participated in faculty workshops on Calculus in Context.

We profited a great deal from the comments and reactions of early users of the text. We extend our thanks to Marian Barry at Aquinas College, Peter Dolan and Mark Halsey at Bard College, Donald Goldberg and his colleagues at Occidental College, Benjamin Levy at Beverly High School, Joan Reinthaler at Sidwell Friends School, Keith Stroyan at the University of Iowa, and Paul Zorn at St. Olaf College. Later users who have helped us are Judith Grabiner and Jim Hoste at Pitzer College; Allen Killpatrick, Mary Scherer, and Janet Beery at the University of Redlands; and Barry Simon at Caltech.

Dissemination grants from the NSF have funded regional workshops for faculty planning to adopt Calculus in Context. We are grateful to Donald Goldberg, Marian Barry, Janet Beery, and to Henry Warchall of the University of North Texas for coordinating workshops.

We owe a special debt to our students over the years, especially those who assisted us in teaching, but also those who gave us the benefit of their thoughtful reactions to the course and the text. Seeing what they were learning encouraged us at every step.

We continue to find it remarkable that our text is to be published the way we want it, not softened or ground down under the pressure of anonymous reviewers seeking a return to the mean. All of this is due to the bold and generous stance of W. H. Freeman. Robert Biewen, its president, understands—more than we could ever hope—what we are trying to do, and he has given us his unstinting support. Our aquisitions editors, Jeremiah Lyons and Holly Hodder, have inspired us with their passionate conviction that our book has something new and valuable to offer science education. Christine Hastings, our production editor, has shown heroic patience and grace in shaping the book itself against our often contrary views. We thank them all.

To the Student

In a typical high school math text, each section has a "technique" which you practice in a series of exercises very like the examples in the text. This book is different. In this course you will be learning to use calculus both as a tool and as a language in which you can think coherently about the problems you will be studying. As with any other language, a certain amount of time will need to be spent learning and practicing the formal rules. For instance, the conjugation of *être* must be almost second nature to you if you are to be able to read a novel—or even a newspaper—in French. In calculus, too, there are a number of manipulations which must become automatic so that you can focus clearly on the content of what is being said. It is important to realize, however, that becoming good at these manipulations is not the goal of learning calculus any more than becoming good at declensions and conjugations is the goal of learning French.

Up to now, most of the problems you have met in math classes have had definite answers such as "17," or "the circle with radius 1.75 and center at (2,3)." Such definite answers are satisfying (and even comforting). However, many interesting and important questions, like "How far is it to the planet Pluto," or "How many people are there with sickle-cell anemia," or "What are the solutions to the equation $x^5 + x + 1 = 0$" can't be answered exactly. Instead, we have ways to **approximate** the answers, and the more time and/or money we are willing to expend, the better our approximations may be. While many calculus problems do have exact answers, such problems often tend to be special or atypical in some way. Therefore, while you will be learning how to deal with these "nice" problems, you will also be developing ways of making good approximations to the solutions of the less well-behaved (and more common!) problems.

The computer or the graphing calculator is a tool that that you will need for this course, along with a clear head and a willing hand. We don't assume that you know anything about this technology ahead of time. Everything necessary is covered completely as we go along.

You can't learn mathematics simply by reading or watching others. The only way you can internalize the material is to work on problems yourself. It is by grappling with the problems that you will come to see what it is you do understand, and to see where your understanding is incomplete or fuzzy.

One of the most important intellectual skills you can develop is that of exploring questions on your own. Don't simply shut your mind down when you come to the end of an assigned problem. These problems have been designed not so much to capture the essence of calculus as to prod your thinking, to get you wondering about the concepts being explored. See if you can think up and answer variations on the problem. Does the problem suggest other questions? The ability to ask good questions of your own is at least as important as being able to answer questions posed by others.

We encourage you to work with others on the exercises. Two or three of you of roughly equal ability working on a problem will often accomplish much more than would any of you working alone. You will stimulate one another's imaginations, combine differing insights into a greater whole, and keep up each other's spirits in the frustrating times. This is particularly effective if you first spend time individually working on the material. Many students find it helpful to schedule a regular time to get together to work on problems.

Above all, take time to pause and admire the beauty and power of what you are learning. Aside from its utility, calculus is one of the most elegant and richly structured creations of the human mind and deserves to be profoundly admired on those grounds alone. Enjoy!

Contents