Chapter 4

Differential Equations

The rate equations with which we began our study of calculus are called **differential equations** when we identify the rates of change that appear within them as derivatives of functions. Differential equations are essential tools in many area of mathematics and the sciences. In this chapter we explore three of their important uses:

- **Modelling** problems using differential equations;
- **Solving** differential equations, both through numerical techniques like Euler's method and, where possible, through finding formulas which make the equations true;
- **Defining** new functions by differential equations.

We also introduce two important functions—the **exponential function** and the **logarithmic function**—which play central roles in the theory of solving differential equations. Finally, we introduce the operation of **antidifferentiation** as an important tool for solving some special kinds of differential equations.

4.1 Modelling with Differential Equations

To analyze the way an infectious disease spreads through a population, we asked how three quantities S, I, and R would vary over time. This was difficult to answer; we found no simple, direct relation between S (or I or R) and t. What we *did* find, though, was a relation between the variables

S, I, and R and their rates S', I', and R'. We expressed the relation as a set of rate equations. Then, given the rate equations and initial values for S, I, and R, we used Euler's method to estimate the values at any time in the future. By constructing a sequence of successive approximations, we were able to make these estimates as accurate as we wished.

There are two ideas here. The first is that we could write down equations for the rates of change that reflected important features of the process we sought to model. The second is that these equations *determined* the variables as functions of time, so we could make predictions about the real process we were modelling. Can we apply these ideas to other processes?

Differential equations and initial value problems

To answer this question, it will be helpful to introduce some new terms. What we have been calling rate equations are more commonly called **differential equations**. (The name is something of an historical accident. Since the equations involve functions and their derivatives, we might better call them *derivative* equations.) Euler's method treats the differential equations for a set of variables as a prescription for finding future values of those variables. However, in order to get started, we must always specify the initial values of the variables—their values at some given time. We call this specification an **initial condition**. The differential equations together with an initial condition is called an **initial value problem**. Each initial value problem determines a set of functions which we find by using Euler's method.

If we use Leibniz's notation for derivatives, a differential equation like $S' = -aSI$ takes the form $dS/dt = -aSI$. If we then treat dS/dt as a quotient of the individual *differentials* dS and dt (see page 123), we can even write the equation as $dS = -aSI\,dt$. Since this expresses the differential dS in terms of the differential dt, it was natural to call it a differential equation. Our approach is similar to Leibniz's, except that we don't need to introduce infinitesimally small quantities, which differentials were for Leibniz. Instead, we write $\Delta S \approx -aSI\,\Delta t$ and rely on the fact that the accumulated error of the resulting approximations can be made as small as we like.

To illustrate how differential equations can be used to describe a wide range of processes in the physical, biological, and social sciences, we'll devote this section to a number of ways to model and analyze the long-term behavior of animal populations. To be specific, we will talk about rabbits and foxes, but the ideas can be adapted to the population dynamics of virtually all living things (and many non-living systems as well, such as chemical reactions).

In each model, we will begin by identifying variables that describe what is happening. Then, we will try to establish how those variables change over time. Of course, no model can hope to capture every feature of the process we seek to describe, so we begin simply. We choose just one or two elements that seem particularly important. After examining the predictions of our simple model and checking how well they correspond to reality, we make modifications. We might include more features of the population dynamics, or we might describe the same features in different ways. Gradually, through a succession of refinements of our original simple model, we hope for descriptions that come closer and closer to the real situation we are studying.

Models can provide successive approximations to reality

Single-species Models: Rabbits

The problem. If we turn 2000 rabbits loose on a large, unpopulated island that has plenty of food for the rabbits, how might the number of rabbits vary over time? If we let $R = R(t)$ be the number of rabbits at time t (measured in months, let us say), we would like to be able to make some predictions about the function $R(t)$. It would be ideal to have a formula for $R(t)$—but this is not usually possible. Nevertheless, there may still be a great deal we can say about the behavior of R. To begin our explorations we will construct a model of the rabbit population that is obviously too simple. After we analyze the predictions it makes, we'll look at various ways to modify the model so that it approximates reality more closely.

The first model. Let's assume that, at any time t, the rate at which the rabbit population changes is simply proportional to the number of rabbits present at that time. For instance, if there were twice as many rabbits, then the *rate* at which new rabbits appear will also double. In mathematical terms, our assumption takes the form of the differential equation

Constant per capita growth

$$\frac{dR}{dt} = k\,R\,\frac{\text{rabbits}}{\text{month}}. \tag{1}$$

The multiplier k is called the **per capita growth rate** (or the **reproductive rate**), and its units are rabbits per month per rabbit. Per capita growth is discussed in exercise 22 in chapter 1, section 2.

For the sake of discussion, let's suppose that $k = .1$ rabbits per month per rabbit. This assumption means that, on the average, one rabbit will produce .1 new rabbits every month. In the S-I-R model of chapter 1, the reciprocals of the coefficients in the differential equations had natural interpretations. The same is true here for the per capita growth rate. Specifically, we can say that $1/k = 10$ months is the average length of time required for a rabbit to produce one new rabbit.

Since there are 2000 rabbits at the start, we can now state a clearly defined initial value problem for the function $R(t)$:

$$\frac{dR}{dt} = .1\,R \qquad\qquad R(0) = 2000.$$

Use Euler's method to find $R(t)$

By modifying the program SIRPLOT, we can readily produce the graph of the function that is determined by this problem. Before we do that, though, let's first consider some of the implications that we can draw out of the problem without the graph.

Since $R'(t) = .1\,R(t)$ rabbits per month and $R(0) = 2000$ rabbits, we see that the initial rate of growth is $R'(0) = 200$ rabbits per month. If this rate were to persist for 20 years ($=$ 240 months), R would have increased by

$$\Delta R = 240 \text{ months} \times 200\ \frac{\text{rabbits}}{\text{month}} = 48000 \text{ rabbits},$$

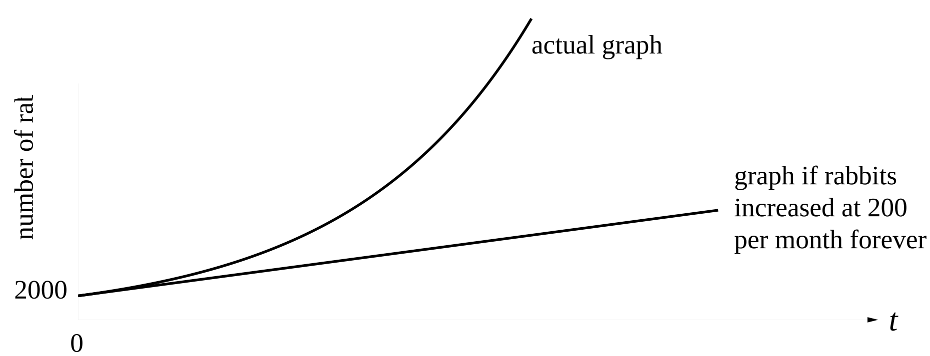
yielding altogether

$$R(240) = R(0) + \Delta R = 2000 + 48000 = 50000 \text{ rabbits}$$

at the end of the 20 years. However, since the population R is always getting larger, the differential equation tells us that the growth rate R' will *also* always be getting larger. Consequently, 50,000 is actually an underestimate of the number of rabbits predicted by this model.

The graph of R curves up

Let's restate our conclusions in a graphical form. If R' were always 200 rabbits per month, the graph of R plotted against t would just be a straight line whose slope is 200 rabbits/month. But R' is always getting bigger, so the slope of the graph should increase from left to right. This will make the graph curve upward. In fact, SIRPLOT will produce the following graph of $R(t)$:



Later, we will see that the function $R(t)$ determined by this initial value problem is actually an exponential function of t, and we will even be able to write down a formula for $R(t)$, namely

$$R(t) = 2000\,(1.10517)^t.$$

This model is too simple to be able to describe what happens to a rabbit population very well. One of the obvious difficulties is that it predicts the rabbit population just keeps growing—forever. For example, if we used the formula for $R(t)$ given above, our model would predict that after 20 years ($t = 240$) there will be more than 50 *trillion* rabbits! While rabbit populations can, under good conditions, grow at a nearly constant per capita rate for a surprisingly long time (this happened in Australia during the 19th century), our model is ultimately unrealistic.

It is a good idea to think qualitatively about the functions determined by a differential equation and make some rough estimates before doing extensive calculations. Your sketches may help you see ways in which the model doesn't correspond to reality. Or, you may be able to catch errors in your computations if they differ noticeably from what your estimates led you to expect.

The second model. One way out of the problem of unlimited growth is to modify equation (1) to take into account the fact that any given ecological system can support only some finite number of creatures over the long term. This number is called the **carrying capacity** of the system. We expect that when a population has reached the carrying capacity of the system, the population should neither grow nor shrink. At carrying capacity, a population should hold steady—its rate of change should be zero. For the sake of specificity, let's suppose that in our example the carrying capacity of the island is 25,000 rabbits.

The carrying capacity of the environment

What we would like to do, then, is to find an expression for R' which is similar to equation (1) when the number of rabbits R is near 2000, but which approaches 0 as R approaches 25,000. One model which captures these features is the **logistic equation**, first proposed by the Belgian mathematician Otto Verhulst in 1845:

Logistic growth

$$R' = k\,R\left(1 - \frac{R}{b}\right)\,\frac{\text{rabbits}}{\text{month}}. \tag{2}$$

In this equation, the coefficient k is called the **natural growth rate**. It plays the same role as the per capita growth rate in equation (1), and it has the same units—rabbits per month per rabbit. The number b is the **carrying capacity**; it is measured in rabbits. (We first saw the logistic equation on pages 80–86.) Notice also that we have written the derivative of R in the simpler form R', a practice we will continue for the rest of the section.

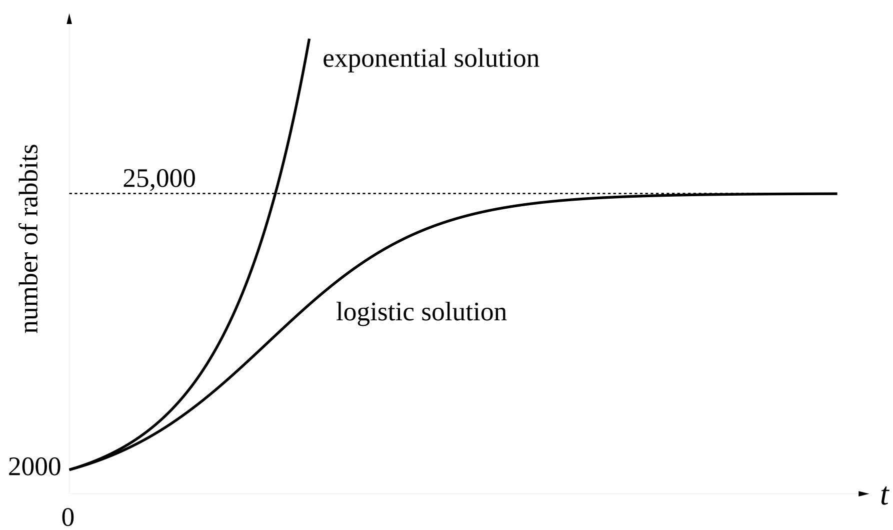
If the carrying capacity of the island is 25,000 rabbits, and if we keep the natural growth rate at .1 rabbits per month per rabbit, then the logistic equation for the rabbit population is

$$R' = .1\,R\left(1 - \frac{R}{25000}\right)\ \frac{\text{rabbits}}{\text{month}}.$$

Check to see that this equation really does have the behavior claimed for it—namely, that a population of 25,000 rabbits neither grows or declines. Notice also that R' is positive as long as R is less than 25000, so the population increases. However, as R approaches 25000, R' will get closer and closer to 0, so the graph will become nearly horizontal. (What would happen if the island ever had more than 25,000 rabbits?)

The graph of $R(t)$ levels off near $R = 25000$

These observations about the qualitative behavior of $R(t)$ are consistent with the following graph, produced by a modified version of the program SIRPLOT. For comparison, we have also graphed the exponential function produced by the first model. Notice that the two graphs "share ink" when R near 2000, but diverge later on.



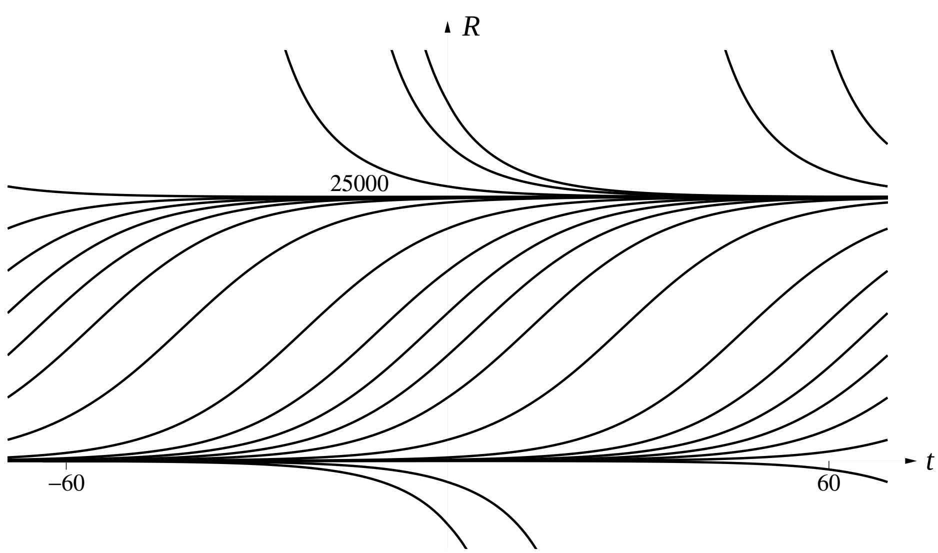
By modifying the program SIRVALUE, we can even get numerical answers to specific questions about the two models. For example, after 30 months under constant per capita growth, the rabbit population will be more than 40,000—well beyond the carrying capacity of the island. Under logistic growth, though, the population will be only about 16,000.

In the following figure we display several functions that are determined by the logistic equation

$$R' = .1\,R\left(1 - \frac{R}{25000}\right)$$

when different initial conditions are given. Each graph therefore predicts the future for a different initial population $R(0)$. One of the graphs is just the t-axis itself. What does this graph predict about the rabbit population? What other graph is just a straight line, and what initial population will lead to this line?

While the logistic equation above was developed to model a physical problem in which only values of R with $R \geq 0$ have any meaning, the mathematical problem of finding solutions for the resulting differential equation makes sense for all values of R. We have drawn three graphs resulting from initial values $R(0) < 0$. While this growth behavior of 'anti-rabbits' is of little practical interest in this case, there may well be other physical problems of an entirely different sort which lead to the same mathematical model, and in which the solutions below the t–axis are crucial.



Solutions to the logistic equation $R' = .1R\,(1 - R/25000)$

Two-species Models: Rabbits and Foxes

Introduce predators

No species lives alone in an environment, and the same is true of the rabbits on our island. The rabbit population will probably have to deal with predators of various sorts. Some are microscopic—disease organisms, for example—while others loom as obvious threats. We will enrich our population model by adding a second species—foxes—that will prey on the rabbits. We will continue to suppose that the rabbits live on abundant native vegetation, and we will now assume that the rabbits are the sole food supply of the foxes. Can we say what will happen? Will the number of foxes and rabbits level off and reach a "steady state" where their numbers don't vary? Or will one species perhaps become extinct?

Let F denote the number of foxes, and R the number of rabbits. As before, measure the time t in months. Then F and R are functions of t: $F(t)$ and $R(t)$. We seek differential equations that describe how the growth rates F' and R' are related to the population sizes F and R. We make the following assumptions.

- In the absence of foxes, the rabbit population grows logistically.
- The population of rabbits declines at a rate proportional to the product $R \cdot F$. This is reasonable if we assume rabbits never die of old age—they just get a little too slow. Their death rate, which depends on the number of fatal encounters between rabbits and foxes, will then be approximately proportional to both R and F—and thus to their product. (This is the same kind of interaction effect we used in our epidemic model to predict the rate at which susceptibles become infected.)
- In the absence of rabbits, the foxes die off at a rate proportional to the number of foxes present.
- The fox population increases at a rate proportional to the number of encounters between rabbits and foxes. To a first approximation, this says that the birth rate in the fox population depends on maternal fox nutrition, and this depends on the number of rabbit-fox encounters, which is proportional to $R \cdot F$.

Our assumptions are about birth and death rates, so we can convert them quite naturally into differential equations. Pause here and check that the assumptions translate into these differential equations:

$$R' = a\,R\left(1 - \frac{R}{b}\right) - cRF = a\,R - \frac{a}{b}\,R^2 - c\,RF$$
$$F' = d\,RF - e\,F$$

These are the **Lotka–Volterra equations with bounded growth**. The coefficients a, b, c, d, and e are **parameters**—constants that have to be determined through field observations in particular circumstances.

Lotka–Volterra equations with bounded growth

An example. To see what kind of predictions the Lotka–Volterra equations make, we'll work through an example with specific values for the parameters. Let

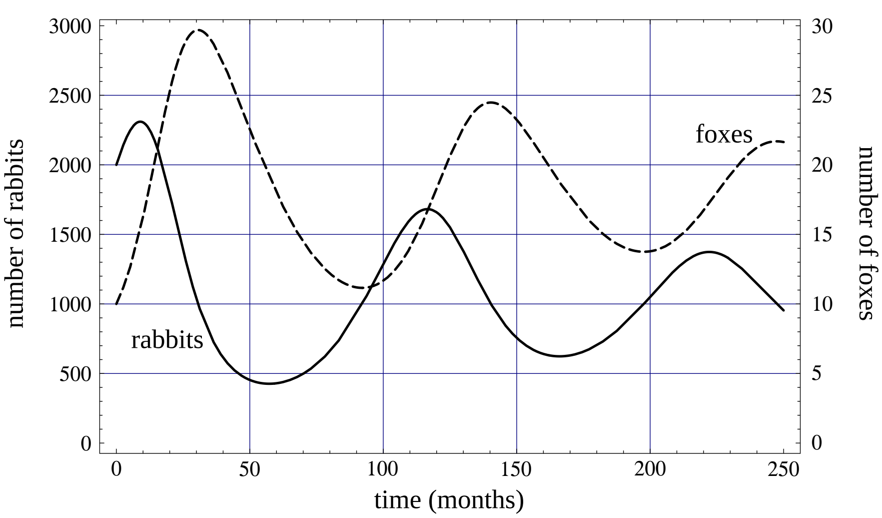
$$\begin{aligned} a &= .1 && \text{rabbits per month per rabbit} \\ b &= 10000 && \text{rabbits} \\ c &= .005 && \text{rabbits per month per rabbit-fox} \\ d &= .00004 && \text{foxes per month per rabbit-fox} \\ e &= .04 && \text{foxes per month per fox} \end{aligned}$$

(Check that these five parameters have the right units.) These choices give us the specific differential equations

$$R' = .1\,R - .00001\,R^2 - .005\,RF$$
$$F' = .00004\,RF - .04\,F$$

To use this model to follow R and F into the future, we need to know the initial sizes of the two populations. Let's suppose that there are 2000 rabbits and 10 foxes at time $t = 0$. Then the two populations will vary in the following way over the next 250 months.

The graphs of R and F

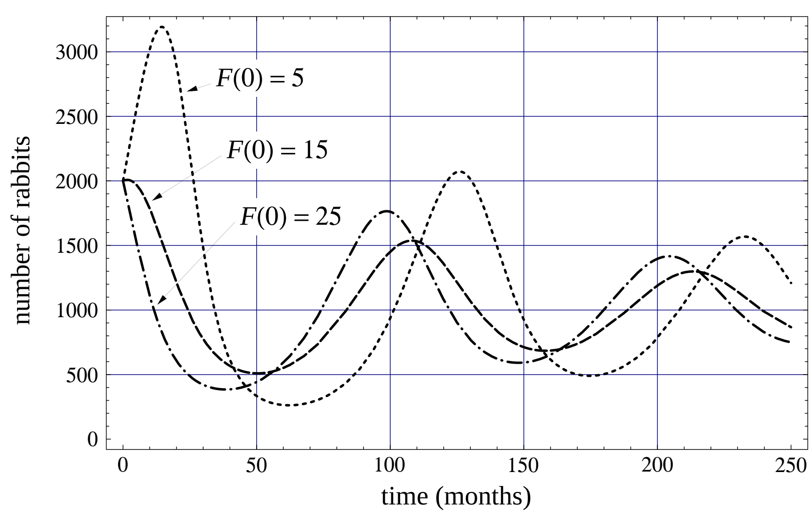


A variant of the program SIRPLOT was used to produce these graphs. Notice that it plots $100F$ rather than F itself. This is because the number of foxes is about 100 times smaller than the number of rabbits. Consequently, $100F$ and R are about the same size, so their graphs fit nicely together on the same screen.

The graphs have several interesting features. There are different scales for the R and the F values, because the program plots $100F$ instead of F. The peak fox population is about 30, while the peak rabbit population is about 2300. The rabbit and fox populations rise and fall in a regular manner. They rise and fall less with each repeat, though, and if the graphs were continued far enough into the future we would see R and F level off to nearly constant values.

How rabbits respond to changes in the initial fox population

The illustration below shows what happens to an initial rabbit population of 2000 in the presence of three different initial fox populations $F(0)$. Note that the peak rabbit populations are different, and they occur at different times. The size of the intervals between peaks also depends on $F(0)$.



Rabbit populations for different initial fox populations

We have looked at three models, each a refinement of the preceding one. The first was the simplest. It accounted only for the rabbits, and it assumed the rabbit population grew at a constant per capita rate. The second was also restricted to rabbits, but it assumed logistic growth to take into account the carrying capacity of the environment. The third introduced the complexity of a second species preying on the rabbits. In the exercises you will have an opportunity to explore these and other models. Remember that when you use Euler's method to find the functions determined by an initial value problem, you must construct a sequence of successive approximations, until you obtain the level of accuracy desired.

Exercises

Single-species models

1. **Constant per capita Growth**. This question considers the initial value problem given in the text:

$$R' = 0.1\,R \text{ rabbits per month}; \qquad R(0) = 2000 \text{ rabbits.}$$

a) Use Euler's method to determine how many rabbits there are after 6 months. Present a table of successive approximations from which you can read the exact value to whole-number accuracy.

b) Determine, to whole-number accuracy, how many rabbits there are after 24 months.

c) How many months does it take for the rabbit population to reach 25,000?

2. **Logistic Growth**. The following questions concern a rabbit population described by the logistic model

$$R' = 0.1\,R\left(1 - \frac{R}{25000}\right) \text{ rabbits per month.}$$

a) What happens to a population of 2000 rabbits after 6 months, after 24 months, and after 5 years? To answer each question, present a table of successive approximations that allows you to give the exact value to the nearest whole number.

b) Sketch the functions determined by the logistic equation if you start with either 2000 or 40000 rabbits. (Suggestion: you can modify the program SIRPLOT to answer this question.) Compare the two functions. How do they differ? In what ways are they similar?

3. **Seasonal Factors** Living conditions for most wild populations are not constant throughout the year—due to factors like drought or cold, the environment is less supportive during some parts of the year than at others. Partially in response to this, most animals don't reproduce uniformly throughout the year. This problem explores ways of modifying the logistic model to reflect these facts.

a) For the eastern cottontail rabbit, most young are born during the months of March–May, with reduced reproduction during June–August, and virtually no reproduction during the other six months of the year. Write a program to generate the solution to the differential equation $R' = k(1 - R/25000)$, where $k = .2$ during March, April, and May; $k = .05$ during June, July, and August; and $k = 0$ the rest of the year. Start with an initial population of 2000 rabbits on January 1. You may find that using the `IF ... THEN` construction in your program is a convenient way to incorporate the varying reproductive rate.

b) How would you modify the model to take into account the fact that rabbits don't reproduce during their first season?

4. **World population**. The world's population in 1990 was about 5 billion, and data show that birth rates range from 35 to 40 per thousand per year and death rates from 15 to 20. Take this to imply a net annual growth rate of 20 per thousand. One model for world population assumes constant per capita growth, with a per capita growth rate of $20/1000 = 0.02$.

a) Write a differential equation for P that expresses this assumption. Use P to denote the world population, measured in billions.

b) According to the differential equation in (a), at what rate (in billions of persons per year) was the world population growing in 1990?

c) By applying Euler's method to this model, using the initial value of 5 billion in 1990, estimate the world population in the years 1980, 2000, 2040, and 2230. Present a table of successive approximations that stabilizes with one decimal place of accuracy (in billions). What step size did you have to use to obtain this accuracy?

5. **Supergrowth**. Another model for the world population, one that actually seems to fit recent population data fairly well, assumes "supergrowth"—the rate P' is proportional to a *higher power* of P, rather than to P itself. The model is

$$P' = .015\, P^{1.2}.$$

As in the previous exercise, assume that P is measured in billions, and the population in 1990 was about 5 billion.

a) According to this model, at what rate (in billions of persons per year) was the population growing in 1990?

b) Using Euler's method, estimate the world population in the years 1980, 2000 and 2040. Use successive approximations until you have one decimal place of accuracy (in billions). What step size did you have to use to obtain this accuracy?

c) Use an Euler approximation with a step size of 0.1 to estimate the world population in the year 2230. What happens if you repeat your calculation with a step size of 0.01? [Comment: Something strange is going on here. We will look again at this model in the next section.]

Two-species models

Here are some other differential equations that model a predator-prey interaction between two species.

6. **The May Model**. This model has been proposed by the contemporary ecologist, R.M. May, to incorporate more realistic assumptions about the encounters between predators (foxes) and their prey (rabbits). So that you can work with quantities that are about the same size (and therefore plot them on the same graph), let y be the number of foxes and let x be the number of rabbits *divided by* 100—we are thus measuring rabbits in units of "hectorabbits".

> While a term like "hectorabbits" is deliberately whimsical, it echoes the common and sensible practice of choosing units that allow us to measure things with numbers that are neither too small nor too large. For example, we wouldn't describe the distance from the earth to the moon in millimeters, and we wouldn't describe the mass of a raindrop in kilograms.

In his model, May makes the following assumptions.

- In the absence of foxes, the rabbits grow logistically.
- The number of rabbits a single fox eats in a given time period is a function $D(x)$ of the number of rabbits available. $D(x)$ varies from 0 if there are no rabbits available to some value c (the **saturation value**) if there is an unlimited supply of rabbits. The total number of rabbits consumed in the given time period will thus be $D(x) \cdot y$.

- The fox population is governed by the logistic equation, and the carrying capacity is proportional to the number of rabbits.

a) Explain why $D(x) = \dfrac{c\,x}{x+d}$ (d some constant) might be a reasonable model for the function $D(x)$. Include a sketch of the graph of D in your discussion. What is the role of the parameter d? That is, what feature of rabbit – fox interactions is reflected by making d smaller or larger?

b) Explain how the following system of equations incorporates May's assumptions.

$$x' = a\,x\left(1 - \frac{x}{b}\right) - \frac{c\,xy}{x+d}$$
$$y' = e\,y\left(1 - \frac{y}{fx}\right)$$

The parameters a, b, c, d, e and f are all positive.

c) Assume you begin with 2000 rabbits and 10 foxes. (Be careful: $x(0) \neq 2000$.) What does May's model predict will happen to the rabbits and foxes over time if the values of the parameters are $a = .6$, $b = 10$, $c = .5$, $d = 1$, $e = .1$ and $f = 2$? Use a suitable modification of the program SIRPLOT.

d) Using the same parameters, describe what happens if you begin with 2000 rabbits and 20 foxes; with 1000 rabbits and 10 foxes; with 1000 rabbits and 20 foxes. Does the eventual long-term behavior depend on the initial condition? How does the long-term behavior here compare with the long-term behavior of the two populations in the Lotka–Volterra model of the text?

e) Using 2000 rabbits and 20 foxes as the initial values, let's see how the behavior of the solutions is affected by changing the values of the parameter c, the saturation value for the number of rabbits (measured in centirabbits, remember) a single fox can eat in a month. Keeping all the other parameters (a, b, d, $\ldots$) fixed at the values given above, get solution curves for $c = .5$, $c = .45$, $c = .4$, $\ldots$, $c = .15$, and $c = .1$. The solutions undergo a qualitative change somewhere between $c = .3$ and $c = .25$. Describe this change. Can you pinpoint the crucial value of c more closely? This phenomenon is an example of **Hopf bifurcation**, which we will look at more closely in chapter 8. The May model undergoes a Hopf bifurcation as you vary each of the other parameters as well. Choose a couple of them and locate approximately the associated bifurcation values.

7. **The Lotka–Volterra Equations**. This model for predator and prey interactions is slightly simpler than the “bounded growth” version we consider in the text. It is important historically, though, because it was one of the first mathematical population models, proposed as a way of understanding why the harvests of certain species of fish in the Adriatic Sea exhibited cyclical behavior over the years. For the sake of variety, let's take the prey to be hares and the predators to be lynx.

Let $H(t)$ denote the number of hares at time t and $L(t)$ the number of lynx. This model, the basic Lotka–Volterra model, differs from the bounded growth model in only one respect: it assumes the hares would experience constant per capita growth if there were no lynx.

a) Explain why the following system of equations incorporates the assumptions of the basic model. (The parameters a, b, c, and d are all positive.)

$$\begin{aligned} H' &= a\,H - b\,HL \\ L' &= c\,HL - d\,L \end{aligned}$$

(These are called the **Lotka–Volterra equations**. They were developed independently by the Italian mathematical physicist Vito Volterra in 1925–26, and by the mathematical ecologist and demographer Alfred James Lotka a few years earlier. Though simplistic, they form one of the principal starting points in ecological modeling.)

b) Explain why a and b have the units hares per month per hare and hares per month per hare-lynx, respectively. What are the units of c and d? Explain why.

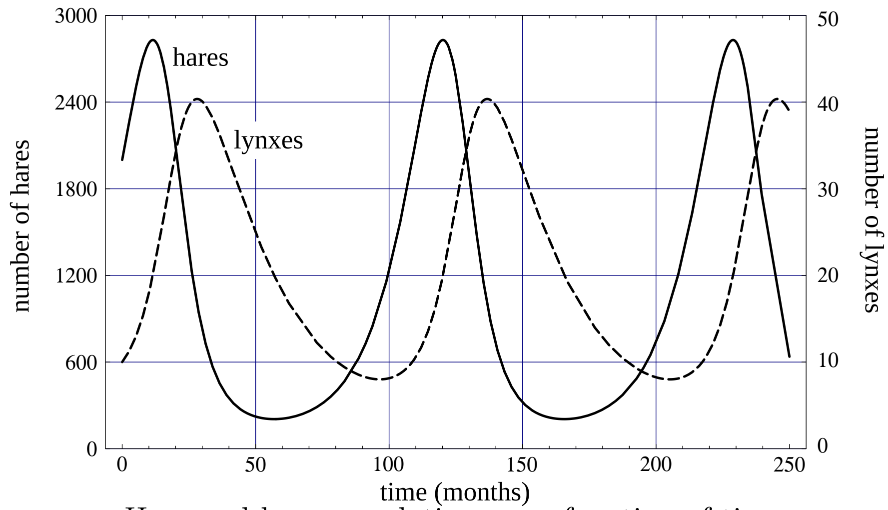
Suppose time t is measured in months, and suppose the parameters have values

$a = .1$	hares per month per hare
$b = .005$	hares per month per hare-lynx
$c = .00004$	lynx per month per hare-lynx
$d = .04$	lynx per month per lynx

This leads to the system of differential equations

$$\begin{aligned} H' &= .1\,H - .005\,HL \\ L' &= .00004\,HL - .04\,L. \end{aligned}$$

c) Suppose that you start with 2000 hares and 10 lynx—that is, $H(0) = 2000$ and $L(0) = 10$. Describe what happens to the two populations. A good way to do this is to draw graphs of the functions $H(t)$ and $L(t)$. It will be convenient to have the Hare scale run from 0 to 3000, and the Lynx scale from 0 to 50. If you modify the program SIRPLOT, have it plot H and $60L$.



Hare and lynx populations as a function of time

You should get graphs like those above. Notice that the hare and lynx populations rise and fall in a fashion similar to the rabbits and foxes, but here they oscillate—returning *periodically* to their original values.

d) What happens if you keep the same initial hare population of 2000, but use different initial lynx populations? Try $L(0) = 20$ and $L(0) = 50$. (In each case, use a step size of .1 month.)

e) Start with 2000 hares and 10 lynx. From part (c), you know the solutions are periodic. The goal of this part is to analyze this periodic behavior. You can do this with your program in part (c), but you may prefer to replace the FOR-NEXT loop in your program by a variety of DO-WHILE loops (see page 77). First find the maximum number of hares. What is the length of one **period** for the hare population? That is, how long does it take the hare population to complete one cycle (e.g., to go from one maximum to the next)? Find the length of one period for the lynx. Do the hare and lynx populations have the same periods?

f) Plot the hare populations over time when you start with 2000 hares and, successively, 10, 20, and 50 lynx. Is the hare population periodic in each case? What is the period? Does it vary with the size of the initial lynx population?

Fermentation

Wine is made by yeast; yeast digests the sugars in grape juice and produces alcohol as a waste product. This process is called fermentation. The alcohol is toxic to the yeast, though, and the yeast is eventually killed by the alcohol. This stops fermentation, and the liquid has become wine, with about 8–12% alcohol.

Although alcohol isn't a "species," it acts like a predator on yeast. Unlike the other predator-prey problems we have considered, though, the yeast does not have an unlimited food supply. The following exercises develop a sequence of models to take into account the interactions between sugar, yeast, and alcohol.

8. a) In the first model assume that the sugar supply is not depleted, that no alcohol appears, and that the yeast simply grows *logistically*. Begin by adding 0.5 lb of yeast to a large vat of grape juice whose carrying capacity is 10 lbs of yeast. Assume that the natural growth rate of the yeast is 0.2 lbs of yeast per hour, per pound of yeast. Let $Y(t)$ be the number of pounds of live yeast present after t hours; what differential equation describes the growth of Y?

b) Graph the solution $Y(t)$, for example by using a suitable modification of the program SIRPLOT. Indicate on your graph approximately when the yeast reaches one-half the carrying capacity of the vat, and when it gets to within 1% of the carrying capacity.

c) Suppose you use a second strain of yeast whose natural growth rate is only half that of the first strain of yeast. If you put 0.5 lb of *this* yeast into the vat of grape juice, when will it reach one-half the carrying capacity of the vat, and when will it get to within 1% of the carrying capacity? Compare these values to the values produced by the first strain of yeast: are they larger, or smaller? Sketch, on the same graph as in part (b), the way this yeast grows over time.

9. a) Now consider how the yeast produces alcohol. Suppose that waste products are generated at a rate proportional to the amount of yeast present; specifically, suppose each pound of yeast produces 0.05 lbs of alcohol per hour.

(The other major waste product is carbon dioxide gas, which bubbles out of the liquid.) Let $A(t)$ denote the amount of alcohol generated after t hours. Write a differential equation that describes the growth of A.

b) Consider the toxic effect of the alcohol on the yeast. Assume that yeast cells die at a rate proportional to the amount of alcohol present, and also to the amount of yeast present. Specifically, assume that, in each pound of yeast, a pound of alcohol will kill 0.1 lb of yeast per hour. Then, if there are Y lbs of yeast and A lbs of alcohol, how many pounds of yeast will die in one hour? Modify the original logistic equation for Y (strain 1) to take this effect into account. The modification involves subtracting off a new term that describes the rate at which alcohol kills yeast. What is the new differential equation?

c) You should now have two differential equations describing the rates of growth of yeast and alcohol. The equations are **coupled**, in the sense that the yeast equation involves alcohol, and the alcohol equation involves yeast. Assuming that the vat contains, initially, 0.5 lb of yeast and no alcohol, describe by means of a graph what happens to the yeast. How close does the yeast get to carrying capacity, and when does this happen? Does the fermentation end? If so, when; and how much alcohol has been produced by that time? (Note that since Y will never get all the way to 0, you will need to adopt some convention like $Y \leq .01$ to specify the end of fermentation.)

10. What happens if the rates of toxicity and alcohol production are different? Specifically, increase the rate of alcohol production by a factor of five—from 0.05 to 0.25 lbs of alcohol per hour, per pound of yeast—and at the same time reduce the toxicity rate by the same factor—from 0.10 to 0.02 lb of yeast per hour, per pound of alcohol and pound of yeast. How do these changes affect the time it takes for fermentation to end? How do they affect the amount of alcohol produced? What happens if only the rate of alcohol production is changed? What happens if only the toxicity rate is reduced?

11. a) The third model will take into account that the sugar in the grape juice is consumed. Suppose the yeast consumes .15 lb of sugar per hour, per lb of yeast. Let $S(t)$ be the amount of sugar in the vat after t hours. Write a differential equation that describes what happens to S over time.

b) Since the carrying capacity of the vat depends on the amount of sugar in it, the carrying capacity must now vary. Assume that the carrying capacity of S lbs of sugar is $.4\,S$ lbs of yeast. How much sugar is needed to maintain a carrying capacity of 10 lbs of yeast? How much is needed to maintain a carrying capacity of 1 lb of yeast? Rewrite the logistic equation for yeast so that the carrying capacity is $.4\,S$ lbs, instead of 10 lbs, of yeast. Retain the term you developed in 9.b to reflect the toxic impact of alcohol on the yeast.

c) There are now three differential equations. Using them, describe what happens to .5 lbs of yeast that is put into a vat of grape juice that contains 25 lbs of sugar at the start. Does all the sugar disappear? Does all the yeast disappear? How long does it take before there is only .01 lb of yeast? How much sugar is left then? How much alcohol has been produced by that time?

Newton's law of cooling

Suppose that we start off with a freshly brewed cup of coffee at 90°C and set it down in a room where the temperature is 20°C. What will the temperature of the coffee be in 20 minutes? How long will it take the coffee to cool to 30°C?

If we let the temperature of the coffee be Q (in °C), then Q is a function of the time t, measured in minutes. We have $Q(0) = 90$°C, and we would like to find the value t_1 for which $Q(t_1) = 30$°C.

It is not immediately apparent how to give Q as a function of t. However, we can describe the *rate* at which a liquid cools off, using **Newton's law of cooling**: the rate at which an object cools (or warms up, if it's cooler than its surroundings) is proportional to the *difference* between its temperature and that of its surroundings.

12. In our example, the temperature of the room is 20°C, so Newton's law of cooling states that $Q'(t)$ is proportional to $Q - 20$, the difference between the temperature of the liquid and the room. In symbols, we have

$$Q' = -k\,(Q - 20)$$

where k is some positive constant.

a) Why is there a minus sign in the equation?

The particular value of k would need to be determined experimentally. It will depend on things like the size and shape of the cup, how much sugar and cream you use, and whether you stir the liquid. Suppose that k has the value of .1° per minute per °C of temperature difference. Then the differential equation becomes:

$$Q' = -.1(Q - 20) \quad {}^\circ\text{C per minute.}$$

b) Use Euler's method to determine the temperature Q after 20 minutes. Write a table of successive approximations with smaller and smaller step sizes. The values in your table should stabilize to the second decimal place.

c) How long does it take for the temperature Q to drop to 30°C? Use a DO-WHILE loop to construct a table of successive approximations that stabilize to the second decimal place.

13. On a hot day, a cold drink warms up at a rate approximately proportional to the difference in temperature between the drink and its surroundings. Suppose the air temperature is 90°F and the drink is initially at 36°F. If Q is the temperature of the drink at any time, we shall suppose that it warms up at the rate

$$Q' = -0.2(Q - 90) \quad {}^\circ\text{F per minute.}$$

According to this model, what will the temperature of the drink be after 5 minutes, and after 10 minutes. In both cases, produce values that are accurate to two decimal places.

14. In our discussion of cooling coffee, we assumed that the coffee did not heat up the room. This is reasonable because the room is large, compared to the cup of coffee. Suppose, in an effort to keep it warmer, we put the coffee into a small insulated container—such as a microwave oven (which is turned off). We must assume that the coffee *does* heat up the air inside the container. Let A be the air temperature in the container and Q the temperature of the coffee. Then both A and Q change over time, and Newton's law of cooling tells us the *rates* at which they change. In fact, the law says that both Q' and A' are proportional to $Q - A$. Thus,

$$\begin{aligned} Q' &= -k_1(Q - A) \\ A' &= k_2(Q - A), \end{aligned}$$

where k_1 and k_2 are positive constants.

a) Explain the signs that appear in these differential equations.

b) Suppose $k_1 = .3$ and $k_2 = .1$. If $Q(0) = 90$°C and $A(0) = 20$°C, when will the temperature of the coffee be 40°C? What is the temperature of the air at this time? Your answers should be accurate to one decimal place.

c) What does the temperature of the coffee become eventually? How long does it take to reach that temperature?

S-I-R revisited

Consider the spread of an infectious disease that is modelled by the S-I-R differential equations

$$\begin{aligned} S' &= -.00001\,SI, \\ I' &= .00001\,SI - .08\,I, \\ R' &= \phantom{-.00001\,SI - {}} .08\,I. \end{aligned}$$

Take the initial condition of the three populations to be

$$\begin{aligned} S(0) &= 35{,}400 \text{ persons}, \\ I(0) &= 13{,}500 \text{ persons}, \\ R(0) &= 22{,}100 \text{ persons}. \end{aligned}$$

15. How many susceptibles are left after 40 days? When is the largest number of people infected? How many susceptibles are there at that time? Explain how you could determine the last number *without* using Euler's method.

16. What happens as the epidemic "runs its course"? That is, as more and more time goes by, what happens to the numbers of infecteds and susceptibles?

17. One of the principal uses of a mathematical model is to get a qualitative idea how a system will behave with different initial conditions. For instance, suppose we introduce 100 infected individuals into a population. How will the spread of the infection depend on the size of the population? Assume the same S-I-R differential equations that were used in the previous exercise, and draw the graphs of $S(t)$ for initial susceptible population sizes $S(0)$ ranging from 0 to 45,000 in increments of 5000 (that is, take $S(0) = 0$, 5000, 10000, …, 45000). In each case assume that $R(0) = 0$ and $I(0) = 100$. Use these

graphs to argue that the larger the initial susceptible population, the more rapidly the epidemic runs its course.

18. Draw the graphs of $I(t)$ for the same initial conditions as in the previous problem. Using these graphs you can demonstrate that the larger the susceptible population, the larger will be the fraction of the population that is infected during the worst stages of the epidemic. Do this by constructing a table displaying $I_{\max}$, $t_{\max}$, and $P_{\max}$, where $I_{\max}$ is the maximum value of $I(t)$, $t_{\max}$ is the time at which this maximum occurs (that is, $I_{\max} = I(t_{\max})$), and $P_{\max}$ is the ratio of $I_{\max}$ to the initial susceptible population: $P_{\max} = I_{\max}/S(0)$. The table below gives the first three sets of values.

$S(0)$	$I_{\max}$	$P_{\max}$	$t_{\max}$
5 000	100	0.02	0
10 000	315	0.03	> 100
15 000	2071	0.14	66
⋮	⋮	⋮	⋮

Your table should show that there is a time when over half the population is infected if $S(0) = 45000$, while there is never a time when more than one-fourth of the population is infected if $S(0) = 20000$.

Constructing models

Systems in which we know a number of quantities at a given time and would like to know their values at a future time (or know at what future time they will attain given values) occur in many different contexts. The following are some systems for discussion. Can any of these be modelled as initial value problems? What information would you need to resolve the question? Make some reasonable assumptions about the missing information and write down an initial value problem which would model the system.

19. We deposit a fixed sum of money in a bank, and we'd like to know how much will be there in ten years.

20. We know the diameter of the mold spot growing on a cheese sandwich is 1/4 inch, and we'd like to know when its diameter will be one inch.

21. We know the fecal bacterial and coliform concentrations in a local swimming hole, and we'd like to know when they fall below certain prescribed levels (which the Board of Health deems safe).

22. We know what the temperature and rainfall is today, and we'd like to know what both will be one week from today.

23. We know what the winning lottery number was yesterday, and we'd like to know what the winning number will be the day after tomorrow.

24. We know where the earth, sun, and moon are in relation to each other now, and how fast and in what direction they are moving. We would like to be able to predict where they are going to be at any time in the future. We know the gravity of each affects the motions of the others by determining the way their velocities are changing.

4.2 Solutions of Differential Equations

Differential Equations are Equations

Differential equations give instructions for Euler's method

Until now, we have viewed a system of differential equations as a set of *instructions* for "stepping into the future" (or the past). Put another way, an initial value problem was treated as a prescription for using Euler's method to determine a set of functions which were then given either graphically or in tabular form.

In this section we take a new point of view: we will think of differential equations as *equations* for which we would like to find *solutions* in terms of functions which can be given by explicit formulas. While it is unfortunately the case that most differential equations do not have solutions which can be given by formulas, there are enough important classes of equations where such solutions do exist to make them worth studying. When such solutions can be found, we have a very powerful tool for examining the behavior of the phenomenon being modelled.

Equations have *solutions*

To see what this means, let's look first at equations in algebra. Consider the equation $x^2 = x + 6$. As it stands, this is neither true nor false. We make it true or false, though, when we substitute a particular number for x. For example, $x = 3$ makes the equation true, because $3^2 = 3 + 6$. On the other hand, $x = 1$ makes the equation false, because $1^2 \neq 1+6$. Any number that makes an equation true is called a *solution* to that equation. In fact, $x^2 = x + 6$ has exactly two solutions: $x = 3$ and $x = -2$.

We can view differential equations the same way. Consider, for example, the differential equation

$$\frac{dy}{dt} = \frac{1}{2y}.$$

Because it involves the expression dy/dt, we understand that y is a function of t. As it stands, the differential equation is neither true nor false. We make it true or false, though, when we substitute a particular *function* for y. For example, $y = \sqrt{t} = t^{1/2}$ makes the differential equation true. To see this, first look at the left-hand side of the equation:

Substitute $y = \sqrt{t}$ into the differential equation

$$\frac{dy}{dt} = \tfrac{1}{2}t^{-1/2} = \frac{1}{2\sqrt{t}}.$$

Now look at the right-hand side:

$$\frac{1}{2y} = \frac{1}{2\sqrt{t}}.$$

The two sides of the equation are equal, so the substitution $y = \sqrt{t}$ makes the equation true.

The function $y = t^2$, however, makes the differential equation *false*. The left-hand side is

$$\frac{dy}{dt} = 2t,$$

but the right-hand side is

$$\frac{1}{2y} = \frac{1}{2t^2}.$$

Since $2t$ and $1/2t^2$ are different functions, the two sides are unequal and the equation is therefore false.

A *solution* makes the equation true

We say that $y = \sqrt{t}$ is a **solution** to this differential equation. The function $y = t^2$ is *not* a solution. To decide whether a particular function is a solution when the function is given by a formula, notice that we need to be able to differentiate the formula.

If we view differential equations simply as instructions for carrying out Euler's method, we need only the microscope equation $\Delta y \approx y' \cdot \Delta t$ in order to find functions. However, if we want to find functions that are solutions to differential equations from our new point of view, we first need to introduce the idea of the derivative and the rules for differentiating functions.

Just as an algebraic equation can have more than one solution, so can a differential equation. In fact, we can show that $y = \sqrt{t + C}$ is a solution to the differential equation

$$\frac{dy}{dt} = \frac{1}{2y},$$

for any value of the constant C. To evaluate the left-hand side dy/dt, we need the chain rule (chapter 3.6). Let's write

$$y = \sqrt{u} \qquad \text{where} \qquad u = t + C.$$

Then the left-hand side is the function

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = \frac{1}{2\sqrt{u}} \cdot 1 = \frac{1}{2\sqrt{t + C}}.$$

Since the right-hand side of the differential equation is

$$\frac{1}{2y} = \frac{1}{2\sqrt{t + C}},$$

the two sides are equal—no matter what value C happens to have. This proves that every function of the form $y = \sqrt{t + C}$ is a solution to the differential equation. Since there are infinitely many values that C can take, the differential equation has infinitely many different solutions!

A differential equation can have infinitely many solutions

If a differential equation arises in modelling a physical or biological process, the variables involved must also satisfy an initial condition. Suppose we add an initial condition to our differential equation:

$$\frac{dy}{dt} = \frac{1}{2y} \qquad \text{and} \qquad y(0) = 5.$$

Does *this* problem have a solution—that is, can we find a function $y(t)$ that is a solution to the differential equation and also satisfies the condition $y(0) = 5$?

Notice $y = \sqrt{t}$ is not a solution to this new problem. Although it satisfies the differential equation, it fails to satisfy the initial condition:

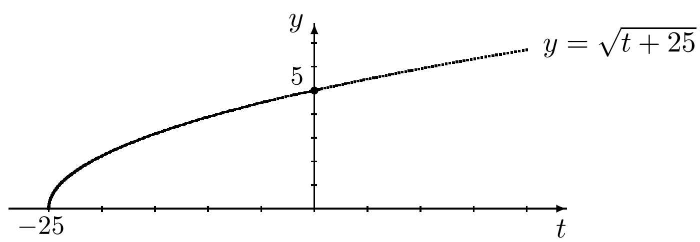
$$y(0) = \sqrt{0} = 0 \neq 5.$$

Perhaps one of the other solutions to the differential equation will work. When we evaluate the solution $y = \sqrt{t + C}$ at $t = 0$ we get

$$y(0) = \sqrt{0 + C} = \sqrt{C}.$$

An initial value problem has only one solution

We want this to equal 5, and it will if $C = 25$. Thus, $y = \sqrt{t+25}$ is a solution to the initial value problem. Furthermore, the only value of C which will make $y(0) = 5$ is $C = 25$, so the initial value problem has only one solution of the form $\sqrt{t+C}$. Here is the graph of this solution:



As always, you can use Euler's method to find the function determined by an initial value problem, and you can graph that function using the program SIRPLOT, for example. How will *that* graph compare with this one? In the exercises you can explore this question.

Checking solutions versus finding solutions. Notice that we have only *checked* whether a given function solves an initial value problem; we have not *constructed* a formula to solve the problem. By this point you are probably wondering where the given solutions came from.

It is helpful to continue exploring the parallels with solutions of algebraic equations. In the case of the equation $x^2 = x + 6$, there are, of course, methods to find solutions. One possibility is to rewrite $x^2 = x + 6$ in the form $x^2 - x - 6 = 0$. By factoring $x^2 - x - 6$ as

$$x^2 - x - 6 = (x - 3)(x + 2)$$

we can see that either $x - 3 = 0$ (so $x = 3$), or $x + 2 = 0$ (so $x = -2$). Another method is to use the **quadratic formula**

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

for the roots of the quadratic function $ax^2 + bx + c$. In our case, the quadratic formula yields

$$x = \frac{-(-1) \pm \sqrt{1 - 4 \cdot 1 \cdot (-6)}}{2 \cdot 1} = \frac{1 \pm \sqrt{1 + 24}}{2} = \frac{1 \pm 5}{2},$$

so again we find that x must be either 3 or -2.

Thus we have at least two different methods for finding solutions to this particular equation. The methods we use to solve an algebraic equation depend very much on the equation we face. For example, there is no way to find a solution to $\sin x = 2^x$ by factoring, or by using a "magic formula" like the quadratic formula. Nevertheless, there are methods that *do* work. In chapters 1 and 2 we dealt with similar problems by using a computer graphing utility that could zoom in on the point of intersection of two graphs. In chapter 6 we will introduce another tool, the Newton–Raphson method, for finding roots. These are both powerful methods, because they will work with nearly all algebraic equations. It is important to recognize that these numerical methods really do solve the problem, even though they do not give solutions in closed form the way the quadratic formula does.

There are special methods to solve particular equations, but other methods work generally

The situation is entirely analogous in dealing with differential equations. The methods we use to solve a differential equation depend on the equation we face. A course in differential equations provides methods for finding formulas that solve many different kinds of differential equations. The methods are like the quadratic formula in algebra, though—they give a complete solution, but they work only with differential equations that have a very specific form. This course will not attempt to survey the methods that find such formulas, although in the next sections we will see effective methods for dealing with some special subcases.

It is important to realize, though, that Euler's method is always there if we can't think of anything cleverer, and it really does provide solutions. In fact, most initial value problems have one, *and only one*, solution, and Euler's method can be used to determine this unique solution. If we can also find a formula for the solution, then it must be the same solution as that produced by Euler's method. In more advanced courses you will see a proof that this is true in general, provided some mild conditions are satisfied. To emphasize the importance of this idea, we give it a name:

Most initial value problem have one, and only one, solution. Euler's method will find it.

Existence and Uniqueness Principle
Under most conditions, an initial value problem has one and only one solution.

The existence and uniqueness principle is one of the most important mathematical results in the theory of differential equations.

We will continue to rely primarily on Euler's method, which generates solutions for nearly all differential equations.

However, there are clear benefits to having a formula for the solution to a differential equation, allowing us to investigate questions that we can't answer very well if we only have solutions given by Euler's method. In this section, we will look at some of those benefits.

World Population Growth

Two models

In the exercises in the last section, we looked at two different models that seek to describe how the world population will grow. One model assumed constant per capita growth—rate of change proportional to population size. The other assumed "supergrowth"—rate of change proportional to a *higher* power of the population size. Let's write P for the population size in the constant per capita growth model and Q for the population size in the supergrowth model. In both cases, the population is expressed in billions of persons and time is measured in years, with $t = 0$ in 1990. In this notation, the two models are

Models for world population growth

$$\text{constant per capita:} \quad \frac{dP}{dt} = .02\,P \qquad P(0) = 5;$$

$$\text{supergrowth:} \quad \frac{dQ}{dt} = .015\,Q^{1.2} \quad Q(0) = 5.$$

By using Euler's method, we discover that the two models predict fairly similar results over sixty years, although the supergrowth model lives up to its name by predicting larger populations than the constant per capita growth model as time passes:

t	P	Q
− 10	4.09	4.08
0	5.00	5.00
10	6.11	6.18
50	13.59	15.94

These estimates are accurate to one decimal place, and that level of accuracy was obtained with the step size $\Delta t = .1$.

However, the predictions made by the models differ widely over longer time spans. If we use Euler's method to estimate the populations after 240 years, we get

Δt	$P(240)$	$Q(240)$
.1	6.046×10^2	1.979×10^{10}
.01	6.073×10^2	2.573×10^{11}
.001	6.075×10^2	3.825×10^{11}

As the step size decreases from 0.1 to 0.01 to 0.001, the estimates of the constant per capita growth model $P(240)$ behave as we have come to expect: already three digits have stabilized. But in the estimates of the supergrowth model, not even one digit of $Q(240)$ has stabilized.

In this section we will see that there are actually *formulas* for the functions $P(t)$ and $Q(t)$. These formulas will illuminate the reason behind the differences in speed of stabilization in the estimates.

A formula for the supergrowth model

Without asking how the following formula might have been derived, let's check that it is indeed a solution to the supergrowth initial value problem.

$$Q(t) = \left(\frac{1}{\sqrt[5]{5}} - .003\,t \right)^{-5}$$

First of all, the formula satisfies the initial condition $Q(0) = 5$:

Checking the initial condition

$$Q(0) = \left(\frac{1}{\sqrt[5]{5}} \right)^{-5} = (\sqrt[5]{5})^5 = 5.$$

To check that it also satisfies the differential equation, we must evaluate the two sides of the differential equation

Checking the differential equation

$$\frac{dQ}{dt} = .015\,Q^{1.2}.$$

Let's begin by evaluating the left-hand side. To differentiate $Q(t)$, we will write Q as a *chain* of functions:

Left-hand side

$$Q = u^{-5} \qquad \text{where} \qquad u = \frac{1}{\sqrt[5]{5}} - .003\,t.$$

Since $Q = u^{-5}$, $dQ/du = (-5)u^{-6}$. Also, since u is just a linear function of t in which the multiplier is $-.003$, we have $du/dt = -.003$. Consequently,

$$\begin{aligned} \frac{dQ}{dt} &= \frac{dQ}{du} \cdot \frac{du}{dt} \\ &= (-5)\, u^{-6} \cdot (-.003) \\ &= .015\, u^{-6} \end{aligned}$$

Ordinarily, we would "finish the job" by substituting for u its formula in terms of t. However, in this case it is clearer to just leave the left-hand side in this form.

Right-hand side

To evaluate the right-hand side of the differential equation (which is the expression $.015\, Q^{1.2}$), we would expect to substitute for Q its formula in terms of t. But since in evaluating the left-hand side, we expressed things in terms of u, let's do the same thing here. Since $Q = u^{-5}$,

$$Q^{1.2} = Q^{6/5} = (u^{-5})^{6/5} = u^{-5 \cdot 6/5} = u^{-6}.$$

Therefore, the right-hand side is equal to $.015\, u^{-6}$. But so is the left-hand side, so $Q(t)$ is indeed a solution to the differential equation

$$\frac{dQ}{dt} = .015\, Q^{1.2}.$$

Notice two things about this result. First, when we work with formulas we have greater need for algebra to manipulate them. For example, we needed one of the laws of exponents, $(a^b)^c = a^{bc}$, to evaluate the right-hand side. Second, we found it simpler to express Q in terms of the intermediate variable u, instead of the original input variable t. In another computation, it might be preferable to replace u by its formula in terms of t. You need to choose your algebraic strategy to fit the circumstances.

Behavior of the supergrowth solution

It was convenient to use a negative exponent in the formula for $Q(t)$ when we wanted to differentiate Q. However, to understand what the formula tells us about supergrowth, it will be more useful to write Q as

$$Q(t) = \left(\frac{1}{1/\sqrt[5]{5} - .003\, t}\right)^5.$$

This way makes it clear that Q is a fraction, and we can see its denominator. In particular, this fraction is not defined when the denominator is zero—that is, when

$$\frac{1}{\sqrt[5]{5}} - .003\,t = 0, \quad \text{or} \quad t = \frac{1/\sqrt[5]{5}}{.003} = 241.6\ldots \quad \text{years after 1990.}$$

Consider what happens, though, as t approaches this special value $241.6\ldots$. The denominator isn't *yet* zero, but it is *approaching* zero, so the fraction Q is becoming infinite. This means that the supergrowth model predicts the world population will become infinite in about 240 years!

Q "blows up" as $t \to 241.6\ldots$

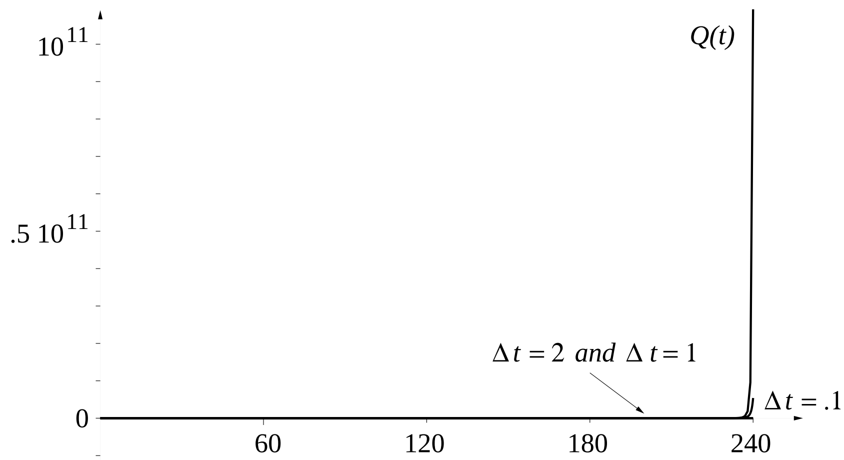
Let's see what the predicted population size is when $t = 240$ (which is the year 2230 A.D.), shortly before Q becomes infinite. We have

$$Q(240) = \left(\frac{1}{\sqrt[5]{5}} - (.003)(240)\right)^{-5} \approx 4.0088 \times 10^{11}.$$

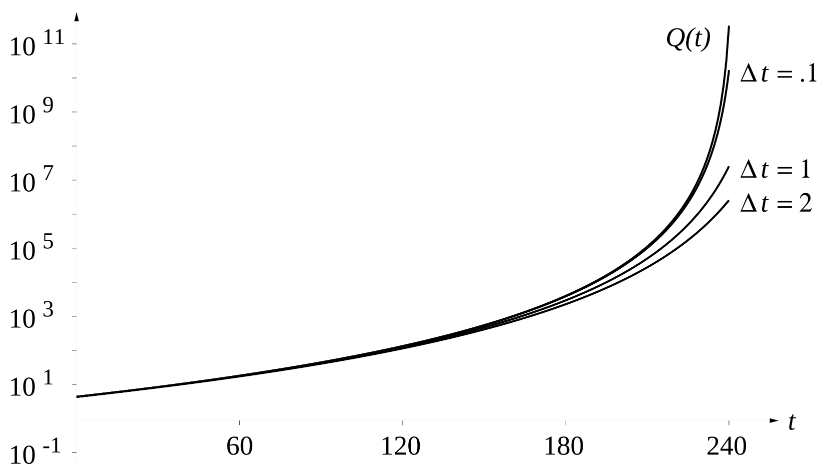
Remember that Q expresses the population in *billions* of people, so the supergrowth model predicts about 4×10^{20} people (i.e. 400 quintillion!) in the year 2230. Refer back to our estimates of $Q(240)$ using Euler's method (page 207). Although not even one digit of the estimates had stabilized, at least the final one (with a step size of .001) had reached the right power of ten. In fact, estimates made with still smaller step sizes do eventually approach the value given by the formula for Q:

step size	$Q(240)$
.1	0.1979×10^{11}
.01	2.5727×10^{11}
.001	2.8249×10^{11}
.0001	3.9999×10^{11}
.00001	4.0069×10^{11}

Let's look at the relationship between the Euler approximations of Q and the formula for Q graphically. Here are graphs produced by a modification of the program SEQUENCE.



The range of values of Q for $0 \leq t \leq 240$ is so immense that these graphs are useless. In a case like this, it is helpful to rescale the vertical axis so that the space between one power of 10 and the next is the same. In other words, instead of seeing 1, 2, 3, ..., we see 10^1, 10^2, 10^3, ... This is called a **logarithmic** scale. Here's what happens to the graphs if we put a logarithmic scale on the vertical axis:



Euler approximations and the formula for Q

The second graph makes it clearer that the Euler approximations do indeed approach the graph of the function given by our formula, but they approach more and more slowly, the closer t approaches 241.6....

Graphs are made with logarithmic scales particularly when the numbers being plotted cover a wide range of values. When just one axis is logarithmic, the result is called a *semi-log* plot; when both axes are logarithmic, the result is called a *log–log* plot.

Since $Q(t)$ becomes infinite when $t = 241.6\ldots$, we must conclude that the solution to the original initial value problem is meaningful only for $t < 241.6\ldots$. Of course, the formula for Q works quite well when $t > 241.6\ldots$. It just has no meaning as the size of a population. For instance, when $t = 260$ we get

The initial value problem has a solution only for $t < 241.6\ldots$

$$Q(260) = \left(\frac{1}{\sqrt[5]{5}} - (.003)(260)\right)^{-5} \approx (-.05522)^{-5} \approx -1.948 \times 10^6.$$

In other words, the function determined by the initial value problem is defined only on intervals around $t = 0$ that do not contain $t = 241.6\ldots$.

The formula for $Q'(t)$ is informative too:

$$Q'(t) = .015\left(\frac{1}{1/\sqrt[5]{5} - .003t}\right)^6$$

Since Q' has the same denominator as Q, it becomes infinite the same way Q does: $Q'(t) \to \infty$ as $t \to 241.6\ldots$. Because Euler's method uses the microscope equation $\Delta Q \approx Q' \cdot \Delta t$ to predict the next value of Q, we can now understand why the estimates of $Q(240)$ were so slow to stabilize: as $Q' \to \infty$, $\Delta Q \to \infty$, too.

Q' blows up as $t \to 241.6\ldots$

A formula for constant per capita growth

The constant per capita growth model for the world population that we are considering is

$$\frac{dP}{dt} = .02\,P \qquad P(0) = 5.$$

This differential equation has a very simple form; if $P(t)$ is a solution, then the derivative of P is just a multiple of P. We have already seen in chapter 3 that *exponential* functions behave this way (exercises 5–7 in section 3). For example, if $P(t) = 2^t$, then

$$\frac{dP}{dt} = .69 \cdot 2^t = .69\,P.$$

Of course, the multiplier that appears here is .69, not .02, so $P(t) = 2^t$ is not a solution to our problem.

Exponential functions satisfy $dy/dt = ky$

However, the multiplier that appears when we differentiate an exponential function changes when we change the base. That is, if $P(t) = b^t$, then $P'(t) = k_b \cdot b^t$, where k_b depends on b. Here is a sample of values of k_b for different bases b:

b	k_b
.5	$-.693147\ldots$
2	$.693147\ldots$
3	$1.098612\ldots$
10	$2.302585\ldots$

Notice that k_b gets larger as b does. Since .02 lies between $-.693147$ and $+.693147$, the table suggests that the value of b we want lies somewhere between .5 and 2.

We can say even more about the multiplier. Since $P'(t) = k_b \cdot P(t)$ and $P(t) = b^t$, we find

$$P'(0) = k_b \cdot P(0) = k_b \cdot b^0 = k_b \cdot 1 = k_b.$$

In other words, *k_b is the slope of the graph of $P(t) = b^t$ at the origin.*

The correct exponential function has slope .02 at the origin

Thus, we will be able to solve the differential equation $dP/dt = .02\,P$ if we can find an exponential function $P(t) = b^t$ whose graph has slope .02 at the origin. This is a problem that we can solve with a computer microscope. Pick a value of b and graph b^t. Zoom in on the graph at the origin and measure the slope. If the slope is more than .02, choose a smaller value for b; if the slope is less than .02, choose a larger value for b. Repeat this process, narrowing down the possibilities for b until the slope is as close to .02 as you wish. Eventually, we get

$$P(t) = (1.0202)^t.$$

You should check that $P'(0) = .02000\ldots$; see the exercises.

Thus $P(t) = (1.0202)^t$ solves the differential equation $P' = .02\,P$. However, it does not satisfy the initial condition, because

$$P(0) = (1.0202)^0 = 1 \neq 5.$$

This is easy to fix; $P(t) = 5 \cdot (1.0202)^t$ satisfies both conditions. More generally, $P(t) = C\,(1.0202)^t$ satisfies the initial condition $P(0) = C$ as well as the differential equation $P'(t) = .02\,P(t)$. To check the initial condition, we compute

$$P(0) = C\,(1.0202)^0 = C \cdot 1 = C.$$

The differential equation is also satisfied:

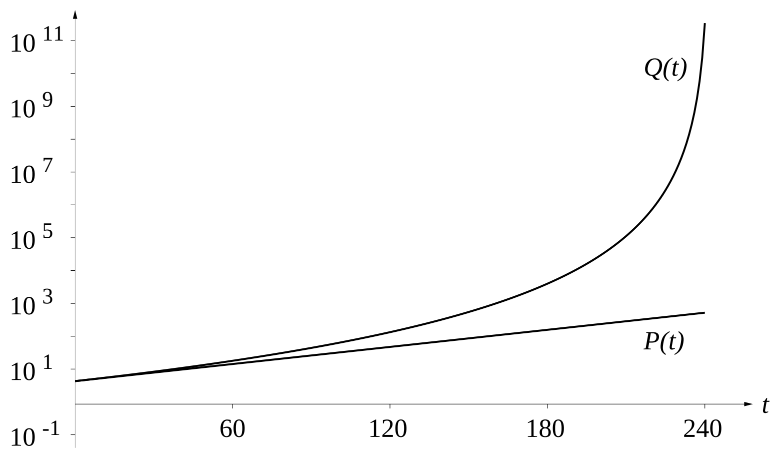
$$P'(t) = (C\,(1.0202)^t)' = C \cdot ((1.0202)^t)' = C \cdot (.02\,(1.0202)^t) = .02\,P(t).$$

So we have verified that the solution to our problem is

$$P(t) = 5\,(1.0202)^t.$$

The formula for P

Because exponential functions are involved, constant per capita growth is commonly called **exponential growth**. In the figure below we compare exponential growth $P(t)$ to "supergrowth" $Q(t)$. The two graphs agree quite well when $t < 50$. Notice that population is plotted on a logarithmic scale (a *semi–log* plot). This makes the graph of P a straight line!



The graphs of $P(t)$ and $Q(t)$

Differential Equations Involving Parameters

How do parameters affect solutions?

The S-I-R model contained two parameters—the transmission and recovery coefficients a and b. When we used Euler's method to analyze S, I, and R, we were working numerically. To do the computations, we had to give the parameters definite numerical values. That made it more difficult to deal with our questions about the effects of *changing* the parameters. As a result, we took other approaches to explore those questions. For example, we used algebra to see that there was a threshold for the spread of the disease: if there were fewer than b/a people in the susceptible population, the infection would fade away.

This is the situation generally. Euler's method can be used to produce solutions to a very broad range of initial value problems. However, if the model includes parameters, then we usually want to know how the solutions are affected when the parameters change. Euler's method is a rather clumsy tool for investigating this question. Other methods—ones that don't require the values of the parameters to be fixed—work better. One possibility is to start with a formula.

Supergrowth parameters

The supergrowth problem illustrates both how questions about parameters can arise and how useful a formula for the solution can be to answer the questions. One of the most striking features of the supergrowth model is that it predicts the population becomes infinite in 241.6... years. That prediction was based on an initial population of 5 billion and a growth constant of .015. Suppose those values turn out to be incorrect, and we need to start with different values. Will that change the prediction? If so, how?

We should treat the initial population and the growth constant as parameters—that is, as quantities that *can* vary, although they will have fixed values in any specific situation that we consider. Suppose we let A denote the size of the initial population, and k the growth constant. If we incorporate these parameters into the supergrowth model, the initial value problem takes this form:

$$\frac{dQ}{dt} = k\,Q^{1.2} \qquad Q(0) = A$$

Here is the formula for a function that solves this problem:

The supergrowth solution with parameters

$$Q(t) = \left(\frac{1}{\sqrt[5]{A}} - .2kt\right)^{-5}$$

Notice that, when $A = 5$ and $k = .015$, this formula reduces to the one we considered earlier.

Checking the formula

Let's check that the formula does indeed solve the initial value problem. First, the initial condition:

$$Q(0) = \left(\frac{1}{\sqrt[5]{A}} - .2k\cdot 0\right)^{-5} = \left(\frac{1}{\sqrt[5]{A}}\right)^{-5} = (\sqrt[5]{A})^5 = A.$$

Next, the differential equation. To differentiate $Q(t)$ we introduce the chain

$$Q = u^{-5} \qquad \text{where} \qquad u = \frac{1}{\sqrt[5]{A}} - .2kt.$$

We see that $dQ/du = -5\,u^{-6}$. Since u is a linear function of t in which the multiplier is $-.2k$, we also have $du/dt = -.2k$. Thus, by the chain rule,

$$\frac{dQ}{dt} = \frac{dQ}{du} \cdot \frac{du}{dt} = -5\,u^{-6} \cdot (-.2k) = k\,u^{-6}.$$

That is the left-hand side of the differential equation. To evaluate the right-hand side, we use the fact that $Q = u^{-5}$. Thus

$$k\,Q^{1.2} = k\,Q^{6/5} = k(u^{-5})^{6/5} = k\,u^{-5 \cdot 6/5} = k\,u^{-6}.$$

Since both sides equal $k\,u^{-6}$, they equal each other, proving that $Q(t)$ is a solution to the differential equation.

Next, we ask when the population becomes infinite. Exactly as before, this will happen when the denominator of the formula for $Q(t)$ becomes zero:

The time to infinity

$$\frac{1}{\sqrt[5]{A}} - .2\,kt = 0, \qquad \text{or} \qquad t = \frac{1}{.2\,k\sqrt[5]{A}}.$$

Here, in fact, is a *formula* that tells us how each of the parameters A and k affects the time it takes for the population to become infinite.

Let's use τ (the Greek letter "tau") to denote the "time to infinity." For example, if we double the initial population, so $A = 10$ billion people, while keeping the original growth constant $k = .015$, then the time to infinity is

$$\tau = \frac{1}{.003 \times \sqrt[5]{10}} \approx 210.3 \text{ years.}$$

By contrast, if we double the growth rate, to $k = .030$, while keeping the original $A = 5$, then the time to infinity is only

$$\tau = \frac{1}{.006 \times \sqrt[5]{5}} \approx 120.8 \text{ years}$$

Conclusion: doubling the growth rate has a much greater impact than doubling the initial population.

Uncertainty in the size of τ

For any specific growth rate and initial population, we can always calculate the time to infinity. But we can actually do more; the formula for τ allows us to do an *error analysis* along the patterns described in chapter 3, section 4. For example, suppose we are uncertain of our value of the growth rate k; there may be an error of size Δk. How uncertain does that make us about the calculated value of τ? Likewise, if the current world population A is known only with an error of ΔA, how uncertain does *that* make τ? Also, how are the *relative* errors related? Let's do this analysis, assuming that $k = .015$ and $A = 5$.

How an error in k propagates

Our tool is the error propagation equation—which is the microscope equation. If we deal with k first, then

$$\Delta\tau \approx \frac{\partial\tau}{\partial k} \cdot \Delta k.$$

We have used partial derivatives because τ is a function of *two* variables, A as well as k. If we write

$$\tau = \frac{1}{.2\sqrt[5]{A}}\, k^{-1},$$

then the differentiation rules yield

$$\frac{\partial\tau}{\partial k} = -1 \cdot \frac{1}{.2\sqrt[5]{A}}\, k^{-2} = \frac{-1}{.2\sqrt[5]{5}} \times (.015)^{-2} \approx -16106.$$

Thus $\Delta\tau \approx -16106 \cdot \Delta k$. For example, if the uncertainty in the value of $k = .015$ is $\Delta k = \pm.001$, then the uncertainty in τ is about ∓ 16 years.

How an error in A propagates

To determine how an error in A propagates to τ, we first write

$$\tau = \frac{1}{.2\,k} A^{-1/5}.$$

Then

$$\frac{\partial\tau}{\partial A} = -\frac{1}{5} \cdot \frac{1}{.2\,k} A^{-6/5} = \frac{-1}{5 \times .2 \times .015} \times 5^{-6/5} \approx -9.7.$$

The error propagation equation is thus $\Delta\tau \approx -9.7 \cdot \Delta A$. If the uncertainty in the world population is about 100 million persons, so $\Delta A = \pm.1$, then the uncertainty in τ is less than 1 year.

Relative errors

To complete the analysis, let's compare relative errors. This involves a lot of algebra. To see how an error in k propagates, we have

$$\Delta\tau \approx -\frac{\Delta k}{.2\,k^2\sqrt[5]{A}} \qquad \text{and} \qquad \tau = \frac{1}{.2\,k\sqrt[5]{A}}.$$

We can therefore compute that a given relative error in k propagates as

$$\frac{\Delta\tau}{\tau} \approx -\frac{\Delta k}{.2\,k^2\sqrt[5]{A}} \cdot \frac{.2\,k\sqrt[5]{A}}{1} = -\frac{\Delta k}{k}.$$

Thus, a 1% error in k leads to a 1% error in τ, although the sign is reversed.

To analyze how a given relative error in A propagates, we start with

$$\Delta\tau \approx -\frac{1}{5} \cdot \frac{\Delta A}{.2\,kA^{6/5}}.$$

Then

$$\frac{\Delta\tau}{\tau} \approx -\frac{1}{5} \cdot \frac{\Delta A}{.2\,kA^{6/5}} \cdot \frac{.2\,k\sqrt[5]{A}}{1} = -\frac{1}{5} \cdot \frac{\Delta A}{A}.$$

This says that it takes a 5% error in A to produce a 1% error in τ. Consequently, the time to infinity τ *is 5 times more sensitive to errors in k than to errors in A.*

How sensitive τ is to errors in k and A

The exercises in this section will give you an opportunity to check that a particular formula is a solution to an initial value problem that arises in a variety of contexts. Later in this chapter, we will make a modest beginning on the much harder task of *finding* solutions given by formulas for special initial value problems. There are more sophisticated methods for finding formulas, when the formulas exist, and they provide powerful tools for some important problems, especially in physics. However, most initial value problems we encounter cannot be solved by formulas. This is particularly true when two or more variables are needed to describe the process being modelled. The tool of widest applicability is Euler's method. This isn't so different from the situation in algebra, where exact solutions given by formulas (e.g. the quadratic formula) are also relatively rare, and numerical methods play an important role. (Chapter 5.5, presents the Newton–Raphson method for solving algebraic equations by successive approximation.) In most cases that will interest us, there are simply no formulas to be found—the limitation lies in the mathematics, not the mathematicians.

Special methods give formulas; general methods are numerical

Exercises

In exercises 1–4, verify that the given formula is a *solution* to the initial value problem.

1. **Powers of y.**

a) $y' = y^2$, $y(0) = 5$: $\quad y(t) = 1/(\frac{1}{5} - t)$

b) $y' = y^3$, $y(0) = 5$: $\quad y(t) = 1/\sqrt{\frac{1}{25} - 2t}$

c) $y' = y^4$, $y(0) = 5$: $\quad y(t) = 1/\sqrt[3]{\frac{1}{125} - 3t}$

d) Write a general formula for the solution of the initial value problem $y' = y^n$, $y(0) = 5$, for any integer $n > 1$.

e) Write a general formula for the solution of the initial value problem $y' = y^n$, $y(0) = C$, for any integer $n > 1$ and any constant $C \geq 0$.

2. **Powers of t.**

a) $y' = t^2$, $y(0) = 5$: $\quad y(t) = \frac{1}{3}t^3 + 5$

b) $y' = t^3$, $y(0) = 5$: $\quad y(t) = \frac{1}{4}t^4 + 5$

c) $y' = t^4$, $y(0) = 5$: $\quad y(t) = \frac{1}{5}t^5 + 5$

d) Write a general formula for the solution of the initial value problem $y' = t^n$, $y(0) = 5$ for any integer $n > 1$.

e) Write a general formula for the solution of the initial value problem $y' = t^n$, $y(0) = C$ for any integer $n > 1$ and any constant C.

3. **Sines and cosines.**

a) $x' = -y$, $y' = x$, $x(0) = 1$, $y(0) = 0$: $\quad x(t) = \cos t$, $y(t) = \sin t$

b) $x' = -y$, $y' = x$, $x(0) = 0$, $y(0) = 1$: $\quad x(t) = \cos(t + \pi/2)$, $y(t) = \sin(t + \pi/2)$

4. **Exponential functions.**

a) $y' = 2.3\,y$, $y(0) = 5$: $\quad y(t) = 5 \cdot 10^t$

b) $y' = 2.3\,y$, $y(0) = C$: $\quad y(t) = C \cdot 10^t$

c) $y' = -2.3\,y$, $y(0) = 5$: $\quad y(t) = 5 \cdot 10^{-t}$

d) $y' = 4.6\,ty$, $y(0) = 5$: $\quad y(t) = 5 \cdot 10^{t^2}$

5. **Initial Conditions.**

a) Choose C so that $y(t) = \sqrt{t + C}$ is a solution to the initial value problem

$$y' = \frac{1}{2y} \quad y(3) = 17.$$

b) Choose C so that $y(t) = -1/(t + C)$ is a solution to the initial value problem

$$y' = y^2 \quad y(0) = -5.$$

c) Choose C so that $y(t) = -1/(t + C)$ is a solution to the initial value problem

$$y' = y^2 \quad y(2) = 3.$$

World population growth with parameters

6. a) Using a graphing utility or a calculator, show that the derivative of $P(t) = (1.0202)^t$ at the origin is approximately .02: $P'(0) \approx .02$. Since quick convergence is desirable, use

$$\frac{\Delta P}{\Delta t} = \frac{P(0+h) - P(0-h)}{2h} = \frac{(1.0202)^h - (1.02020)^{-h}}{2h}$$

b) By using more decimal places to get higher precision, show that $P(t) = (1.0202013)^t$ satisfies $P'(0) = .02$ even more exactly.

7. a) Show that the function $y = 2^{t/.69}$ satisfies the differential equation $dy/dt = y$. Use the chain rule: $y = 2^u$, $u = t/.69$. (Recall that $k_2 = .69\ldots$.)
b) Show that the function $y = 2^{kt/.69}$ satisfies the differential equation $dy/dt = k\,y$.
c) Show that the function $P(t) = A \cdot 2^{kt/.69}$ is a solution to the initial value problem

$$\frac{dP}{dt} = k\,P \qquad P(0) = A.$$

Note that this describes a population that grows at the constant per capita rate k from an initial size of A.

8. a) Show that the function $y = 10^{t/2.3}$ satisfies the differential equation $dy/dt = y$. Use the chain rule: $y = 10^u$, $u = t/2.3$. (Recall that $k_{10} = 2.3\ldots$.)
b) Show that the function $y = 10^{kt/2.3}$ satisfies the differential equation $dy/dt = k\,y$.
c) Show that the function $P(t) = A \cdot 10^{kt/2.3}$ is a solution to the initial value problem

$$\frac{dP}{dt} = k\,P \qquad P(0) = A.$$

This formula provides an alternative way to describe a population that grows at the constant per capita rate k from an initial size of A.

9. a) The formula $P(t) = 5 \cdot 2^{k\,t/.69}$ describes how an initial population of 5 billion will grow at a constant per capita rate of k persons per year per person. Use this formula to determine how many years t it will take for the population to double, to 10 billion persons.

b) Suppose the initial population is A billion, instead of 5 billion. What is the doubling time then?

c) Suppose the initial population is 5 billion, and the per capita growth rate is .02, but that value is certain only with an error of Δk. How much uncertainty is there in the doubling time that you found in part (a)?

Newton's law of cooling

There are formulas that describe how a body cools, or heats up, to match the temperature of its surroundings. See the exercises on Newton's law of cooling in section 1. Consider first the model

$$\frac{dT}{dt} = -.1(T - 20) \qquad T(0) = 90,$$

introduced on page 197 to describe how a cup of coffee cools.

10. Show that the function $y = 2^{-.1\,t/.69}$ is a solution to the differential equation $dy/dt = -.1\,y$. (Use the chain rule: $y = 2^u$, $u = -.1\,t/.69$.)

11. a) Show that the function

$$T = 70 \cdot 2^{-.1\,t/.69} + 20$$

is a solution to the initial value problem $dT/dt = -.1(T - 20)$, $T(0) = 90$. This is the temperature T of a cup of coffee, initially at 90°C, after t minutes have passed in a room whose temperature is 20°C.

b) Use the formula in part (a) to find the temperature of the coffee after 20 minutes. Compare this result with the value you found in exercise 12 (b), page 198.

c) Use the formula in part (a) to determine how many minutes it takes for the coffee to cool to 30°C. In doing the calculations you will find it helpful to know that $1/7 = 2^{-2.8}$. Compare this result with the value you found in exercise 11 (c), page 198.

12. a) A cold drink is initially at $Q = 36$°F when the air temperature is 90°F. If the temperature changes according to the differential equation

$$\frac{dQ}{dt} = -.2(Q - 90)°\text{F per minute},$$

show that the function $Q(t) = 90 - 54 \cdot 2^{-.2\,t/.69}$ describes the temperature after t minutes.

b) Use the formula to find the temperature of the drink after 5 minutes and after 10 minutes. Compare your results with the values you found in exercise 12, page 198.

13. Find a formula for a function that solves the initial value problem

$$\frac{dQ}{dt} = -k(Q - A) \qquad Q(0) = B.$$

A leaking tank

The rate at which water leaks from a small hole at the bottom of a tank is proportional to the square root of the height of the water surface above the bottom of the tank. Consider a cylindrical tank that is 10 feet tall and stands on one of its circular ends, which is 3 feet in diameter. Suppose the tank is currently half full, and is leaking at a rate of 2 cubic feet per hour.

14. a) Let $V(t)$ be the volume of water in the tank t hours from now. Explain why the leakage rate can be written as the differential equation

$$V'(t) = -k\sqrt{V(t)},$$

for some positive constant k. (The issue to deal with is this: why is it permissible to use the square root of the *volume* here, when the rate is known to depend on the square root of the *height*?)

b) Determine the value of k. [Answer: $k \approx .3364$; you need to explain *why* this is the value.]

15. a) How much water leaks out of the tank in 12 hours; in 24 hours? Use Euler's method, and compute successive approximations until your results stabilize.

b) How many hours does it take for the tank to empty?

16. a) Use the differentiation rules to show that any function of the form

$$V(t) = \begin{cases} \dfrac{k^2}{4}(C - t)^2 & \text{if } 0 \leq t \leq C \\ 0 & \text{if } C < t \end{cases}$$

satisfies the differential equation.

b) For the situation we are considering, what is the value of C? According to this solution, how long does it take for the tank to empty? Compare this result with your answer using Euler's method.

c) Sketch the graph of $V(t)$ for $0 \leq t \leq 2C$, taking particular care to display the value $V(0)$ in terms of k and C.

Motion

Newton created the calculus to study the motion of the planets. He said that all motion obeys certain basic laws. One law says that the velocity of an object changes only if a force acts on the body. Furthermore, the *rate* at which the velocity changes is proportional to the force. By knowing the forces that act on a body we can construct—and then solve—a differential equation for the velocity.

Falling bodies—with gravity. A body falling through the air starts up slowly but picks up speed as it falls. Its velocity is thus changing, so there must be a force acting. We call the force that pulls objects to the earth **gravity**. At the surface of the earth, the rate of change of velocity caused by gravity is essentially the same for all objects.

Suppose an object is x meters above the surface of the earth after t seconds have passed. Then, by definition, its velocity is

$$v = \frac{dx}{dt} \text{ meters/second.}$$

According to Newton's laws of motion, the force of gravity causes the velocity to change, and we can write

$$\frac{dv}{dt} = -g.$$

Here g is a constant whose numerical value is about 9.8 meters/second per second. Since x and v are positive when measured *upwards*, but gravity acts *downwards*, a minus sign is needed in the equation for dv/dt. (The derivative of velocity is commonly called **acceleration**, and g is called the **acceleration due to gravity**.)

17. Verify that $v(t) = -g\,t + v_0$ is a solution to the differential equation $dv/dt = -g$ with initial velocity v_0.

18. Since $dx/dt = v$, and since $v(t) = -g\,t + v_0$, the *position* x of the body satisfies the differential equation

$$\frac{dx}{dt} = -g\,t + v_0 \text{ meters/second.}$$

Find a formula for $x(t)$ that solves this differential equation. This function describes how a body moves under the force of gravity.

19. Suppose the initial position of the body is x_0, so that the position x is a solution to the initial value problem

$$\frac{dx}{dt} = -g\,t + v_0 \qquad x(0) = x_0 \text{ meters.}$$

Find a formula for $x(t)$.

20. a) Suppose a body is held motionless 200 meters above the ground, and then released. What values do x_0 and v_0 have? What is the formula for the motion of this body as it falls to the ground?

b) How far has the body fallen in 1 second? In 2 seconds?

c) How long does it take for the body to reach the ground?

Falling bodies—with gravity and air resistance. As a body falls, air pushes against it. Air resistance is thus another force acting on a falling body. Since air resistance is slight when an object moves slowly but increases as the object speeds up, the simplest model we can make is that the force of air resistance is proportional to the velocity: force $= -bv$ (reality turns out to be somewhat more complicated than this). The multiplier b is positive, and the minus sign tells us that the direction of the force is always opposite the velocity. The forces of gravity and air resistance combine to change the velocity:

$$\frac{dv}{dt} = -g - b\,v \text{ meters/second per second.}$$

21. Show that

$$v(t) = \frac{g}{b}\left(2^{-b\,t/.69} - 1\right) \text{ meters/second}$$

is a solution to this differential equation that also satisfies the initial condition $v(0) = 0$ meters/second.

22. a) Show that the position $x(t)$ of a body that falls against air resistance from an initial height of x_0 meters is given by the formula

$$x(t) = x_0 - \frac{g}{b}\,t - \frac{g}{b^2}\left(2^{-b\,t/.69} - 1\right) \text{ meters.}$$

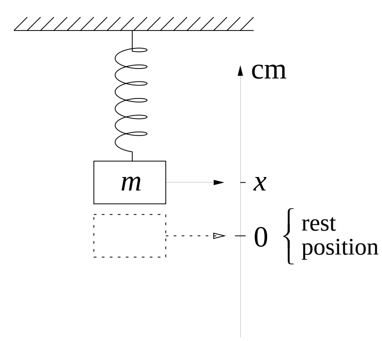
b) Suppose the coefficient of air resistance is $b = .2$ per second. If a body is held motionless 200 meters above the ground, and then released, how far will it fall in 1 second? In 2 seconds? Compare these values with those you obtained assuming there was no air resistance.

c) How long does it take for the body to reach the ground? (Use a computer graphing package to get this answer.) Compare this value with the one you obtained assuming these was no air resistance. How much does air resistance add to the time?

23. a) According to the equation $dv/dt = -g - b\,v$, there is a velocity v_T at which the force of air resistance exactly balances the force of gravity, and the velocity doesn't change. What is v_T, expressed as a function of g and b. Note: v_T is called the **terminal velocity** of the body. Once the body reaches its terminal velocity, it continues to fall at that velocity.

b) What is the terminal velocity of the body in the previous exercise?

The oscillating spring. Springs can smooth out life's little irregularities (as in the suspension of a car) or amplify and measure them (as in earthquake detection devices). Suppose a spring that hangs from a hook has a weight at its end. Let the weight come to rest. Then, when the weight moves, let x denote the position of the weight above the rest position. (If x is negative, this means the weight is below the rest position.) If you pull down on the weight, the spring pulls it back up. If you push up on the weight, the spring (and gravity) push it back down. This push is the **spring force**.



The simplest assumption is that the spring force is proportional to the amount x that the spring has been stretched: force $= -c^2x$. The constant c^2 is customarily written as a square to emphasize that it is positive. The minus sign tells us the force pushes down if $x > 0$ (so the weight is above the rest position), but it pushes up if $x < 0$.

If $v = dx/dt$ is the velocity of the weight, then Newton's law of motion says

$$\frac{dv}{dt} = -c^2 x.$$

Suppose we move the weight to the point $x = a$ on the scale, hold it motionless momentarily, and then release it at time $t = 0$. This determines the initial value problem

$$\begin{array}{ll} x' = v & x(0) = a \\ v' = -c^2 x & v(0) = 0. \end{array}$$

24. a) Show that

$$x(t) = a\cos(ct) \qquad v(t) = -ac\sin(ct)$$

is a solution to the initial value problem.

b) What range of values does x take on; that is, how far does the weight move from its rest position?

25. a) Use a graphing utility to compare the graphs of $y = \cos(x)$, $y = \cos(2x)$, $y = \cos(3x)$, and $y = \cos(.5x)$. Based on your observations, explain how the value of c affects the nature of the motion $x(t) = a\cos(ct)$ for a fixed value of a.

b) How long does it take the weight to complete one cycle (from $x = a$ back to $x = a$) when $c = 1$? The motion of the weight is said to be **periodic**, and the time it takes to complete one cycle is called its **period**.

c) What is the period of the motion when $c = 2$? When $c = 3$? Does the period depend on the initial position a?

d) Write a formula that expresses the period of the motion in terms of the parameters a and c.

26. a) The parameter c depends on two things: the mass m of the weight, and the stiffness k of the spring:

$$c = \sqrt{\frac{k}{m}}.$$

Write a formula that expresses the period of the motion of the weight in terms of m and k.

b) Suppose you double the weight on the spring. Does that *increase* or *decrease* the period of the motion? Does your answer agree with your intuitions?

c) Suppose you put the first weight on a second spring that is twice as stiff as the first (i.e., double the value of k). Does that *increase* or *decrease* the period of the motion? Does your answer agree with your intuitions?

d) When you calculate the period of the motion using your formula form part (a), suppose you know the actual value of the mass only to within 5%. How accurately do you know the period—as a percentage of the calculated value?

4.3 The Exponential Function

The Equation $y' = ky$

As we have seen, initial value problems *define* functions—as their solutions. They therefore provide us with a vast, if somewhat bewildering, array of new functions. Fortunately, a few differential equations—in fact, the very simplest—arise over and over again in an astonishing variety of contexts. The functions they define are among the most important in mathematics.

A simple and natural model of growth and decay

One of the simplest differential equations is $dy/dt = ky$, where k is a constant. It is also one of the most useful. We used it in chapter 1 to model the populations of Poland and Afghanistan, as well as bacterial growth and radioactive decay. In this chapter, it was our initial model of a rabbit population and one of our models of the world population. Later, we will use it to describe how money accrues interest in a bank and how radiation penetrates solid objects.

In this section we will look at the solutions to differential equations of this form from two different vantage points. On the one hand, we already have named functions which solve such equations—the exponential functions. On the other hand, the fact that Euler's method produces the same functions will allow us to prove properties of such functions and to compute their values effectively.

Exponential solutions—with different bases ...

In chapter 3 we established that the solutions to $dy/dt = ky$ are exponential functions. Specifically, for each base b, the exponential function $y = b^t$ was a solution to $dy/dt = k_b \cdot b^t = k_b \cdot y$, where k_b was the slope of the graph of $y = b^t$ at the origin. In this approach, if the constant k changes, we must change the base b so that $k_b = k$.

. . . and a fixed base

Exercise 7 in the previous section (page 219) opened up a new possibility: for the *fixed* base 2, the function

$$y = 2^{kt/.6931...}$$

was a solution to the differential equation $dy/dt = kt$, no matter what value k took. There was nothing special about the base 2, of course. In the next exercise, we saw that the functions

$$y = 10^{kt/2.3025...}$$

would serve equally well as solutions.

In fact, we can show that, for any base b, the functions

$$y = b^{kt/k_b}$$

are also solutions to $dy/dt = ky$. Construct the chain

$$y = b^u \qquad \text{where} \qquad u = kt/k_b.$$

Then $dy/du = k_b \cdot b^u = k_b \cdot y$, while $du/dt = k/k_b$. Thus, by the chain rule we have

$$\frac{dy}{dt} = \frac{dy}{du} \cdot \frac{du}{dt} = k_b\, y \cdot \frac{k}{k_b} = ky.$$

Advantages of a fixed base

If we express solutions to $dy/dt = ky$ by exponential functions with a *fixed* base, it is easy to alter the solution if the growth constant k changes. We just change the value of k in the exponent of b^{kt/k_b}. Let's see how this works when $b = 2$ and $b = 10$:

differential equation	solution base 2	solution base 10
$\dfrac{dy}{dt} = .16\, y$	$2^{.231\,t}$	$10^{.069\,t}$
$\dfrac{dy}{dt} = .18\, y$	$2^{.260\,t}$	$10^{.078\,t}$

Notice that the growth constant k gets "swallowed up" in the exponent of the solution when k has a specific numerical value. The number that appears in the exponent is k divided by $k_2 = .6931\ldots$ (when the base is 2) and by $k_{10} = 2.3025\ldots$ (when the base is 10).

The base e

The most vivid solution to $dy/dt = ky$ would use the base b for which $k_b = 1$ *exactly*. There is such a base, and it is always denoted e. (We will determine the value of e in a moment.) Since $k_e = 1$, k would stand out in the exponent:

differential equation	solution base e
$\dfrac{dy}{dt} = .16\,y$	$e^{.16\,t}$
$\dfrac{dy}{dt} = .18\,y$	$e^{.18\,t}$

The simplicity and clarity of this expression have led to the universal adoption of the base e for describing exponential growth and decay—that is, for describing solutions to $dy/dt = ky$.

The use of the symbol e to denote the base dates back to a paper that Euler wrote at age 21, entitled *Meditatio in experimenta explosione tormentorum nuper instituta* (Meditation upon recent experiments on the firing of cannons), where the symbol e was used sixteen times. It is now in universal use. The number e is, like π, one of the most important and ubiquitous in mathematics.

By design, $y = e^t$ is a solution to the differential equation $dy/dt = y$. In particular, the slope of the graph of $y = e^t$ at the origin is exactly 1. As we have just seen, the function $y = e^{kt}$ is a solution to the differential equations $dy/dt = ky$ whose growth constant is k. Finally:

The general initial value problem for exponential functions

$$\boxed{\begin{array}{c} y = C\cdot e^{kt} \text{ is the solution to the initial value problem} \\ \dfrac{dy}{dt} = ky \qquad y(0) = C. \end{array}}$$

We can check this quickly. The initial condition is satisfied because $e^0 = 1$, so $y(0) = C\cdot e^{k\cdot 0} = C\cdot 1 = C$. The differential equation is satisfied because

$$\left(C\cdot e^{kt}\right)' = C\cdot\left(e^{kt}\right)' = C\cdot k\,e^{kt} = k\,y.$$

We used the differentiation rule for a constant multiple of a function, and we used the fact that the derivative of e^{kt} was already established to be $k\,e^{kt}$.

The Number e

The number e is determined by the property that $k_e = 1$. Since this number is the slope of the graph of $y = e^t$ at the origin, one way to find e is with a computer microscope. Pick an approximation E for e and graph $y = f^t$. Zoom in the graph at the origin, and measure the slope. If the slope is more than 1, choose a smaller approximation; if the slope is less than 1, choose a larger value. Repeat this process, narrowing down the value of e until you know its value to as many decimal places as you wish.

Finding e with a computer microscope

We already know $E = 2$ is too small, because the slope of $y = 2^t$ at the origin is .69. Likewise, $E = 3$ is too large, because the slope of $y = 3^t$ at the origin is 1.09. Thus $2 < e < 3$, and is closer to 3 than to 2. At the next stage we learn that 2.7 is too small (slope = .9933) but 2.8 is too large (slope = 1.0296). Thus, at least we know $e = 2.7\ldots$. Several stages later we would learn $e = 2.71828\ldots$.

Under successive magnifications: $e = 2.71828\ldots$

While the method just described for finding the value of e works, it is somewhat ponderous. We can take a very different approach to finding the numerical value of e by using the fact that e is defined by an initial value problem. Here is the idea: e is the value of the function e^t when $t = 1$, and $y(t) = e^t$ is the solution to the initial value problem

Finding e by Euler's method

$$y' = y \qquad y(0) = 1.$$

We can then find $e = y(1)$ in the usual way by solving this initial value problem using Euler's method. Due to some convenient algebraic simplifications, this approach yields powerful insights about the nature of e.

Suppose we take n steps to go from $t = 0$ to $t = 1$. Then the step size is $\Delta t = 1/n$. The following table shows the calculations:

Finding $y(1)$ by Euler's method when $y' = y$ and $y(0) = 1$

t	y	$y' = y$	$\Delta y = y' \cdot \Delta t$
0	1	1	$1 \cdot 1/n$
$1/n$	$1 + 1/n$	$1 + 1/n$	$(1 + 1/n) \cdot 1/n$
$2/n$	$(1 + 1/n)^2$	$(1 + 1/n)^2$	$(1 + 1/n)^2 \cdot 1/n$
$3/n$	$(1 + 1/n)^3$	$(1 + 1/n)^3$	$(1 + 1/n)^3 \cdot 1/n$
$\vdots$	$\vdots$	$\vdots$	$\vdots$
n/n	$(1 + 1/n)^n$		

The entries in the y column need to be explained. The first two should be clear: $y(0)$ is the initial value 1, and $y(1/n) = y(0) + \Delta y = 1 + 1/n$. To get from any entry to the next we must do the following:

$$\begin{aligned} \text{new } y &= \text{current } y + \Delta y \\ &= \text{current } y + y' \cdot \Delta t \\ &= \text{current } y + \text{current } y \cdot \Delta t \\ &= \text{current } y \cdot (1 + \Delta t) \\ &= \text{current } y \cdot (1 + 1/n) \end{aligned}$$

The new y is the current y multiplied by $(1 + 1/n)$. Since the second y is itself $(1 + 1/n)$, the third will be $(1 + 1/n)^2$, the fourth will be $(1 + 1/n)^3$, and so on.

Euler's method with n steps therefore gives us the following estimate for $e = y(1) = y(n/n)$:

$$e \approx (1 + 1/n)^n$$

We can calculate these numbers on a computer. In the following table we give values of $(1 + 1/n)^n$ for increasing values of n. By the time $n = 2^{40}$ (about 10^{12}), eleven digits of e have stabilized.

n	$(1 + 1/n)^n$
2^0	2.0
2^4	2.638
2^8	2.712 992
2^{12}	2.717 950 081
2^{16}	2.718 261 089 905
2^{20}	2.718 280 532 282
2^{24}	2.718 281 747 448
2^{28}	2.718 281 823 396
2^{32}	2.718 281 828 142
2^{36}	2.718 281 828 439
2^{40}	2.718 281 828 458

Expressing e as a limit

The *true* value of e is the limit of these approximations as we take n arbitrarily large:

$$e = \lim_{n \to \infty} (1 + 1/n)^n = 2.71828182845904\ldots$$

We can generalize the preceding to get an expression for e^T for any value of T. In exactly the same way as we did above, divide the interval from 0 to T into n pieces, each of width $\Delta t = T/n$. Starting from $t = 0$ and $y(0) = 1$ and applying Euler's method, we find, using the same algebraic simplifications, that after n steps the value for t will be T and y will be $(1 + T/n)^n$. Since these approximations approach the true value of the function as $n \to \infty$, we have that

$$e^T = \lim_{n\to\infty} (1 + T/n)^n \text{ for any value of } T.$$

Differential Equations Define Functions

There is an important point underlying the operations we just performed having to do with the question of **computability**. While it may be appalling to think about doing it by hand, there is nothing conceptually difficult about evaluating an expression like $(1 + 1/1000)^{1000}$—all we need are ordinary addition, division, and multiplication. In fact, for any differential equation, Euler's method generates a solution using only ordinary arithmetic.

Euler's method uses only arithmetic

By contrast, think for a moment about the earlier method for evaluating e by evaluating expressions like $(2.718^{.0001} - 2.718^{-.0001})/.0002$ and seeing whether we get a value bigger than or less than 1. While a calculator or a computer will readily give us a value, how does it "know" what $2.718^{.0001}$ is? The fact is, it doesn't have a built-in exponentiator which lets it know immediately what the value of this expression is any more than we do. A computer—like humans—can essentially only add, subtract, and multiply. Any other operation has to be reduced to these operations somehow. Thus when we use a computer to evaluate something like $2.718^{.0001}$, we actual trigger a fairly elaborate program having little directly to do with raising numbers to powers which produces an approximation to the desired number. It turns out that if you use the $\boxed{x^y}$ key on your calculator to evaluate 2^5 it doesn't come up with the answer by multiplying 2 by itself 5 times, but uses this more complicated program.

There is often a large gap between naming and defining a function, and being able to compute values for it to four or five decimals. Think about the trigonometric functions for a moment. You have probably seen several definitions of the cosine function by now—as the ratio of the adjacent side over the hypotenuse of a right triangle, or as the x-coordinate of a point moving around a circle of radius 1. Yet neither of these definitions would help you calculate cos(2) to five decimals. It turns out that most methods

Defining a function is not the same as being able to evaluate it for evaluating functions are based on the way the derivatives of the functions behave. While we will have much more to say about this in chapter 10, Euler's method is a good first example of this.

Returning to exponents, think what would be involved in evaluating $2^{\sqrt{3}}$ using the pre-calculus concept of exponents. We might first get a series of rational approximations to $\sqrt{3} = 1.73205081\ldots$: $17/10 = 1.7$, $173/100 = 1.73$, $433/250 = 1.732$, and so on. We would then calculate

$$
\begin{aligned}
2^{17/10} &= (\sqrt[10]{2})^{17} \\
2^{173/100} &= (\sqrt[100]{2})^{173} \\
2^{433/250} &= (\sqrt[250]{2})^{433} \\
&\vdots
\end{aligned}
$$

Even evaluating the first of these approximations would involve finding the 10th root of 2 and raising it to the 17th power, which would not be easy. We would continue with these approximations until the desired number of digits remained fixed.

By contrast, evaluating $e^{\sqrt{3}}$ by Euler's method is very straightforward. As we saw above, it reduces to evaluating $(1 + 1.73205\ldots/n)^n$ for increasing values of n until the desired number of digits remains fixed. Moreover, this same process works just as well for any kind of exponent—positive or negative, rational or irrational.

In fact, *all* the properties of the exponential function follow from the fact that it is the solution to its initial value problem, so we could have made this the definition in the first place. This would have given us the benefit of coherence (not having to distinguish among different kinds of exponents) and direct computability. It would also directly reflect the primary reason the exponential function is important, namely that its rate of change is proportional to its value. Since the process of deducing the properties of a function from its defining equation will be important later on, and since it is a good exercise in some of the theoretical ideas we've been developing, let's see how this works.

Defining $\exp(t)$

We will assume *nothing* about the function $y = e^t$. Instead, we begin simply with the observation that each initial value problem defines a function—its solution. Therefore, the specific problem

$$y' = y \qquad y(0) = 1$$

defines a function; we call it $y = \exp(t)$. At the outset, all we know about the function $\exp(t)$ is that

$$\exp'(t) = \exp(t) \qquad \exp(0) = 1.$$

As before, we can use Euler's method to evaluate $\exp(1)$, which we will call e. From these facts alone we want to *deduce* that $\exp(t) = e^t$ for all values of t. We will actually show this only for all *rational* values of t, since there is, as we've seen, a bit of hand-waving about what it means to raise a number to an irrational power. The following theorem is the key to establishing this result.

Theorem 1. *For any real numbers r and s,*

$$\exp(r + s) = \exp(r) \cdot \exp(s).$$

We will prove this result shortly, but let's see what we can deduce from it. First off, note that

$$\exp(2) = \exp(1 + 1) = \exp(1) \cdot \exp(1) = (\exp(1))^2 = e^2.$$

Notice that we invoked Theorem 1 to equate $\exp(1+1)$ with $\exp(1) \cdot \exp(1)$. In a similar way,

$$\exp(3) = \exp(2 + 1) = \exp(2) \cdot \exp(1) = (\exp(1))^2 \cdot \exp(1) = (\exp(1))^3 = e^3.$$

Repeating this argument for any positive integer m, we get

Corollary 1. *For any positive integer m,*

$$\exp(m) = (\exp(1))^m = e^m.$$

Negative integers

We can also express $\exp(t)$ in terms of e when t is a *negative* integer. We begin with another consequence of Theorem 1:

$$1 = \exp(0) = \exp(-1 + 1) = \exp(-1) \cdot \exp(1).$$

This says $e = \exp(1)$ is the reciprocal of $\exp(-1)$:

$$\exp(-1) = (\exp(1))^{-1} = e^{-1}.$$

Since $-2 = -1 - 1$, $-3 = -2 - 1$, and so forth, we can eventually show that

Corollary 2. *For any negative integer* $-m$,

$$\exp(-m) = \exp(-1-1-\ldots-1) = (\exp(-1))^m = (\exp(1))^{-m} = e^{-m}.$$

Rational numbers

We can even do the same thing with fractions. Here's how to deal with $\exp(1/3)$, for example:

$$\begin{aligned}\exp(1) &= \exp\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{3}\right)\\ &= \exp(1/3)\cdot\exp(1/3)\cdot\exp(1/3)\\ &= (\exp(1/3))^3,\end{aligned}$$

so $\exp(1/3)$ is the cube root of $\exp(1)$:

$$\exp(1/3) = (\exp(1))^{1/3} = e^{1/3}.$$

A similar argument will show that

Corollary 3. *For any positive integer* n, $\exp(1/n) = e^{1/n}$.

Finally, we can deal with any rational number m/n:

$$\exp(m/n) = (\exp(1/n))^m = \left(e^{1/n}\right)^m = e^{m/n}.$$

This leads to

Theorem 2. *For any rational number* r, $\exp(r) = (\exp(1))^r = e^r$.

In other words, Theorem 1 implies that the function $\exp(t)$ is the same function as the exponential function e^t—at least when t is a rational number m/n, as claimed. We could now prove that $\exp(t) = e^t$ when t is an irrational number, which would require being clearer about what it means to raise a number to an irrational power than most high school texts are.

We adopt a more attractive option. Since $\exp(t)$ and e^t agree at rational values of t, and since $\exp(t)$ is well-defined for *all* values of t—including irrational numbers—we *define* e^t for irrational values of t by setting it equal to $\exp(t)$.

Proof of Theorem 1

The proof of Theorem 1

The proof uses the Existence and Uniqueness Principle for differential equations we articulated earlier: if two functions satisfy the same differential equation and satisfy the same initial conditions then they have to be the same function.

Theorem 1 involves two fixed real numbers, r and s. We fix one of them, say r, to define two new functions of t:

$$P(t) = \exp(r + t) \qquad Q(t) = \exp(r) \cdot \exp(t).$$

We shall show that *both* of these functions are solutions to the same initial value problem:

$$\frac{dy}{dt} = y \qquad y(0) = \exp(r).$$

(Remember, $\exp(r)$ is a constant, because r is fixed.)

If we show this, it will then follow that $P(t)$ and $Q(t)$ must be the same function. Since

$$\begin{aligned} P(0) &= \exp(r + 0) = \exp(r) \\ Q(0) &= \exp(r) \cdot \exp(0) = \exp(r) \cdot 1 = \exp(r), \end{aligned}$$

$P(t)$ and $Q(t)$ both satisfy the initial condition $y(0) = \exp(r)$. Next we show that they both satisfy the differential equation $y' = y$:

$$Q'(t) = (\exp(r) \cdot \exp(t))' = \exp(r) \cdot (\exp(t))' = \exp(r) \cdot \exp(t) = Q(t),$$

so $Q(t)$ is a solution. To differentiate $P(t)$ we construct a chain:

$$P = \exp(u) \qquad \text{where} \qquad u = r + t.$$

Then $dP/du = \exp(u)$ and $du/dt = 1$, so

$$P'(t) = \frac{dP}{du} \cdot \frac{du}{dt} = \exp(u) \cdot 1 = P(t) \cdot 1 = P(t),$$

so $P(t)$ is also a solution. Therefore $P(t)$ and $Q(t)$ must be the same function. It follows then that $P(t) = Q(t)$ for all values of t, in particular for $t = s$. But this means that

$$\exp(r + s) = P(s) = Q(s) = \exp(r) \cdot \exp(s),$$

which is exactly the statement of Theorem 1, and so completes the proof.

Now that we have established $\exp(x) = e^x$, we shall call $\exp(x)$ ***the* exponential function** and we shall use the forms e^x and $\exp(x)$ interchangeably. The following theorem summarizes several more properties of the exponential function.

Theorem 3. *For any real numbers r and s,*

$$\begin{aligned}
\exp(s) &> 0 \\
\exp(-s) &= \frac{1}{\exp(s)} \\
\exp(r-s) &= \frac{\exp(r)}{\exp(s)} \\
\exp(rs) &= (\exp(r))^{s} = (\exp(s))^{r}.
\end{aligned}$$

To make the statements in this theorem seem more natural, you should stop and translate them from $\exp(x)$ to e^x. Proofs will be covered in the exercises.

Exponential Growth

The function $\exp(x) = e^x$, like polynomials and the sine and cosine functions, is defined for all real numbers. Nevertheless, it behaves in a way that is quite different from any of those functions.

One difference occurs when x is large, either positive or negative. The sine function and the cosine function stay bounded between $+1$ and -1 over their entire domain. By contrast, every polynomial "blows up" as $x \to \pm\infty$. In this regard, the exponential function is a hybrid. As $x \to -\infty$, $\exp(x) \to 0$. As $x \to +\infty$, however, $\exp(x) \to +\infty$.

How fast do x^n and e^x become infinite?

Let's look more closely at what happens to power functions x^n and the exponential function e^x as $x \to \infty$. Both kinds of functions "blow up" but they do so at quite different rates, as we shall see. Before we compare power and exponential functions directly, let's compare one power of x with another—say x^2 with x^5. As $x \to \infty$, both x^2 and x^5 get very large. However, x^2 is only a small fraction of the size of x^5, and that fraction gets smaller, the larger x is. The following table demonstrates this. Even though x^2 grows enormous, we interpret the fact that $x^2/x^5 \to 0$ to mean that x^2 *grows more slowly than* x^5.

x	x^2	x^5	x^2/x^5
10	10^2	10^5	10^{-3}
100	10^4	10^{10}	10^{-6}
1000	10^6	10^{15}	10^{-9}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	∞	∞	0

It should be clear to you that we can compare *any* two powers of x this way. We will find that x^p grows more slowly than x^q if, and only if, $p < q$. To prove this, we must see what happens to the ratio x^p/x^q, as $x \to +\infty$. We can write $x^p/x^q = 1/x^{q-p}$, and the exponent $q - p$ that appears here is positive, because $q > p$. Consequently, as $x \to \infty$, $x^{q-p} \to \infty$ as well, and therefore $1/x^{q-p} \to 0$. This completes the proof.

x^p grows more slowly than x^q if $p < q$

How does e^x compare to x^p? To make it tough on e^x, let's compare it to x^{50}. We know already that x^{50} grows faster than any lower power of x. The table below compares x^{50} to e^x. However, the numbers involved are so large that the table shows only their *order of magnitude*—that is, the number of digits they contain. At the start, x^{50} is *much* larger than e^x. However, by the time $x = 500$, the ratio x^{50}/e^x is so small its first 82 decimal places are zero!

x	x^{50}	e^x	x^{50}/e^x
100	$\sim 10^{100}$	$\sim 10^{43}$	$\sim 10^{56}$
200	10^{115}	10^{86}	10^{28}
300	10^{123}	10^{130}	10^{-7}
400	10^{130}	10^{173}	10^{-44}
500	10^{134}	10^{217}	10^{-83}
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
∞	∞	∞	0

So x^{50} grows more slowly than e^x, and so does any lower power of x. Perhaps a higher power of x would do better. It does, but ultimately the ratio $x^p/e^x \to 0$, no matter how large the power p is. We don't yet have all the tools needed to prove this, but we will after we introduce the logarithm function in the next section.

e^x grows more rapidly than any power of x

The speed of exponential growth has had an impact in computer science. In many cases, the number of operations needed to calculate a particular quantity is a power of the number of digits of precision required in the answer. Sometimes, though, the number of operations is an *exponential* function of the number of digits. When that happens, the number of operations can quickly exceed the capacity of the computer. In this way, some problems that can be solved by an algorithm that is straightforward in theoretical terms are intractable in practical terms.

Exercises

The exponential functions b^t

1. Use a graphing utility or a calculator to approximate the slopes of the following functions at the origin and show:

a) If $f(t) = (2.71)^t$, then $f'(0) < 1$.

b) If $g(t) = (2.72)^t$, then $g'(0) > 1$.

c) Use parts (a) and (b) to explain why $2.71 < e < 2.72$.

2. a) In the same way find the value of the parameter k_b for the bases $b = .5, .75$, and $.9$ accurate to 3 decimal places.

b) What is the shape of the graph of $y = b^t$ when $0 < b < 1$? What does that imply about the *sign* of k_b for $0 < b < 1$? Explain your reasoning.

Differentiating exponential functions

3. Differentiate the following functions.

a) $7e^{3x}$

b) Ce^{kx}, where C and k are constants.

c) $1.5e^{t}$

d) $1.5e^{2t}$

e) $2e^{3x} - 3e^{2x}$

f) $e^{\cos t}$

4. Find partial derivatives of the following functions.

a) e^{xy}

b) $3x^2e^{2y}$

c) $e^{u} \sin v$

d) $e^{u \sin(v)}$

Powers of e

5. Simplify the following and rewrite as powers of e. For each, explain your work, citing any theorems you use.

a) $\exp(2x+3)$

b) $(\exp(x))^2$

c) $\exp(17x)/\exp(5x)$

6. Use the second property in Theorem 3 to explain why

$$\lim_{t\to-\infty} \exp(t) = 0.$$

7. This purpose of this exercise is to prove the fourth property listed in Theorem 3: $\exp(rs) = (\exp(s))^r$, for all real numbers r and s. The idea of the proof is the same as for Theorem 1: show that two different-looking functions solve the same initial value problem, thus demonstrating that the functions must be the same. The initial value problem is

$$y' = ry \qquad y(0) = 1.$$

a) Show that $P(t) = \exp(rt)$ solves the initial value problem. (You need to use the chain rule.)

b) Show that $Q(t) = (\exp(t))^r$ solves the initial value problem. (Here use the chain $Q = u^r$, where $u = \exp(t)$. There is a bit of algebra involved.)

c) From parts (a) and (b), and the fact that an initial value problem has a unique solution, it follows that $P(t) = Q(t)$, for every t. Explain how this establishes the result.

Solving $y' = ky$ using e^t

8. **Poland and Afghanistan**. Refer to problem 25 in chapter 1.2.

a) Write out the initial value problems that summarize the information about the populations P and A given in parts (a) and (b) of problem 21.

b) Write formulas for the solutions P and A of these initial value problems.

c) Use your formulas in part (b) (and a calculator) to find the population of each country in the year 2005. What were the populations in 1965?

9. **Bacterial growth**. Refer to problem 26 in chapter 1.2.

a) Assuming that we begin with the colony of bacteria weighing 32 grams, write out the initial value problem that summarizes the information about the weight P of the colony.

b) Write a formula for the solution P of this initial value problem.

c) How much does the colony weigh after 30 minutes? after 2 hours?

10. **Radioactivity**. Refer to problem 27 in chapter 1.2.

a) Assuming that when we begin the sample of radium weighs 1 gram, write out the initial value problem that summarizes the information about the weight R of the sample.

b) How much did the sample weigh 20 years ago? How much will it weigh 200 years hence?

11. **Intensity of radiation**. As gamma rays travel through an object, their intensity I decreases with the distance x that they have travelled. This is called **absorption**. The absorption rate dI/dx is proportional to the intensity. For some materials the multiplier in this proportion is large; they are used as radiation shields.

a) Write down a differential equation which models the intensity of gamma rays $I(x)$ as a function of distance x.

b) Some materials, such as lead, are better shields than others, such as air. How would this difference be expressed in your differential equation?

c) Assume the unshielded intensity of the gamma rays is I_0. Write a formula for the intensity I in terms of the distance x and verify that it gives a solution of the initial value problem.

12. In this problem you will find a solution for the initial value problem $y' = ky$ and $y(t_0) = C$. (Notice that this isn't the original initial value problem, because t_0 was 0 originally.)

a) Explain why you may assume $y = Ae^{kt}$ for some constant A.

b) Find A in terms of k, C and t_0.

Solving other differential equations

13. a) **Newton's law of cooling**. Verify that

$$Q(t) = 70e^{-.1\,t} + 20$$

is a solution to the initial value problem $Q'(t) = -.1(Q - 20)$, $Q(0) = 90$. What is the relationship between this formula and the one found in problem 11 in section 2?

b) Verify that

$$Q(t) = (Q_0 - A)e^{-kt} + A$$

is a solution to the the initial value problem $Q'(t) = -k(Q - A)$, $Q(0) = B$. What is the relationship between this formula and the one found in problem 13 in section 2?

14. In *An Essay on the Principle of Population*, written in 1798, the British economist Thomas Robert Malthus (1766–1834) argued that food supplies grow at a constant rate, while human populations naturally grow at a constant *per capita* rate. He therefore predicted that human populations would inevitably run out of food (unless population growth was suppressed by unnatural means).

a) Write differential equations for the size P of a human population and the size F of the food supply that reflect Malthus' assumptions about growth rates.

b) Keep track of the population in millions, and measure the food supply in millions of units, where one unit of food feeds one person for one year. Malthus' data suggested to him that the food supply in Great Britain was growing at about .28 million units per year and the per capita growth rate of the population was 2.8% per year. Let $t = 0$ be the year 1798, when Malthus estimated the population of the British Isles was $P = 7$ million people. He assumed his countrymen were on average adequately nourished, so he estimated that the food supply was $F = 7$ million units of food. Using these values, write formulas for the solutions $P = P(t)$ and $F = F(t)$ of the differential equations in (a).

c) Use the formulas in (b) to calculate the amount of food and the population at 25 year intervals for 100 years. Use these values to help you sketch graphs of $P = P(t)$ and $F = F(t)$ on the same axes.

d) The per capita food supply in any year equals the ratio $F(t)/P(t)$. What happens to this ratio as t grows larger and larger? (Use your graphs in (c) to assist your explanation.) Do your results support Malthus's prediction? Explain.

15. a) **Falling bodies**. Using the base e instead of the base 2, modify the solution $v(t)$ to the initial value problem

$$\frac{dv}{dt} = -g - bv \qquad v(0) = 0$$

that appears in exercise 21 on page 223. Show that the modified expression is still a solution.

b) If an object that falls against air resistance is $x(t)$ meters above the ground after t seconds, and it started x_0 meters above the ground, then it is the solution of the initial value problem

$$\frac{dx}{dt} = v(t) \qquad x(0) = x_0,$$

where $v(t)$ is the velocity function from the previous exercise. Find a formula for $x(t)$ using the exponential function with base e. (Compare this formula with the one in exercise 22 (a), page 224.)

c) Suppose the coefficient of air resistance is .2 per second. If a body is held motionless 200 meters above the ground, and then released, how far will it fall in 1 second? In 2 seconds? Use your formula from part (b). Compare these values with those you obtained in exercise 22 (b), page 224.

Interest rates

Bank advertisements sometimes look like this:

Civic Bank and Trust

- Annual rate of interest 6%.
- Compounded monthly.
- Effective rate of interest 6.17%.

The first item seems very straightforward. The bank pays 6% interest per year. Thus if you deposit \$100.00 for one year then at the end of the year you would expect to have \$106.00. Mathematically this is the simplest way to compute interest; each year add 6% to the account. The biggest problem with this is that people often make deposits for odd fractions of a year, so if interest were paid only once each year then a depositor who withdrew her money after 11 months would receive no interest. To avoid this problem banks usually compute and pay interest more frequently. The Civic Bank and Trust advertises interest **compounded** monthly. This means that the bank computes interest each month and credits it (that is, adds it) to the account.

Month	Start	Interest	End
1	\$100.0000	.5000	\$100.5000
2	\$100.5000	.5025	\$101.0025
3	\$101.0025	.5050	\$101.5075
4	\$101.5075	.5075	\$102.0151
5	\$102.0151	.5101	\$102.5251
6	\$102.5251	.5126	\$103.0378
7	\$103.0378	.5152	\$103.5529
8	\$103.5529	.5178	\$104.0707
9	\$104.0707	.5204	\$104.5911
10	\$104.5911	.5230	\$105.1140
11	\$105.1140	.5256	\$105.6396
12	\$105.6396	.5282	\$106.1678

Since this particular account pays interest at the rate of 6% per year and there are 12 months in a year the interest rate is $6\%/12 = 0.5\%$ per month. The following table shows the interest computations for one year for a bank account earning 6% annual interest compounded monthly.

Notice that at the end of the year the account contains \$106.17. It has effectively earned 6.17% interest. This is the meaning of the advertised *effective* rate of interest. The reason that the effective rate of interest is higher than the original rate of interest is that the interest earned each month itself earns interest in each succeeding month. (We first encountered this phenomenon when we were trying to follow the values of S, I, and R into the future.) The difference between the original rate of interest and the effective rate can be very significant. Banks routinely advertise the effective rate to attract depositors. Of course, banks do the same computations for loans. They rarely advertise the effective rate of interest for loans because customers might be repelled by the true cost of borrowing.

The effective rate of interest can be computed much more quickly than we did in the previous table. Let R denote the annual interest rate as a decimal. For example, if the interest rate is 6% then $R = 0.06$. If interest is compounded n times per year then each time it is compounded the interest rate is R/n. Thus each time you compound the interest you compute

$$V + \left(\frac{R}{n}\right)V = \left(1 + \frac{R}{n}\right)V$$

where V is the value of the current deposit. This computation is done n times during the course of a year. So, if the original deposit has value V, after one year it will be worth

$$\left(1 + \frac{R}{n}\right)^n V.$$

For our example above this works out to

$$\left(1 + \frac{0.06}{12}\right)^{12} V = 1.061678\,V$$

and the effective interest rate is 6.1678%.

Many banks now compound interest daily. Some even compound interest *continuously*. The value of a deposit in an account with interest compounded continuously at the rate of 6% per year, for example, grows according to the differential equation

$$V' = 0.06V.$$

16. Many credit cards charge interest at an annual rate of 18%. If this rate were compounded monthly what would the effective annual rate be?

17. In fact many credit cards compound interest daily. What is the effective rate of interest for 18% interest compounded daily? Assume that there are 365 days in a year.

18. The assumption that a year has 365 days is, in fact, *not* made by banks. They figure every one of the 12 months has 30 days, so their year is 360 days long. This practice stem from the time when interest computations were done by hand or by tables, so simplicity won out over precision. Therefore when banks compute interest they find the daily rate of interest by dividing the annual rate of interest by 360. For example, if the annual rate of interest is 18% then the daily rate of interest is 0.05%. Find the effective rate of interest for 18% compounded 360 times per year.

19. In fact, once they've obtained the daily rate as 1/360-th of the annual rate, banks then compute the interest *every* day of the year. They compound the interest 365 times. Find the effective rate of interest if the annual rate of interest is 18% and the computations are done by banks. First, compute the daily rate by dividing the annual rate by 360 and then compute interest using this daily rate 365 times.

20. Consider the following advertisement.

Civic Bank and Trust

- Annual rate of interest 6%.
- Compounded daily.
- Effective rate of interest 6.2716%.

Find the effective rate of interest for an annual rate of 6% compounded daily in the straightforward way—using 1/365-th of the annual rate 365 times. Then do the computations the way they are done in a bank. Compare your two answers.

21. There are two advertisements in the newspaper for savings accounts in two different banks. The first offers 6% interest compounded quarterly (that is, four times per year). The second offers 5.5% interest compounded continuously. Which account is better? Explain.

4.4 The Logarithm Function

Suppose a population is growing at the net rate of 3 births per thousand persons per year. If there are 100,000 persons now, how many will there be 37 years from now? How long will it take the population to double?

Translating into mathematics, we want to find the function $P(t)$ that solves the initial value problem

$$P'(t) = .003P(t) \qquad \text{and} \qquad P(0) = 100000.$$

Using the results of section 3 we know that the solution is the exponential function

$$P(t) = 100000\, e^{.003\, t}.$$

The size of the population 37 years from now will therefore be

$$\begin{aligned} P(37) &= 100000\, e^{.111} \\ &= 100000 \times 1.117395 \\ &\approx 111740 \text{ people} \end{aligned}$$

The doubling time of a population

To find out how long it will take the population to double, we want to find a value for t so that $P(t) = 200000$. In other words, we need to solve for t in the equation

$$100000\, e^{.003\, t} = 200000.$$

Dividing both sides by 100,000, we have

$$e^{.003\, t} = 2.$$

We can't proceed because one side is expressed in exponential form while the other isn't. One remedy is to express 2 in exponential form. In fact, $2 = e^{.693147}$, as you should verify with a calculator. Then

$$e^{.003\, t} = 2 = e^{.693147} \qquad \text{implies} \qquad .003\, t = .693147,$$

so $t = .693147/.003 = 231.049$. Thus it will take about 231 years for the population to double.

To determine the doubling time of the population we had to know the number b for which

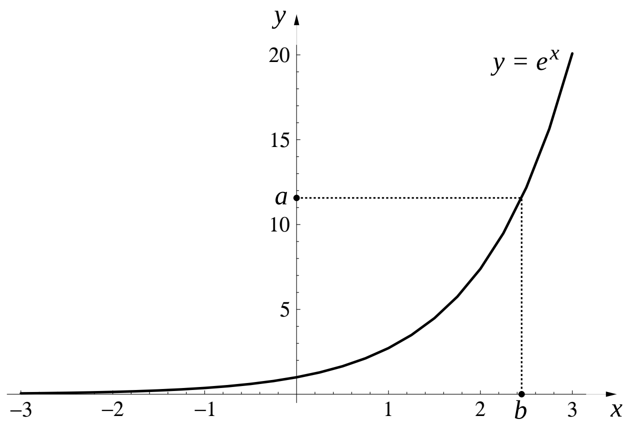
$$\exp(b) = e^{b} = 2.$$

Solving an exponential equation

This is an aspect of a very general question: given a positive number a, find a number b for which

$$e^{b} = a.$$

A glance at the graph of the exponential function below shows that, by working backwards from any point $a > 0$ on the vertical axis, we can indeed find a unique point b on the horizontal axis which gives us $\exp(b) = a$.



This process of obtaining the number b that satisfies $\exp(b) = a$ for any given positive number a is a clear and unambiguous rule. Thus, it defines a function. This function is called **the natural logarithm**, and it is denoted $\ln(a)$, or sometimes $\log(a)$. That is,

The natural logarithm function

$$\ln(a) = \log(a) = \{\text{the number } b \text{ for which } \exp(b) = a\}.$$

In other words, the two statements

$$\ln(a) = b \quad \text{and} \quad \exp(b) = a$$

express exactly the same relation between the quantities a and b.

The question that led to the introduction of the logarithm function was: what number gives the exponent to which e must be raised in order to produce the value 2? This number is $\ln(2)$, and we verified that $\ln(2) = .693147$. Quite generally we can say that the number $\ln(x)$ gives the exponent to which e must be raised in order to produce the value x:

$$e^{\ln(x)} = x.$$

If we set $y = \ln(x)$, then $x = e^y$ and we can restate the last equation as a pair of companion equations:

$$e^{\ln(x)} = x \quad \text{and} \quad \ln(e^y) = y.$$

The first equations says *the exponential function "undoes" the effect of the logarithm function* and the second one says *the logarithm function "undoes" the effect of the exponential function*. For this reason the exponential and logarithm functions are said to be **inverses** of each other.

The logarithm and exponential functions are inverses

Many of the other pairs of functions—sine and arscsine, squareroot and squaring—that share a key on a calculator have this property. There are even functions (at least one can be found on any calculator) that are their own inverses—apply such a function to any number, then apply this same function to the result, and you're back at the original number. What functions do this? We will say more about inverse functions later in this section.

Properties of the Logarithm Function

The inverse relationship allows us to translate each of the properties of the exponential function into a corresponding statement about the logarithm function. We list the major pairs of properties below.

exponential version	**logarithmic version**
$e^0 = 1$	$\ln(1) = 0$
$e^{a+b} = e^a \cdot e^b$	$\ln(m \cdot n) = \ln(m) + \ln(n)$
$e^{a-b} = e^a/e^b$	$\ln(m/n) = \ln(m) - \ln(n)$
$(e^a)^s = e^{as}$	$\ln(m^s) = s \cdot \ln(m)$
range of e^x is all positive reals	domain of $\ln(x)$ is all positive reals
domain of e^x is all real numbers	range of $\ln(x)$ is all real numbers
$e^x \to 0$ as $x \to -\infty$	$\ln(x) \to -\infty$ as $x \to 0$
e^x grows faster than x^n, any $n > 0$	$\ln(x)$ goes to infinity slower than $x^{1/n}$, any $n > 0$

For each pair, we can use the exponential property and the inverse relationship between exp and ln to establish the logarithmic property. As an example, we will establish the second property. You should be able to demonstrate the others.

Proof of the second property. Remember that to show $\ln(a) = b$, we need to show $e^b = a$. In our case a and b are more complicated. We have

$$a = m \cdot n, \qquad b = \ln(m) + \ln(n);$$

thus we need to show

$$e^{\ln(m) + \ln(n)} = m \cdot n\,.$$

But, by the exponential version of property 2,

$$e^{\ln(m) + \ln(n)} = e^{\ln(m)} \cdot e^{\ln(n)} = m \cdot n,$$

and our proof is complete.

The Derivative of the Logarithm Function

Since the natural logarithm is a function in its own right, it is reasonable to ask: what is the derivative of this function? Since the derivative describes the slope of the graph, let us begin by examining the graph of ln. Can we take advantage of the relationship between ln and exp—a function whose graph we know well—as we do this? Indeed we can, by making the following observations.

The graph of $\ln$

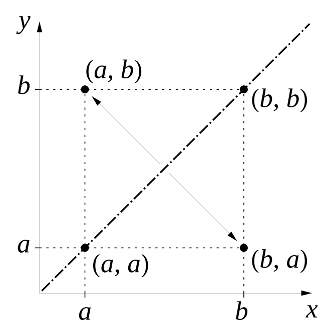
- We know the point (a, b) is on the graph of $y = \ln(x)$ if and only if $b = \ln(a)$.
- We know $b = \ln(a)$ says the same thing as $a = e^b$.
- Finally, we know $a = e^b$ is true if and only if the point (b, a) is on the graph of $y = e^x$.

Putting our observations together, we have

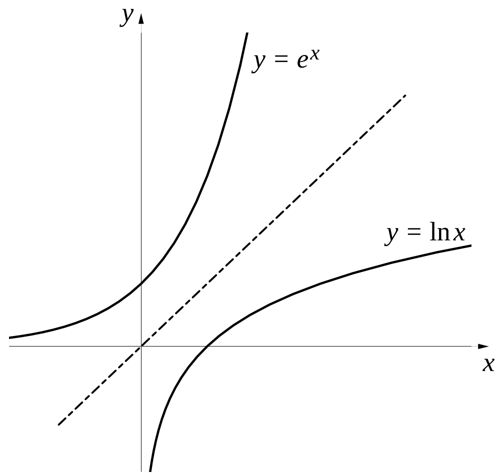
$$\begin{gathered}(a, b) \text{ is on the graph of } y = \ln(x)\\ \text{if and only if}\\ (b, a) \text{ is on the graph of } y = e^x.\end{gathered}$$

The picture below demonstrates that the point (a, b) and the point (b, a) are reflections of each other about the 45° line. (Remember that points on the 45° line have the same x and y coordinates.) This is because these two points are the endpoints of the diagonal of a square whose other diagonal is the line $y = x$.

Reflection across the 45° line

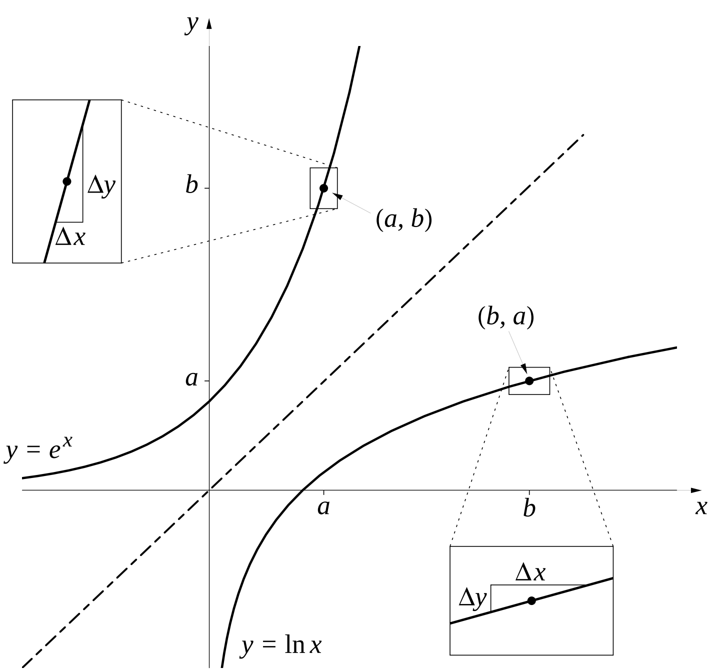


Since we have just seen that every point (a, b) on the graph of $y = \ln(x)$ corresponds to a point (b, a) on the graph of $y = e^x$, we see that *the graphs of $y = \ln(x)$ and $y = e^x$ are the reflections of each other about the line $y = x$.*



Microscopic views at mirror image points

Finally, since the two graphs are reflections of one another, a microscopic view of $\ln(x)$ at any point (b, a) will be the mirror image of the microscopic view of of e^x at the point (a, b). Any change in the y-value on one of these lines will correspond to an equal change in the x-value in its mirror image, and vice versa. The figure below shows what microscopic views of a pair of corresponding points look like, showing how a vertical change in one line equals the horizontal change in the other, and conversely.



It follows that the slopes of the two lines must be reciprocals of each other. This says that the rate of change of $\ln(x)$ at $x = b$ is just the reciprocal of the rate of change of e^x at $x = a$, where $a = \ln(b)$. But the rate of change of e^x at $x = \ln(b)$ is just $e^{\ln(b)} = b$. Therefore the rate of change of $\ln(x)$ at $x = b$ is the reciprocal of this value, namely $1/b$. We have thus proved the following result:

The slopes are reciprocals

Theorem 1. $(\ln(x))' = 1/x$.

Note that one interpretation of this theorem is that the function $\ln(x)$ is the solution to a certain initial value problem, namely

$$\frac{dy}{dx} = \frac{1}{x} \qquad y(1) = 0.$$

As was the case with the exponential function, we can now apply Euler's method to this differential equation as an effective way to compute values of $\ln(x)$. Applications of this idea can be found in the exercises.

Exponential Growth

The logarithm gives us a useful tool for comparing the growth rates of exponential and power functions. In the last section we claimed that e^x grows faster than any power x^p of x, as $x \to +\infty$. We interpreted that to mean

Comparing rates of growth

$$\lim_{x\to+\infty} \frac{x^p}{e^x} = 0,$$

for any number p. Using the natural logarithm, we can now show why it is true.

To analyze the quotient $Q = x^p/e^x$, we first replace it by its logarithm

$$\ln Q = \ln\left(x^p/e^x\right) = \ln\left(x^p\right) - \ln\left(e^x\right) = p\ln x - x.$$

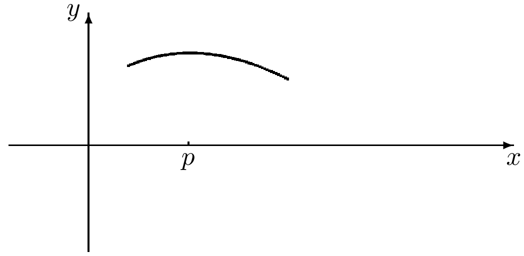
Several properties of the logarithm function were invoked here to reduce $\ln Q$ to $p\ln x - x$. By another property of the logarithm function, if we can show $\ln Q \to -\infty$ we will have established our original claim that $Q \to 0$.

Let $y = \ln Q = p\ln x - x$. We know y is increasing when $dy/dx > 0$ and decreasing where $dy/dx < 0$. Using the rules of differentiation, we find

$$\frac{dy}{dx} = \frac{p}{x} - 1.$$

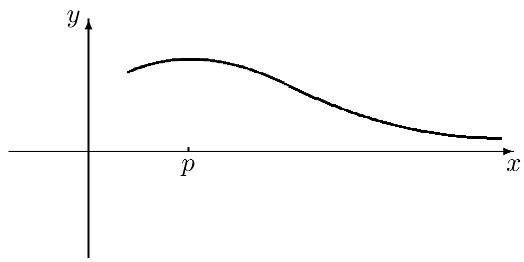
$y = \ln Q$ decreases when $x > p \ldots$

The expression $p/x - 1$ is positive when x is less than p and negative when x is greater than p. For x near p, the graph of y must therefore look like this:



Since dy/dx remains negative as x gets large, y will continue to decrease. This does not, in itself, imply that $y \to -\infty$, however. It's conceivable that y might "level off" even as it continues to decrease—as it does in the next graph.

... but y may still "level off"

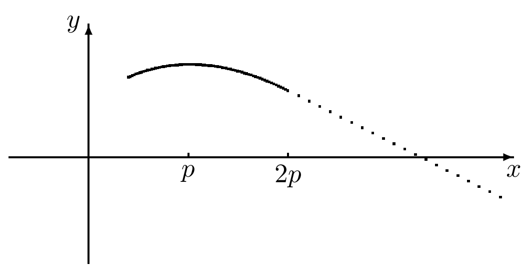


However, we can show that y does *not* "level off" in this way; it continues to plunge down to $-\infty$. We start by assuming that x has already become larger than $2p$: $x > 2p$. Then $1/x < 1/2p$ (the bigger number has the smaller reciprocal), and thus $p/x < p/2p = 1/2$. Thus, when $x > 2p$,

$$\frac{dy}{dx} = \frac{p}{x} - 1 < \frac{1}{2} - 1 = -\frac{1}{2}.$$

y lies below a line that slopes down to $-\infty$

In other words, the slope of the graph of y is more negative than $-1/2$. The graph of y must therefore lie *below* the straight line with slope $-1/2$ that we see below:



This guarantees that $y = \ln Q \to -\infty$ as $x \to \infty$. Hence $Q \to 0$, and since $Q = x^p/e^x$, we have shown that e^x grows faster than any power of x.

The Exponential Functions b^x

We have come to adopt the exponential function $\exp(x) = e^x$ as the natural one for calculus, and especially for dealing with differential equations of the form $dy/dx = ky$. Initially, though, all exponential functions b^x were on an equal footing. With the natural logarithm function, however, a *single* exponential function will meet our needs. Let's see why.

One exponential function is enough

If b is any positive real number, then $b = e^{\ln b}$. Consequently,

$$b^x = \left(e^{\ln b}\right)^x = e^{\ln b \cdot x}.$$

In other words, $b^x = e^{cx}$, where $c = \ln b$. Thus, every exponential function can be expressed in terms of exp in a simple way. This is, in fact, the way computers evaluate exponents, since the computer can raise any number to any power so long as it has a way to evaluate the functions ln and exp. For instance, when you ask a computer or calculator to evaluate `2^5` (2 to the 5th power in most computer languages), it will first calculate $\ln 2$, then multiply this number by 5, then apply exp to the result. That is, it evaluates 2^5 by thinking $e^{5 \ln 2}$! While this may seem a roundabout way to come up with 32, its virtue is that the computer needs only one algorithm to calculate any base to any power, without having to consider different cases.

$b^x = e^{\ln b \cdot x}$

How to calculate 2^5

This expression gives us a new way to find the derivative of b^x. We already know that

$$(e^{cx})' = c \cdot e^{cx},$$

for any constant c. This follows from the chain rule. When $c = \ln b$, we get

$$(b^x)' = (e^{\ln(b) \cdot x})' = \ln(b) \cdot e^{\ln(b) \cdot x} = \ln(b) \cdot b^x.$$

Thus, $y = b^x$ is a solution to the differential equation

$$\frac{dy}{dx} = \ln(b) \cdot y.$$

In chapter 3, we wrote this differential equation as

$$\frac{dy}{dx} = k_b \cdot y.$$

We see now that $k_b = \ln(b)$.

$k_b = \ln(b)$

We can use the connection between k_b and the natural logarithm, and between the natural logarithm and the exponential function, to gain new insights. For example, on page 212 we argued that there must be a value of b for which $k_b = .02$. This simply means

$$\ln b = .02 \qquad \text{or} \qquad b = e^{.02}.$$

In other words, we now have an explicit formula that tells us the value of b for which $k_b = .02$:

$$b = e^{.02} = 1.02020134\ldots.$$

Inverse Functions

Most of what we have said about the exponential and logarithm functions carries over directly to *any* pair of inverse functions. We begin by saying precisely what it means for two functions f and g to be inverses of each other.

Definition. Two functions f and g are **inverses** if

$$\begin{aligned} f(g(a)) &= a \\ \text{and } g(f(b)) &= b \end{aligned}$$

for every a in the domain of g and every b in the domain of f.

Observe that if f and g are inverses of each other, then each one "undoes" the effect of the other by sending any value back to the number it came from via the other function. One implication of this is that neither function can have two different input values going to the same output value. For instance, suppose b_1 and b_2 get sent to the same value by f: $f(b_1) = f(b_2)$. Applying g to both sides of this equation we would get $b_1 = g(f(b_1)) = g(f(b_2)) = b_2$, so b_1 and b_2 were actually the same number. This is an important enough property that there is a name for it:

Definition. We say that a function f is **one-to-one**, usually written as **1–1**, if it is true that whenever $x_1 \neq x_2$ then it is also the case that $f(x_1) \neq f(x_2)$. Equivalently, f is 1–1 if whenever $f(x_1) = f(x_2)$, then it must be true that $x_1 = x_2$.

We have thus seen that only functions which are one-to-one can have inverses. This means that to establish inverses for some functions, we will need to restrict their domains to regions where they are one-to-one. Let's re-examine the examples we mentioned earlier to see how they fit this definition.

Example 1. Suppose $f(x) = \exp(x)$ and $g(x) = \ln(x)$. Then the equations

$$f(g(a)) = \exp(\ln(a)) = e^{\ln(a)} = a \text{ for } a > 0$$
$$\text{and } g(f(b)) = \ln(\exp(b)) = \ln(e^{b}) = b$$

hold for all real numbers b and for all *positive* real numbers a. The domain of the exponential function is all real numbers and the domain of the natural logarithm function is all positive real numbers.

x^2 is invertible on $x \geq 0$

Example 2. Suppose $f(x) = x^2$ and $g(x) = \sqrt{x}$. The squaring function is not invertible on its natural domain because it is not one-to-one. Since a number and its negative have the same square, we wouldn't know which one to send the square back to when we took the square root. We can't avoid the problem by saying that $g(4) = \pm 2$, since a function has to have only one output for each input. The squaring function is invertible, though, if we restrict it to non-negative real numbers. Then

$$f(g(a)) = (\sqrt{a})^2 = a \ \text{ (for } \ a \geq 0)$$
$$\text{and } g(f(b)) = \sqrt{b^2} = b \ \text{ (for } \ b \geq 0).$$

The domain of the square root function is all $b \geq 0$.

Note that we could have restricted the domain of f in another way to make it one-to-one by considering only non-positive real numbers. Now g is no longer the inverse of this restricted f. For instance, $g(f(-3)) = g(9) = 3 \neq -3$. What would the inverse of f be in this case?

$\sin x$ is invertible on $-\pi/2 \leq x \leq \pi/2$

Example 3. Suppose $f(x) = \sin(x)$ and $g(x) = \arcsin(x)$. Since f is not one-to-one on its natural domain, we again need to restrict it in order for it to have an inverse. By convention, the domain of $\sin(x)$ is taken to be $-\pi/2 \leq x \leq \pi/2$.

$$f(g(a)) = \sin(\arcsin(a)) = a \ \text{ (for } \ -1 \leq a \leq 1) \ \text{ and}$$
$$g(f(b)) = \arcsin(\sin(b)) = b \ \text{ (for } \ -\pi/2 \leq b \leq \pi/2).$$

Each pair of inverse functions share corresponding properties, just as the logarithm and exponential functions do—the particular properties depending on the particular functions. But two they all share are

- The range of f is the domain of g.
- The domain of f is the range of g.

The exercises check this for examples 2 and 3.

Finally, the graphs—and therefore the derivatives—of a function and of its inverse are mirror images, exactly like those of the exponential and logarithm functions. We begin with the same list of observations.

- We know the point (a, b) is on the graph of $y = g(x)$ if and only if $b = g(a)$.
- We know $b = g(a)$ says the same thing as $a = f(b)$.
- Finally, we know $a = f(b)$ is true if and only if the point (b, a) is on the graph of $y = f(x)$.

As before, putting our observations together, we have

(a, b) is on the graph of $y = g(x)$
if and only if
(b, a) is on the graph of $y = f(x)$.

The graphs of inverse functions are mirror images ...

Exactly as before, we have that the point (a, b) and the point (b, a) are reflections of each other about the line $y = x$. Since we have just seen that every point (a, b) on the graph of $y = g(x)$ corresponds to a point (b, a) on the graph of $y = f(x)$, we again see that *the graphs of $y = g(x)$ and $y = f(x)$ are the reflections of each other about the line $y = x$.*

... and their derivatives are reciprocals

Finally, since the two graphs are reflections of one another, the local linear approximation of $g(x)$ at any point (a, b) will be the mirror image of the local linear approximation of $f(x)$ at the point (b, a). Any change in the y-value on one of these local lines will correspond to an equal change in the x-value in its mirror image, and vice versa. Just as before, it follows that the slopes of the two lines must be reciprocals of each other. This says that the rate of change of $g(x)$ at $x = b$ is the reciprocal of the rate of change of $f(x)$ at $x = a$, where $a = g(b)$. But the rate of change of $f(x)$ at $x = g(b)$ is just $f'(g(b))$. Therefore the rate of change of $g(x)$ at $x = b$ is the reciprocal of this value, namely $1/f'(g(b))$. We have thus proved the following result:

Theorem 2. *If the functions f and g are inverses, then g is locally linear at (b, a) if and only if f is locally linear at (a, b). When local linearity holds,*

$$g'(b) = \frac{1}{f'(a)}.$$

Exercises

1. Determine the numerical value of each of the following.

a) $\ln(2e)$	b) $\ln(e^3)$	c) e^{-1}	d) $\ln(\sqrt{e})$
e) $e^{\ln 2}$	f) $e^{3\ln 2}$	g) $(e^{\ln 2})^3$	h) $e^{2\ln 3}$
i) $\ln 10$	j) $\ln 10^3$	k) $e^{\ln 10}$	l) $e^{\ln 1000}$
m) $\ln(1/e)$	n) $\ln(1/2)$	o) $e^{-\ln 2}$	p) $e^{-3\ln 2}$

2. a) In the text we noted that the function $\ln x$ is the solution to the initial-value problem

$$\frac{dy}{dx} = \frac{1}{x} \qquad y(1) = 0,$$

so that we can use Euler's method to compute values for $\ln x$. Use this method to evaluate $\ln 2$ to 3 decimal places. What value of Δx gives the desired accuracy?

b) If you now wanted to calculate $\ln 6$ to 3 decimals, can you think of a better way to do it than simply starting at $x = 1$ and running Euler's method out to $x = 6$? Remember the basic properties of logarithms, and figure out a way to use the results of part (a).

c) Suppose you had figured out that $\ln 2 = 0.693147\ldots$. How would you use Euler's method to calculate $\ln 1300$ quickly? You might find the fact that $2^{10} = 1024$ helpful.

3. The rate of growth of the population of a particular country is proportional to the population. The last two censuses determined that the population in 1980 was 40,000,000, and in 1985 it was 45,000,000. What will the population be in 1995?

4. Find the derivatives of the following functions.

a) $\ln(3x)$

b) $17\ln(x)$

c) $\ln(e^w)$

d) $\ln(2^t)$

e) $\pi\ln(3e^{4s})$

5. Suppose a bacterial population grows so that its mass is

$$P(t) = 200e^{.12t} \quad \text{grams}$$

after t hours. Its initial mass is $P(0) = 200$ grams. When will its mass double, to 400 grams? How much longer will it take to double again, to 800 grams? After the population reaches 800 grams, how long will it take for yet another doubling to happen? What is the *doubling time* of this population?

6. Suppose a beam of X-rays whose intensity is A rads (the "rad" is a unit of radiation) falls perpendicularly on a heavy concrete wall. After the rays have penetrated s feet of the wall, the radiation intensity has fallen to

$$R(s) = Ae^{-.35s} \quad \text{rads.}$$

What is the radiation intensity 3 inches inside the wall; 18 inches? (Your answers will be expressed in terms of A.) How far into the wall must the rays travel before their intensity is cut in half, to $A/2$? How much further before the intensity is $A/4$?

7. Virtually all living things take up carbon as they grow. This carbon comes in two principal forms: normal, stable carbon—C^{12}—and radioactive carbon—C^{14}. C^{14} decays into C^{12} at a rate proportional to the amount of C^{14} remaining. While the organism is alive, this lost C^{14} is continually replenished. After the organism dies, though, the C^{14} is no longer replaced, so the percentage of C^{14} decreases exponentially over time. It is found that after 5730 years, half the original C^{14} remains. If an archaeologist finds a bone with only 20% of the original C^{14} present, how old is it?

8. The human population of the world appears to be growing exponentially. If there were 2.5 billion people in 1960, and 3.5 billion in 1980, how many will there be in 2010?

9. If bacteria increase at a rate proportional to the current number, how long will it take 1000 bacteria to increase to 10,000 if it takes them 17 minutes to increase to 2000?

10. Suppose sugar in water dissolves at a rate proportional to the amount left undissolved. If 40 lb. of sugar reduces to 12 lb. in 4 hours, how long will you have to wait until 99% of the sugar is dissolved?

11. Atmospheric pressure is a function of altitude. Assume that at any given altitude the rate of change of pressure with altitude is proportional to the pressure there. If the barometer reads 30 psi (pounds per square inch) at sea level and 24 psi at 6000 feet above sea level, how high are you when the barometer reads 20 psi?

12. a) An important concept in many economic analyses is the idea of **present value**. It is used to compare the values of different possible payments made at different times. As a simple example, suppose you had a small wood lot and had the choice of selling the timber on it now for \$5,000 or waiting 10 years for the trees to get larger, at which point you estimate the timber could be sold for \$8,000. To compare these two options, you need to convert the prospect of \$8,000 ten years from now into an equivalent amount of money now—its present value. This is the amount of money you would need to invest now to have \$8,000 in 10 years. Suppose you thought you could invest money at an annual interest rate of 4% compounded continuously. If you invested \$5,000 now at this rate, then in 10 years you would have $5000\,e^{.4} = \$7,459.12$. That is, \$5,000 now is worth \$7,459.12 in 10 years—both amounts have the same present value. Clearly \$8,000 in 10 years must have a slightly greater present value under the assumption of a 4% annual interest rate. What is it?

b) On the other hand, if you can get a higher interest rate than 4%, the present value of the \$8,000 will be much less. What is the present value of a payment of \$8,000 ten years from now if the annual interest rate is 8%?

c) At what interest rate do \$5,000 now and \$8,000 in ten years have the same present value?

13. Use properties of exp to prove the following properties of the logarithm. (Remember that $\ln a = b$ means $a = \exp b$.)

a) $\ln(1) = 0$.

b) $\ln(m/n) = \ln(m) - \ln(n)$.

c) $\ln(m^n) = n\ln(m)$.

14. a) Use a graphing program to find a good numerical approximation to $(\ln x)'$ at $x = 2$. Make a short table, for decreasing interval sizes Δx, of the quantity $\Delta(\ln x)/\Delta x$.

b) Use a graphing program to find a good numerical approximation to $(e^x)'$ at $x = \ln(2) = 0.6931\ldots$. Make a short table for decreasing interval sizes Δx, of the quantity $\Delta(e^x)/\Delta x$.

c) What is the relationship between the values you got in parts (a) and (b)?

15. Find a solution (using $\ln x$) to the differential equation

$$f'(x) = 3/x \quad \text{satisfying} \quad f(1) = 2.$$

16. a) Find a formula using the natural logarithm function giving the solution of $y' = a/x$ with $y(1) = b$.

b) Solve $P' = 2/t$ with $P(1) = 5$.

17. Find the domain and range of each of the following pairs of inverse functions.

a) $f(x) = x^2$ (restricted to $x \geq 0$) and $g(x) = \sqrt{x}$.

b) $f(x) = \sin(x)$ (restricted to $-\pi/2 \leq x \leq \pi/2$) and $g(x) = \arcsin(x)$.

18. Show that $f(x) = 1/x$ equals its own inverse. What are the domain and range of f?

19. Let n be a positive integer. and let $f(x) = x^n$. What is an inverse of f? How do we need to restrict the domain of f for it to have an inverse? Caution: the answer depends on n.

20. a) What is the inverse g of the function $f(x) = 1 - 3x$?

b) Do f and g satisfy Theorem 2?

21. What is an inverse of $f(x) = x^2 - 4$?

22. Use the relationship between the derivatives of a function and its inverse to find the indicated derivatives.

a) $g'(100)$ for $g(x) = \sqrt{x}$.

b) $g'(\sqrt{2}/2)$ for $g(x) = \arcsin(x)$.

c) $g'(1/2)$ for $g(x) = 1/x$.

23. a) Use Theorem 2 and the fact that $(x^2)' = 2x$ to derive the formula for the derivative of $\sqrt{x}$.

b) Use Theorem 2 and the fact that $(x^n)' = nx^{n-1}$ to derive the formula for the derivative of $\sqrt[n]{x}$

24. Compare the rates of growth of e^x and b^x for both $e < b$ and $1 < b < e$.

4.5 The Equation $y' = f(t)$

Most differential equations we have encountered express the rate of growth of a quantity *in terms of the quantity itself.* The simplest models for biological growth had this form: $y' = ky$ and $y' = ky^p$. Even when several variables were present—as in the S-I-R model and the predator-prey models—it was most natural to express the rates at which those variables change in terms of the variables themselves. Even the motion of a spring (pages 224–226) was described that way: the rate of change of position equalled the velocity, and the rate of change of velocity was proportional to the position.

The motion of a falling body ...

Sometimes, though, a differential equation will express the rate of change of a variable *directly in terms of the input variable.* For example, on page 222 we saw that the velocity dx/dt of a body falling under the sole influence of gravity is a linear function of the time:

$$\frac{dx}{dt} = -gt + v_0.$$

Here x is the height of the body above the ground, g is the acceleration due to gravity, and v_0 is the velocity at time $t = 0$. This equation has the general form

$$\frac{dx}{dt} = f(t),$$

where $f(t)$ is a given function of t. We will now consider special methods that can be used to study differential equations of this special form.

Antiderivatives

... and its solution

To solve the equation of motion of a body falling under gravity, we must find a function $x(t)$ whose derivative is given as

$$x' = -gt + v_0.$$

$-gt$ is the derivative of $-gt^2/2$

We can call upon our knowledge of the rules of differentiation to find x. Consider $-gt$ first. What function has $-gt$ as its derivative? We can start with t^2, whose derivative is $2t$. Since we want the derivative to turn out to be $-gt$, we can reason this way:

$$-gt = -\frac{g}{2} \cdot 2t = -\frac{g}{2} \times \text{the derivative of } t^2.$$

This leads us to identify $-gt^2/2$ as a function whose derivative is $-gt$. Check for yourself that this is correct by differentiating $-gt^2/2$.

v_0 is the derivative of v_0t and of v_0t+b

Now consider v_0, the other part of dx/dt. What function has the constant v_0 as its derivative? A derivative is a rate of growth, and we know that the linear functions are precisely the ones that have constant growth rates. Furthermore, the rate is the multiplier for a linear function, so we conclude that any linear function of the form v_0t+b has derivative v_0.

If we put the two pieces together, we find that

$$x(t) = -\frac{g}{2}\,t^2 + v_0t + b$$

The antiderivative of a function

is a solution to the differential equation, for any value of b. (Recall from section 2 that a differential equation can have many solutions.) We constructed this formula for $x(t)$ by "undoing" the process of differentiation, a process sometimes called **antidifferentiation**. The function produced is called an **antiderivative**. Thus:

$$-\frac{g}{2}t^2 + v_0t + b \text{ is an } \textit{antiderivative} \text{ of } -gt + v_0$$

$$\text{because } -gt + v_0 \text{ is the } \textit{derivative} \text{ of } -\frac{g}{2}t^2 + v_0t + b.$$

All the functions $F(x)+C$ are antiderivatives of $F'(x)$

Note that a function has only one derivative, but it has many antiderivatives. If $F(t)$ is an antiderivative of $f(t)$, then so is $F(t)+C$, where C is any constant.

The list of functions and their derivatives that we compiled in chapter 3 (see page 148) can be "turned around" to become a list of functions and their antiderivatives. Note that the antiderivative column should really be labelled "an antiderivative" since we could add a constant to any of the listed functions and still have an antiderivative for the function in the first column.

function	antiderivative
0	c
x^p	$\frac{1}{p+1}\,x^{p+1}$ (if $p \neq -1$)
x^{-1}	$\ln x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
$\exp x = e^x$	$\exp x = e^x$
b^x	$\frac{1}{\ln b}\,b^x$

Every power of x has an antiderivative

Notice the formula for the antiderivative of x^p requires $p+1 \neq 0$, that is, $p \neq -1$. This leaves out x^{-1}. However, the antiderivative of x^{-1} is $\ln x$, so no power of x is excluded from the table.

We also had differentiation rules that told us how to deal with different *combinations* of functions. Each of these rules has an analogue in antidifferentiation. The simplest combinations are a sum and a constant multiple.

function	antiderivative
$f(x)$	$F(x)$
$g(x)$	$G(x)$
$c \cdot f(x)$	$c \cdot F(x)$
$f(x) + g(x)$	$F(x) + G(x)$

We defer a discussion of the analogue of the chain rule to chapter 11.

With just these rules we can find the antiderivative of any polynomial, for instance. (Recall that a polynomial is a sum of constant multiples of powers of the input variable.) Here is a collection of sample antiderivatives that illustrate the various rules. To emphasize the fact that antiderivatives are determined only up to an additive constant, various constants have been tacked on—any other constant would work just as well. You should compare this table with the one on page 150.

function	antiderivative
$5x^4 - 2x^3$	$x^5 - \frac{1}{2}x^4 + 17$
$5x^4 - 2x^3 + 17x$	$x^5 - \frac{1}{2}x^4 + \frac{17}{2}x^2 - 243.77$
$6 \cdot 10^z + 17/z^7$	$6 \cdot 10^z/\ln 10 - 17/6z^6 + .002$
$3\sin t - 2t^3$	$-3\cos t - \frac{1}{2}t^4 + 5\ln 7$
$\pi\cos x + \pi^2$	$\pi\sin x + \pi^2 x - 12e^{7.21}$

Euler's Method Revisited

If we know the formula for an antiderivative of $f(t)$, then we can write down a solution to the differential equation $dy/dt = f(t)$. For example, the general solution to

$$\frac{dy}{dt} = 12t^2 + \sin t$$

is $y = 4t^3 - \cos t + C$. In such a case we have a shortcut to solving the differential equation without needing to use Euler's method. Often, though, there is no formula for an antiderivative of $f(t)$—even when $f(t)$ itself has a simple formula. There is no formula for the antiderivative of $\cos(t^2)$, or $\sin t/t$, or $\sqrt{1+t^3}$, for instance. In other cases, $f(t)$ may not even be given by a formula. It may be a data function, given as a graph made by a pen tracing on a moving sheet of graph paper.

Whether we can find a formula for an antiderivative of $f(t)$ or not, we can still solve the differential equation $dy/dt = f(t)$ by Euler's method. It turns out that Euler's method takes on a relatively simple form in such cases. Let's investigate this in the following context.

The volume of a reservoir varies over time

Let V be the volume of water in a reservoir serving a small town, measured in millions of gallons. Then V is a function of the time t, measured in days. Rainfall adds water to the reservoir, while evaporation and consumption by the townspeople take it away. Let f be the *net rate* at which water is flowing into the reservoir, in millions of gallons per day. Sometimes f will be positive—when rainfall exceeds evaporation and consumption—and sometimes f will be negative. The net inflow rate varies from day to day; that is, f is a function of time: $f = f(t)$. Our model of the reservoir is the differential equation

$$\frac{dV}{dt} = f(t) \text{ millions of gallons per day.}$$

The net inflow rate

Suppose $f(t)$ is measured every two days, and those measurements are recorded in the following table.

time t (days)	rate $f(t)$ ($10^6 \times$ gals. per day)
0	.34
2	.11
4	$-.07$
6	$-.23$
8	$-.14$
10	.03
12	.08

Note that in this table we are able to write down the rate for all values of t immediately, without having to calculate the intermediate values of the dependent variable V. This is in marked contrast with most of the examples we've looked at in this course where we had to know the values of all the variables for any time t before we could calculate the new rate value at that time. It is this simplification that gives differential equations of the form $y' = f(t)$ their special structure.

If we assume the value of $f(t)$ remains constant for the two days after each measurement is made, we can approximate the total change in V over these 14 days. The following table does this; it tells us how much V changes over each two-day period, and also the total *accumulated* change in V by the end that period. Since $\Delta t = 2$ days, we calculate ΔV by

The accumulated change in V

$$\Delta V = V' \cdot \Delta t = f(t) \cdot \Delta t = 2 \cdot f(t).$$

starting t	current ΔV	accumulated ΔV	ending t
0	.68	.68	2
2	.22	.90	4
4	$-.14$	.76	6
6	$-.46$	.30	8
8	$-.28$	.02	10
10	.06	.08	12
12	.16	.24	14

At the end of the 14 days, V has accumulated a total change of .24 million gallons. Notice this does not depend on the initial size of V. If V had been 92.64 million gallons at the start, it would be 92.64 + .24 = 92.88 million gallons at the end. If it had been only 2 million gallons at the start, it would be 2 + .24 = 2.24 million gallons at the end. Other models do not behave this way: in two weeks, a rabbit population of 900 will change much more than a population of 90. The total change in V is independent of V because the *rate* at which V changes is independent of V.

We can therefore use Euler's method to solve any differential equation of the form $dy/dt = f(t)$ *independently* of an initial value for y. We just calculate the total accumulated change in y, and add that total to any given initial y. Here is how it works when the initial value of t is a, and the time step is Δt.

Calculating just the accumulated change in y

starting t	current Δy	accumulated Δy	ending t
a	$f(a) \cdot \Delta t$	previously	$a + \Delta t$
$a + \Delta t$	$f(a + \Delta t) \cdot \Delta t$	accumulated	$a + 2\Delta t$
$a + 2\Delta t$	$f(a + 2\Delta t) \cdot \Delta t$	Δy	$a + 3\Delta t$
$a + 3\Delta t$	$f(a + 3\Delta t) \cdot \Delta t$	+	$a + 4\Delta t$
$\vdots$	$\vdots$	current	$\vdots$
$a + (n-1)\Delta t$	$f(a + (n-1)\Delta t) \cdot \Delta t$	Δy	$a + n\,\Delta t$

The accumulated change in y

The third column is too small to hold the values of "accumulated Δy." Instead, it contains the instructions for obtaining those values. It says: to get the current value of "accumulated Δy," add the "current Δy" to the previous value of "accumulated Δy."

Let's use Euler's method to find the accumulated Δy when $t = 4$, given that

$$\frac{dy}{dt} = \cos(t^2)$$

and t is initially 0. If we use 8 steps, then $\Delta t = .5$ and we obtain the following:

starting t	current Δy	accumulated Δy	ending t
0	.5000	.5000	.5
.5	.4845	.9845	1.0
1.0	.2702	1.2546	1.5
1.5	−.3141	.9405	2.0
2.0	−.3268	.6137	2.5
2.5	.4997	1.1134	3.0
3.0	−.4556	.6579	3.5
3.5	.4752	1.1330	4.0

The following program generated the last three columns of this table.

Program: TABLE

```
DEF fnf (t) = COS(t ^ 2)
tinitial = 0
tfinal = 4
numberofsteps = 2 ^ 3
deltat = (tfinal - tinitial) / numberofsteps
t = tinitial
accumulation = 0
FOR k = 1 TO numberofsteps
        deltay = fnf(t) * deltat
        accumulation = accumulation + deltay
        t = t + deltat
        PRINT deltay, accumulation, t
NEXT k
```

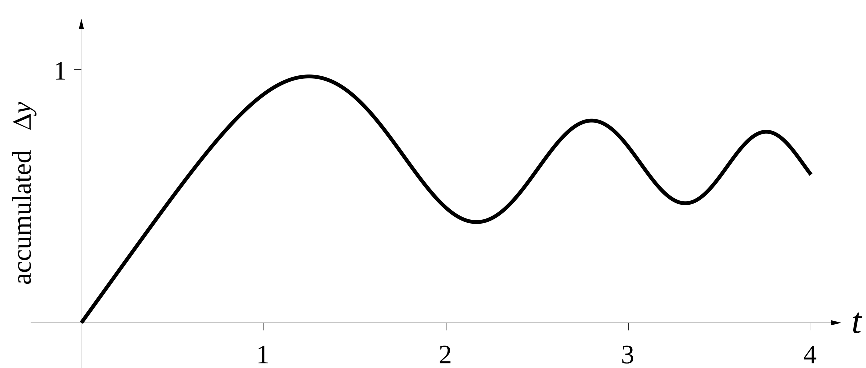
TABLE is a modification of the program SIRVALUE (page 65). To emphasize the fact that it is the accumulated change that matters rather than the actual value of y, we have modified the program accordingly. Note that `accumulation` always starts at 0, no matter what the initial value of y is. The first line of the program takes advantage of a capacity most programming languages have to define functions which can then be referred to elsewhere in the program.

As usual, to find the exact value of the accumulated Δy, it is necessary to recalculate, using more steps and smaller step sizes Δt. If we use TABLE to do this, we find

number of steps	accumulated Δy
2^3	1.13304
2^6	.65639
2^9	.60212
2^{12}	.59542
2^{15}	.59458
2^{18}	.59448

Thus we can say that if $dy/dt = \cos(t^2)$, then y increases by .594... when t increases from 0 to 4.

In the same way we changed SIRVALUE to produce the program SIRPLOT (page 69), we can change the program TABLE into one that will *plot* the values of y. In the following program all those changes are made, and one more besides: we have increased the number of steps to 400 to get a closer approximation to the true values of y. The output of PLOT is shown immediately below.



The accumulated Δy when $dy/dt = \cos(t^2)$

Program: PLOT

```
Set up GRAPHICS
DEF fnf (t) = COS(t ^ 2)
tinitial = 0
tfinal = 4
numberofsteps = 400
deltat = (tfinal - tinitial) / numberofsteps
t = tinitial
accumulation = 0
FOR k = 1 TO numberofsteps
      deltay = fnf(t) * deltat
      Plot the line from (t, accumulation)
         to (t + deltat, accumulation + deltay)
      accumulation = accumulation + deltay
      t = t + deltat
NEXT k
```

Exogenous and endogenous factors

Let's compare our reservoir model with population growth. The rate at which a population grows depends, in an obvious way, on the size of the population. By contrast, the rate at which the reservoir fills does *not* depend on how much water there is in the reservoir. It depends on factors *outside* the reservoir: rainfall and consumption. These factors are said to be **exogenous** (from the Greek *exo*-, "outside" and *-gen*, "produced," or "born"). The opposite is called an **endogenous** factor (from the Greek *endo*-, "within"). Evaporation is an endogenous factor for the reservoir model; population size is certainly an endogenous factor for a population model.

Precisely because exogenous factors are "outside the system," we need to be *given* the information on how they vary over time. In the reservoir model, this information appears in the function $f(t)$ that describes the rate at which V changes. In general, if y depends on exogenous factors that vary over time, we can expect the differential equation for y to involve a function of time:

$$\frac{dy}{dt} = f(t)$$

Thus, we can view this section as dealing with models that involve exogenous factors.

The differential equation of motion for a falling body, $dx/dt = -gt + v_0$, indicates that gravity is an exogenous factor. In Greek and medieval European science, the reason an object fell to the ground was assumed to lie within the object itself—it was the object's "heaviness." By making the cause of motion exogenous, rather than endogenous, Galileo and Newton started a scientific revolution.

Exercises

1. Find a formula $y = F(t)$ for a solution to the differential equation $dy/dt = f(t)$ when $f(t)$ is

a) $5t - 3$

b) $t^6 - 8t^5 + 22\pi^3$

c) $5e^t - 3\sin t$

d) $12\sqrt{t}$

e) $2^t + 7/t^9$

f) $5e^{4t} - 1/t$

2. Find $G(5)$ if $y = G(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = \frac{1}{x^2} \qquad y(2) = 3.$$

3. Find $F(2)$ if $y = F(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = \frac{1}{x} \qquad y(1) = 5.$$

4. Find $H(3)$ if $y = H(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = x^3 - 7x^2 + 19 \qquad y(-1) = 5.$$

5. Find $L(-2)$ if $y = L(x)$ is the solution to the initial value problem

$$\frac{dy}{dx} = e^{3x} \qquad y(1) = 6.$$

6. a) Sketch the graph of the solution to the initial value problem

$$\frac{dy}{dx} = \sin x \qquad y(0) = 1$$

over the interval $0 \leq x \leq 4\pi$.

b) By finding a suitable antiderivative, evaluate $y(2)$.

7. a) Sketch the graph of the solution to the initial value problem

$$\frac{dy}{dx} = \sin(x^2) \qquad y(0) = 0$$

over the interval $0 \leq x \leq 5$. (This one can't be done by finding a formula for the antiderivative.)

b) What is the slope of the solution graph at $x = 0$? Does your graph show this?

c) How many peaks (local maxima) does the solution have on the interval $0 \leq x \leq 5$?

d) What is the maximum value that the solution achieves on the interval $0 \leq x \leq 5$? For which value of x does this happen?

e) What is $y(6)$?

8. a) What is the accumulated change in y if $dy/dt = 3t^2 - 2t$ and t increases from 0 to 1? What if t increases from 1 to 2? What if t increases from 0 to 2?

b) Sketch the graph of the accumulated change in y as a function of t. Let $0 \leq t \leq 2$.

9. a) Here's another problem for which there is no formula for an antiderivative. Sketch the graph of the solution to the initial value problem

$$\frac{dy}{dx} = \frac{\sin x}{x} \qquad y(0) = 0$$

on the interval $0 \leq x \leq 40$. [Note: $\sin x/x$ is not defined when $x = 0$, so take the initial value of x to be .00001. That is, use $y(.00001) = 0$.]

b) How many peaks (local maxima) does the solution have on the interval $0 \leq x \leq 40$?

c) What is the maximum value of the solution on the interval $0 \leq x \leq 40$? For which x is this maximum achieved?

4.6 Chapter Summary

The Main Ideas

- A **system of differential equations** expresses the derivatives of a set of functions in terms of those functions and the input variable.
- An **initial value problem** is a system of differential equations together with values of the functions for some specified value of the input variable.
- Many processes in the physical, biological, and social sciences are **modelled** as initial value problems.
- A **solution to a system of differential equations** is a set of functions which make the equations *true* when they and their derivatives are substituted into the equations.
- A **solution to an initial value problem** is a set of functions that solve the differential equations and satisfy the initial conditions. Typically, a solution is unique.
- **Euler's method** provides a recipe to find the solution to an initial value problem.
- In special circumstances it is possible to find **formulas** for the solution to a system of differential equations. If the differential equations involve **parameters**, the solutions will, too.
- Systems of differential equations define functions as their solutions. Among the most important are the **exponential** and **logarithm functions**.
- The **natural logarithm** function is the inverse of the exponential function.
- The **graphs** and the **derivatives** of a function and its inverse are connected geometrically to each other by **reflection**.
- Exponential functions b^x **grow to infinity** faster than any power of x.
- The solution to $dy/dx = f(x)$ is an **antiderivative** of f—that is, a function whose derivative is f.

Expectations

- You should be able to use computer programs to produce tables and graphs of solutions to initial value problems.
- You should be able to check whether a system of differential equations reflects the hypotheses being made in constructing a model of a process.
- You should be able to verify whether a set of functions given by formulas is a solution to a system of differential equations.
- You should be familiar with the basic properties of the exponential and logarithm functions.
- You should be able to express solutions to initial value problems involving exponential growth or decay in terms of the exponential function.
- You should be able to solve $dy/dx = f(x)$ by antidifferentiation when $f(x)$ is a basic function or a simple combination of them.
- You should be able to analyze and graph the inverse of a given function.