Chapter 11

Techniques of Integration

Chapter 6 introduced the integral. There it was defined numerically, as the limit of approximating Riemann sums. Evaluating integrals by applying this basic definition tends to take a long time if a high level of accuracy is desired. If one is going to evaluate integrals at all frequently, it is thus important to find **techniques of integration** for doing this efficiently. For instance, if we evaluate a function at the midpoints of the subintervals, we get much faster convergence than if we use either the right or left endpoints of the subintervals.

A powerful class of techniques is based on the observation made at the end of chapter 6, where we saw that the fundamental theorem of calculus gives us a second way to find an integral, using antiderivatives. While a Riemann sum will usually give us only an approximation to the value of an integral, an antiderivative will give us the exact value. The drawback is that antiderivatives often can't be expressed in **closed form**—that is, as a **formula** in terms of named functions. Even when antiderivatives can be so expressed, the formulas are often difficult to find. Nevertheless, such a formula can be so powerful, both computationally and analytically, that it is often worth the effort needed to find it. In this chapter we will explore several techniques for finding the antiderivative of a function given by a formula.

We will conclude the chapter by developing a numerical method—Simpson's rule—that gives a good estimate for the value of an integral with relatively little computation.

11.1 Antiderivatives

Definition

Recall that we say F is an **antiderivative** of f if $F' = f$. Here are some examples.

FUNCTION:	x^2	$1/y$	$\sin u$	$2\sin t\,\cos t$	2^z
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$
ANTIDERIVATIVE:	$\dfrac{x^3}{3}$	$\ln y$	$-\cos u$	$\sin^2 t$	$\dfrac{2^z}{\ln 2}$

Undo a differentiation

Notice that you go up (↑) from the bottom row to the top by carrying out a differentiation. To go down (↓) you must "undo" that differentiation. The process of reversing, or undoing, a differentiation has come to be called **antidifferentiation**. You should differentiate each function on the bottom row to check that it is an antiderivative of the function above it.

A function has many antiderivatives

While a function can have only one derivative, it has many antiderivatives. For example, $1 - \cos u$ and $99 - \cos u$ are also antiderivatives of the function $\sin u$ because

$$(1 - \cos u)' = \sin u = (99 - \cos u)'.$$

In fact, every function $C - \cos u$ is an antiderivative of $\sin u$, for any constant C whatsoever. This observation is true in general. That is, if F is an antiderivative of a function f, then so is $F + C$, for any constant C. This follows from the addition rule for derivatives:

$$(F + C)' = F' + C' = F' + 0 = f.$$

A caution

It is tempting to claim the converse—that *every* antiderivative of f is equal to $F + C$, for some appropriately chosen value of C. In fact, you will often see this statement written. The statement is true, though, only for continuous functions. If the function f has breaks in its domain, then there will be more antiderivatives than those of the form $F + C$ for a *single* constant C—over each piece of the domain of f, F can be modified by a *different* constant and still yield an antiderivative for f. Exercises 18, 19, and 20 at the end of this section explore this for a couple of cases. If f is continuous, though, $F + C$ will cover all the possibilities, and we sometimes say that $F + C$ is *the* antiderivative of f. For the sake of keeping a compact notation, we will even write this when the domain of f consists of more than

What the '$+\,C$' term really means

one interval. You should understand, though, that in such cases, over each piece F can be modified by a different constant

Antiderivatives of basic functions

For future reference we collect a list of basic functions whose antiderivatives we already know. Remember that each antiderivative in the table can have an arbitrary constant added to it.

function	antiderivative
x^p	$\dfrac{x^{p+1}}{p+1}, \quad p \neq -1$
$1/x$	$\ln x$
$\sin x$	$-\cos x$
$\cos x$	$\sin x$
e^x	e^x
b^x	$\dfrac{b^x}{\ln b}$

All of these antiderivatives are easily verified and could have been derived with at most a little trial and error fiddling to get the right constant. You should notice one incongruity: the function $1/x$ is defined for all $x \neq 0$, but its listed antiderivative, $\ln x$, is only defined for $x > 0$. In exercise 18 (page 697) you will see how to find antiderivatives for $1/x$ over its entire domain.

Two of our basic functions—$\ln x$ and $\tan x$—do not appear in the left column of the table. This happens because there is no simple multiple of a basic function whose derivative is equal to either $\ln x$ or $\tan x$. It turns out that these functions *do* have antiderivatives, though, that can be expressed as more complicated combinations of basic functions. In fact, by differentiating $x \ln x - x$ you should be able to verify that it is an antiderivative of $\ln x$. Likewise, $-\ln(\cos x)$ is an antiderivative of $\tan x$. It would take a long time to stumble on these antiderivatives by inspection or by trial and error. It is the purpose of later sections to develop techniques which will enable us to discover antiderivatives like these quickly and efficiently. In particular, the antiderivative of $\ln x$ is derived in chapter 11.3 on page 712, while the antiderivative of $\tan x$ is derived in chapter 11.5 on page 744.

There are a couple of other functions that don't appear in the above table whose antiderivatives are needed frequently enough that they should become part of your repertoire of elementary functions that you recognize immediately:

function	antiderivative
$\dfrac{1}{\sqrt{1-x^2}}$	$\arcsin x$
$\dfrac{1}{1+x^2}$	$\arctan x$

The antiderivatives are inverse trigonometric functions, which we've had no need for until now. We introduce them immediately below. They are examples of functions that occur more often for their antiderivative properties than for themselves. Note that the derivatives of the inverse trigonometric functions have no obvious reference to trigonometric relations. In fact, they often occur in settings where there are no triangles or periodic functions in sight. Let's see how the derivatives of these inverse functions are derived.

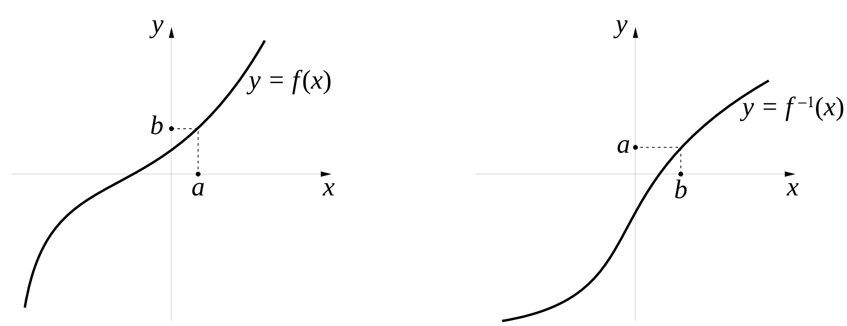
Inverse Functions

We discussed inverse functions in chapter 4.4. Here's a quick summary of the main points made there. Two functions f and g are **inverses** if

$$\begin{aligned} f(g(a)) &= a, \\ \text{and} \quad g(f(b)) &= b, \end{aligned}$$

for every a in the domain of g and every b in the domain of f. It follows that the *range* of f is the same as the *domain* of g, and vice versa. We write $g = f^{-1}$ to indicate that g is the inverse of f, and $f = g^{-1}$ to indicate that f is the inverse of g.

The graphs of $y = f(x)$ and $y = g(x)$ are mirror reflections about the line $y = x$. As the figure below shows, the mirror image of a point with coordinates (a, b) is the point with coordinates (b, a).



This connection between the graphs is a direct translation of the definitions into graphical language, since

$$\begin{aligned}(a, b) \text{ is on the graph of } y = g(x) &\iff g(a) = b \qquad \text{(definition of graph)}\\ &\iff f(b) = a \qquad \text{(definition of inverse)}\\ &\iff (b, a) \text{ is on the graph of } y = f(x).\end{aligned}$$

Because of the connection between the graphs, it follows immediately that if the graph of f is locally linear at the point (b, a) with slope m, then the graph of g will be locally linear at the point (a, b) with slope $1/m$. Algebraically, this is expressed as

$$g'(a) = 1/f'(b),$$

where $a = f(b)$ and $b = g(a)$. We get same result by differentiating the expression $f(g(x)) = x$, using the chain rule:

$$1 = x' = (f(g(x))' = f'(g(x))g'(x),$$

and therefore

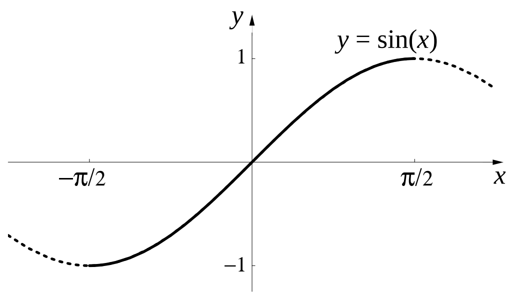
$$g'(x) = \frac{1}{f'(g(x))}$$

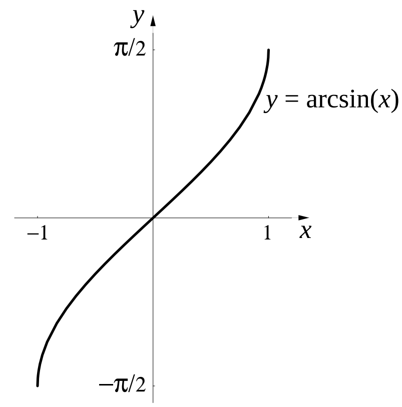
for any value of x for which $g(x)$ is defined.

Inverse trigonometric functions

The arcsine function In the discussion in chapter 4 we saw that a function has an inverse only when it is **one-to-one**, so if we want an inverse, we often have to restrict the domain of a function to a region where it is one-to-one. This is certainly the case with the sine function, which takes the same value infinitely many times. The standard choice of domain on which the sine function is one-to-one is $[-\pi/2, \pi/2]$. Over this interval the sine function increases from -1 to 1. We can then define an inverse function, which we call the **arcsine function**, written $\arcsin x$, whose domain is the interval $[-1, 1]$, and whose range is $[-\pi/2, \pi/2]$. Since the sine function is strictly increasing on its domain, the arcsine function will be strictly increasing on its domain as well—do you see why this has to be?

Another notation for the arcsine function is $\sin^{-1} x$; this form commonly appears on one of the buttons on a calculator.





To find the derivative of the arcsine function, let $f(x) = \sin x$, and let $g(x) = \arcsin x$. Then by the remarks above, we have

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\cos(g(x))} = \frac{1}{\cos(\arcsin x)}.$$

The function $\cos(\arcsin x)$ can be expressed in another form, which we will obtain two ways, one algebraic, the other geometric. Both perspectives are useful.

The algebraic approach. Recall that for any input t, $\sin^2 t + \cos^2 t = 1$. This can be solved for $\cos t$ as $\cos t = \pm\sqrt{1 - \sin^2 t}$. That is, the cosine of anything is the square root of 1 minus the square of the sine of that input, with a possible minus sign needed out front, depending on the context. Since the output of the arcsine function lies in the range $[-\pi/2, \pi/2]$, and the cosine function is positive (or 0) for numbers in this interval, it follows that $\cos(\arcsin x) \geq 0$ for any value of x in the domain of the arcsine function. Therefore,

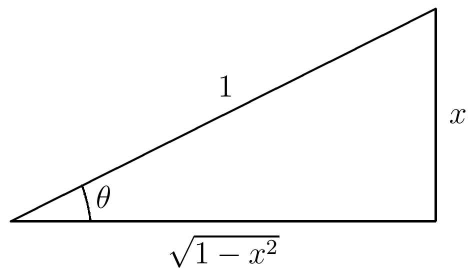
$$\cos(\arcsin x) = \sqrt{1 - \sin^2(\arcsin x)} = \sqrt{1 - (\sin(\arcsin x))^2} = \sqrt{1 - x^2},$$

since $\sin(\arcsin x) = x$ by definition of inverse functions. It follows that

$$(\arcsin x)' = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1 - x^2}},$$

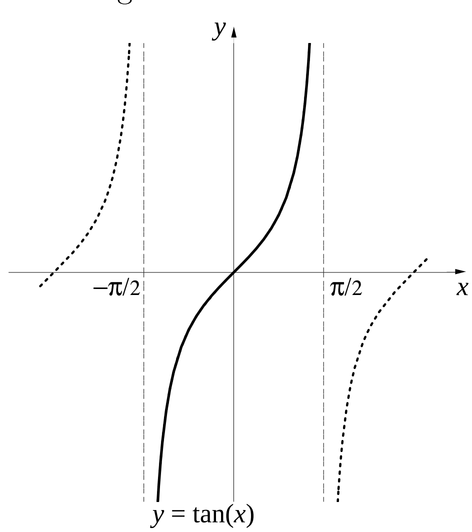
as we indicated above on page 684.

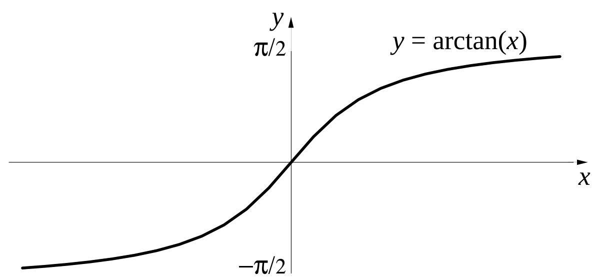
The geometric approach Introduce a new variable $\theta = \arcsin x$, so that $x = \sin\theta$. We can represent these relationships in the following picture:



Notice that we have labelled the side opposite the angle θ as x and the hypotenuse as 1. This ensures that $\sin\theta = x$. By the Pythagorean theorem, the remaining side must then be $\sqrt{1-x^2}$. From this picture it is then obvious that $\cos(\arcsin x) = \cos\theta = \sqrt{1-x^2}/1 = \sqrt{1-x^2}$, as before.

The arctangent function To get an inverse for the tangent function, we again need to limit its domain. Here the standard choice is to restrict it to the interval $(-\pi/2, \pi/2)$ (*not* including the endpoints this time, since $\tan(-\pi/2)$ and $\tan(\pi/2)$ aren't defined). Over this domain the tangent function increases from $-\infty$ to $+\infty$. We can then define the inverse of the tangent, called the **arctangent function**, written $\arctan x$, whose domain is the interval $(-\infty, \infty)$, and whose range is $(-\pi/2, \pi/2)$. Again, both functions are increasing over their domains.





To find the derivative, we proceed as we did with $\arcsin x$, letting $f(x) = \tan x$, and $g(x) = \arctan x$. This time we get

$$g'(x) = \frac{1}{f'(g(x))} = \frac{1}{\sec^2(g(x))} = \frac{1}{\sec^2(\arctan x)}.$$

This expression also has a different form; we obtain it from another trigonometric identity, as we did before, deriving the desired result algebraically and geometrically.

Algebraic We start as before with the identity $\sin^2 t + \cos^2 t = 1$. Dividing through by $\cos^2 t$, we get the equivalent identity $\tan^2 t + 1 = \sec^2 t$. That is, the square of the secant of any input is just 1 plus the square of the tangent applied to the same input. In particular,

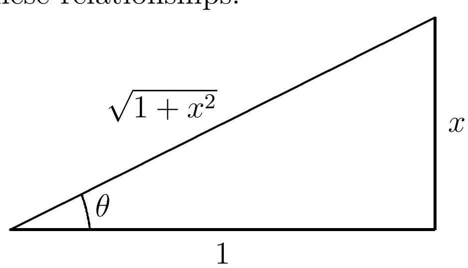
$$\sec^2(\arctan x) = \tan^2(\arctan x) + 1 = (\tan(\arctan x))^2 + 1 = x^2 + 1;$$

since $\tan(\arctan x) = x$. It follows that

$$(\arctan x)' = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1 + x^2},$$

as we indicated above on page 684.

Geometric Let $\theta = \arctan x$, so $x = \tan\theta$. Again we can draw and label a triangle reflecting these relationships:



Notice that this time we have labelled the side opposite the angle θ as x and the adjacent side as 1 to ensure that $\tan\theta = x$. Again by the Pythagorean theorem, the hypotenuse must be $1 + x^2$. From this picture it is then obvious that $\sec(\arctan x) = \sec\theta = (\sqrt{1+x^2})/1 = \sqrt{1+x^2}$, so $\sec^2(\arctan x) = 1 + x^2$ again.

Notation

According to the fundamental theorem of calculus (see chapter 6.4), every accumulation function

$$A(x) = \int_a^x f(t)\,dt$$

is an antiderivative of the function f, no matter at what point $t = a$ the accumulation begins—so long as the function is defined over the entire interval from $t = a$ to $t = x$. (This is an important caution when dealing with functions like $1/x$, for instance, for which the integral from, say, -1 to 1 makes no sense.) In other words, the expression

$$\int_a^x f(t)\,dt$$

represents an antiderivative of f. The influence of the fundamental theorem is so pervasive that this expression—with the "limits of integration" a and x omitted—is used to denote an antiderivative:

Write an antiderivative as an integral

Notation: The antiderivative of f is $\int f(x)\,dx$.

With this new notation the antiderivatives we have listed so far can be written in the following form.

$$\int x^p\,dx = \frac{1}{p+1}\,x^{p+1} + C \qquad (\,p \neq -1)$$
$$\int \frac{1}{x}\,dx = \ln x + C$$
$$\int \sin x\,dx = -\cos x + C$$
$$\int \cos x\,dx = \sin x + C$$
$$\int e^x\,dx = e^x + C$$
$$\int b^x\,dx = \frac{1}{\ln b\,b^x} + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$

The basic antiderivatives again

The integration sign $\int$ now has two distinct meanings. Originally, it was used to describe the *number*

The *definite* integral is a number...

$$\int_a^b f(x)\,dx,$$

which was always calculated as the limit of a sequence of Riemann sums. Because this integral has a definite numerical value, it is called the **definite integral**. In its new meaning, the integration sign is used to describe the antiderivative

$$\int f(x)\,dx,$$

...while the *indefinite* integral is a function

which is a *function*, not a number. To contrast the new use of $\int$ with the old, and to remind us that the new expression is a variable quantity, it is called the **indefinite integral**. The function that appears in either a definite or an indefinite integral is called the **integrand**. The terms "antiderivative" and "indefinite integral" are completely synonymous. We will tend to use the former term in general discussions, using the latter term when focusing on the process of finding the antiderivative.

Because an indefinite integral represents an antiderivative, the process of finding an antiderivative is sometimes called **integration**. Thus the term *integration*, as well as the symbol for it, has two distinct meanings.

Using Antiderivatives

According to the fundamental theorem, we can use an *indefinite* integral to find the value of a *definite* integral—and this largely explains the importance of antiderivatives. In the language of indefinite integrals, the statement of the fundamental theorem in the box on page 410 takes the following form.

$$\int_a^b f(x)\,dx = F(b) - F(a), \text{ where } F(x) = \int f(x)\,dx.$$

Example 1 Find $\int_1^4 x^2\,dx$. We have that

$$\int x^2\,dx = \tfrac{1}{3}x^3 + C;$$

it follows that

$$\int_1^4 x^2\,dx = \tfrac{1}{3}4^3 + C - \left(\tfrac{1}{3}1^3 + C\right) = \frac{64}{3} + C - \frac{1}{3} - C = 21.$$

Example 2 Find $\int_0^{\pi/2} \cos t\, dt$. This time the indefinite integral we need is

$$\int \cos t\, dt = \sin t + C.$$

The value of the definite integral is therefore

$$\int_0^{\pi/2} \cos t\, dt = \sin \pi/2 + C - (\sin 0 + C) = 1 + C - 0 - C = 1.$$

In each example the two appearances of C cancel each other. Thus C does not appear in the final result. This implies that it does not matter which value of C we choose to do the calculation. Usually, we just take $C = 0$.

The calculation doesn't depend on C, so take $C = 0$

Notation: Because expressions like $F(b) - F(a)$ occur often when we are using indefinite integrals, we use the abbreviation

$$F(b) - F(a) = F(x)\Big|_a^b.$$

Thus,

$$\int_1^4 x^2\, dx = \frac{x^3}{3}\Big|_1^4 = \frac{64}{3} - \frac{1}{3} = 21.$$

There are clear advantages to using antiderivatives to evaluate definite integrals: we get exact values and we avoid many lengthy calculations. The difficulty is that the method works only if we can find a formula for the antiderivative.

There are several reasons why we might not find the formula we need. For instance, the antiderivative we want may be a function we have never seen before. The function $\arctan x$ is an example.

We may not recognize the antiderivative. . .

Even if we have a broad acquaintance with functions, we may still not be able to find the formula for a given antiderivative. The reason is simple: for most functions we can write down, the antiderivative is just not among the basic functions of calculus. For example, none of the basic functions, in any form or combination, equals

. . . and there may even be no formula for it

$$\int e^{x^2} dx \qquad \text{or} \qquad \int \frac{\sin x}{x}\, dx.$$

This does not mean that e^{x^2}, for example, has no antiderivative. On the contrary, the accumulation function

$$\int_0^x e^{t^2}\,dt$$

is an antiderivative of e^{x^2}. It can be evaluated, graphed, and analyzed like any other function. What we lack is a *formula* for this antiderivative in terms of the basic functions of calculus.

Finding Antiderivatives

In the rest of this chapter we will be deriving a number of statements involving antiderivatives. It is important to remember what such statements mean.

> The following statements are completely equivalent:
>
> $$\int f(x)\,dx = F(x) + C \qquad \text{and} \qquad F' = f.$$

We differentiate to verify a statement about antiderivatives

In other words, a statement about antiderivatives can be verified by looking at a statement about derivatives. If it is claimed that the antiderivative of f is F, you check the statement by seeing if it is true that $F' = f$. This was how we verified the elementary antiderivatives we've considered so far. Another way of expressing these relationships is

$$\left(\int f(x)\,dx\right)' = f(x) \quad \text{and} \quad \int F'(x)\,dx = F(x) + C$$

for any functions f and F.

This duality between statements about derivatives and statements about antiderivatives also holds when applied to more general statements. Many of the most useful techniques for finding antiderivatives are based on converting the general rules for taking derivatives of sums, products, and chains into equivalent antiderivative form. We'll start with the simplest combinations, which involve a **function multiplied by a constant** and a **sum of two functions**.

The constant multiple and addition rules

derivative form	antiderivative form
$(k \cdot F)' = k \cdot F'$	$\displaystyle\int k \cdot f\,dx = k \cdot \int f\,dx$
$(F + G)' = F' + G'$	$\displaystyle\int (f + g)\,dx = \int f\,dx + \int g\,dx$

Let's verify these rules. Set $\int f = F$ and $\int g = G$. Then the first rule is claiming that $\int (k \cdot f) = k \cdot F$—the antiderivative of a constant times a function equals the constant times the antiderivative of the function. To verify this, we have to show that when we take the derivative of the right-hand side, we get the function under the integral on the left-hand side. But $(k \cdot F)' = k \cdot F'$ (by the derivative rule), which is just $k \cdot f$, which is what we had to show.

Similarly, to show that $\int (f + g) = F + G$—the antiderivative of the sum of two functions is the sum of their separate antiderivatives—we differentiate the right-hand side and find $(F + G)' = F' + G'$ (by the derivative rule for sums) which is just $f + g$, so the rule is true.

Example 3 This example illustrates the use of both the addition and the constant multiple rules.

$$\begin{aligned}\int (7e^x + \cos x)\,dx &= \int 7e^x\,dx + \int \cos x\,dx \\ &= 7\int e^x\,dx + \int \cos x\,dx \\ &= 7e^x + \sin x + C.\end{aligned}$$

To verify this answer, you should take the derivative of the right-hand side to see that it equals the integrand on the left-hand side.

In the following sections we will develop the antidifferentiation rules that correspond to the product rule and to the chain rule. They are called *integration by parts* and *integration by substitution*, respectively.

Trial and error

While antiderivatives can be hard to find, they are easy to check. This makes "trial and error" a good strategy. In other words, if you don't immediately see what the antiderivative should be, but the function doesn't look too bad, try guessing. When you differentiate your guess, what you see may lead you to a better guess.

Example 4 Find
$$F(x) = \int \cos(3x)\,dx.$$
Since the derivative of $\sin u$ is $\cos u$, it is reasonable to try $\sin(3x)$ as an antiderivative for $\cos(3x)$. Therefore:

$$\text{FIRST GUESS: } F(x) = \sin(3x),$$
$$\text{CHECK: } F'(x) = \cos(3x) \cdot 3 \neq \cos(3x).$$

We wanted $F' = \cos(3x)$ but we got $F' = 3\cos(3x)$. The chain rule gave us an extra—and unwanted—factor of 3. We can compensate for that factor by multiplying our first guess by 1/3. Then

$$\text{SECOND GUESS: } F(x) = \tfrac{1}{3}\sin(3x),$$
$$\text{CHECK: } F'(x) = \tfrac{1}{3}\cos(3x) \cdot 3 = \cos(3x).$$

Thus $\int \cos(3x)\,dx = \frac{1}{3}\sin(3x)$.

Tables of Integrals

Because indefinite integrals are difficult to calculate, reference manuals in mathematics and science often include tables of integrals. There are sometimes many hundreds of individual formulas, organized by the type of function being integrated. A modest selection of such formulas can be found at the back of this book. You should take some time to learn how these tables are arranged and get some practice using them. You should also check some of the more unlikely looking formulas by differentiating them to see that they really are the antiderivatives they are claimed to be.

Computers are having a major impact on integration techniques. We saw in the last chapter that any continuous function—even one given by the output from some laboratory recording device or as the result of a numerical technique like Euler's method—can be approximated by a polynomial (usually using lots of computation!), and the antiderivative of a polynomial is easy to find.

Integration can now be done quickly and efficiently by computer software

Moreover, computer software packages which can find any existing formula for a definite integral are becoming widespread and will probably have a profound impact on the importance of integration techniques over the next several years. Just as hand-held calculators have rendered obsolete many traditional arts—like using logarithms for performing multiplications or knowing how to interpolate in trig tables—there is likely to be a decreased importance placed on humans' being adept at some of the more esoteric integration techniques. While some will continue to derive pleasure from becoming proficient in these skills, for most users it will generally be much faster, and more accurate, to use an appropriate software package. Nevertheless, for those going on in mathematics and the physical sciences, it will still to be useful to be able to perform some of the simpler integrations by hand reasonably rapidly. The subsequent sections of this chapter develop the most commonly needed techniques for doing this.

Exercises

1. What is the inverse of the function $y = 1/x$? Sketch the graph of the function and its inverse.

2. What is the inverse of the function $y = 1/x^3$? Sketch the graph of the the function and its inverse. Do the same for $y = 3x - 2$.

3. a) Let $\theta = \arctan x$. Then $\tan\theta = x$. Refer to the picture on page 688 showing the relationship between x and θ. Use this drawing to show that $\arctan(x) + \arctan(1/x)$ is constant—that is, its value doesn't depend on x. What is the value of the constant?

b) Use part (a), and the derivative of $\arctan x$, to find the derivative of $\arctan(1/x)$.

c) Use the chain rule to verify your answer to part (b).

4. The logarithm for the base b is defined as the inverse to the exponential function with base b:

$$y = \log_b x \qquad \text{if} \qquad x = b^y.$$

Using only the fact that $dx/dy = \ln b \cdot b^y$, deduce the formula

$$\frac{dy}{dx} = \frac{1}{\ln b} \cdot \frac{1}{x}.$$

Note: this is purely an algebra problem; you don't need to invoke any differentiation rules.

5. a) Define $\arccos x$, the inverse of the cosine function. Be sure to limit the domain of the cosine function to an interval on which it is one-to-one.

b) Sketch the graph $y = \arccos x$. How did you limit the range of y?

c) Determine dy/dx.

6. a) If $\theta = \arcsin x$, refer to the picture on page 687 reflecting the relation between θ and x. Using this picture, proceed as in problem 3 to to show that the sum $\arcsin x + \arccos x$ is constant. What is the value of the constant?

b) Use part (a), and the derivative of $\arcsin x$, to determine the derivative of $\arccos x$. Does this result agree with what you got in the last exercise?

7. Find a formula for $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$.

8. Verify that the antiderivatives given in the table on page 689 are correct.

9. Find an antiderivative of each of the following functions. Don't hesitate to use the "trial and error" method of Example 4 above.

3	$5t$	$-5t$	$3-5t$
$7x^4$	$\dfrac{1}{y^3}$	e^{2z}	$u+\dfrac{1}{u}$
$(1+w^3)^2$	$\cos(5v)$	$x^9+5x^7-2x^5$	$\sin t\,\cos t$

10. Find a formula for each of the following indefinite integrals.

a) $\displaystyle\int 3x\,dx$

b) $\displaystyle\int 3u\,du$

c) $\displaystyle\int e^z\,dz$

d) $\displaystyle\int 5t^4\,dt$

e) $\displaystyle\int 7y + \frac{1}{y}\,dy$

f) $\displaystyle\int 7y - \frac{4}{y^2}\,dy$

g) $\displaystyle\int 5\sin w - 2\cos w\,dw$

h) $\displaystyle\int dx$

i) $\displaystyle\int e^{z+2}\,dz$

j) $\displaystyle\int \cos(4x)\,dx$

k) $\displaystyle\int \frac{5}{1+r^2}\,dr$

l) $\displaystyle\int \frac{1}{\sqrt{1-4s^2}}\,ds$

11. a) Find an antiderivative $F(x)$ of $f(x) = 7$ for which $F(0) = 12$.

b) Find an antiderivative $G(x)$ of $f(x) = 7$ for which $G(3) = 1$.

c) Do $F(x)$ and $G(x)$ differ by a constant? If so, what is the value of that constant?

12. a) Find an antiderivative $F(t)$ of $f(t) = t + \cos t$ for which $F(0) = 3$.

b) Find an antiderivative $G(t)$ of $f(t) = t + \cos t$ for which $G(\pi/2) = -5$.

c) Do $F(t)$ and $G(t)$ differ by a constant? If so, what is the value of that constant?

13. Find an antiderivative of the function $a + by$ when a and b are fixed constants.

14. a) Verify that $(1 + x^3)^{10}$ is an antiderivative of $30x^2(1 + x^3)^9$.

b) Find an antiderivative of $x^2(1 + x^3)^9$.

c) Find an antiderivative of $x^2 + x^2(1 + x^3)^9$.

15. a) Verify that $x \ln x$ is an antiderivative of $1 + \ln x$.

b) Find an antiderivative of $\ln x$. [Do you see how you can use part (a) to find this antiderivative?]

16. Recall that $F(y) = \ln(y)$ is an antiderivative of $1/y$ for $y > 0$. According to the text, *every* antiderivative of $1/y$ over this domain must be of the form $\ln(y) + C$ for an appropriate value of C.

a) Verify that $G(y) = \ln(2y)$ is also an antiderivative of $1/y$.

b) Find C so that $\ln(2y) = \ln(y) + C$.

17. Verify that $-\cos^2 t$ is an antiderivative of $2 \sin t \, \cos t$. Since you already know $\sin^2 t$ is an antiderivative, you should be able to show

$$-\cos^2 t = \sin^2 t + C$$

for an appropriate value of C. What is C?

18. Since the function $\ln x$ is defined only when $x > 0$, the equation

$$\int \frac{1}{x}\, dx = \ln x + C$$

applies only when $x > 0$. However, the integrand $1/x$ is defined when $x < 0$ as well. Therefore, it makes sense to ask what the integral (i.e., antiderivative) of $1/x$ is when $x < 0$.

a) When $x < 0$ then $-x > 0$ so $\ln(-x)$ is defined. In these circumstances, show that $\ln(-x)$ is an antiderivative of $1/x$.

b) Now put these two "pieces" of antiderivative together by defining the function

$$F(x) = \begin{cases} \ln(-x) & \text{if } x < 0, \\ \ln(x) & \text{if } x > 0 \end{cases}$$

Sketch together the graphs of the functions $F(x)$ and $1/x$ in such a way that it is clear that $F(x)$ is an antiderivative of $1/x$.

c) Explain why $F(x) = \ln|x|$. For this reason a table of integrals often contains the entry

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

d) Every function $\ln|x| + C$ is an antiderivative of $1/x$, but there are even more. As you will see, this can happen because the domain of $1/x$ is broken into two parts. Let

$$G(x) = \begin{cases} \ln(-x) & \text{if } x < 0, \\ \ln(x) + 1 & \text{if } x > 0. \end{cases}$$

Sketch together the graphs of the functions $G(x)$ and $1/x$ in such a way that it is clear that $G(x)$ is an antiderivative of $1/x$.

e) Explain why there is no value of C for which

$$\ln|x| + C = G(x).$$

This shows that the functions $\ln|x| + C$ do not exhaust the set of antiderivatives of $1/x$.

f) Construct two more antiderivatives of $1/x$ and sketch their graphs. What is the general form of the new antiderivatives you have constructed? (A suggestion: you should be able to use two separate constants C_1 and C_2 to describe the general form.)

19. On page 683 of the text there is an antiderivative for the tangent function:

$$\int \tan x\, dx = -\ln(\cos x).$$

However, this is not defined when x makes $\cos x$ either zero or negative.

a) How many separate intervals does the domain of $\tan x$ break down into?

b) For what values of x is $\cos x$ equal to zero, and for what values is it negative?

c) Modify the antiderivative $-\ln(\cos x)$ so that it *is* defined when $\cos x$ is negative. (How is this problem with the logarithm function treated in the previous question?)

d) In a typical table of integrals you will find the statement

$$\int \tan x\, dx = -\ln|\cos x| + C.$$

Explain why this does not cover all the possibilities.

e) Give a more precise expression for $\int \tan x\, dx$, modelled on the way you answered part (f) of problem 18. How many different constants will you need?

f) Find a function G that is an antiderivative for $\tan x$ and that also satisfies the following conditions:

$$G(0) = 5, \qquad G(\pi) = -23, \qquad G(17\pi) = 197.$$

20. In the table on page 689 the antiderivative of x^p is given as

$$\frac{1}{p+1} x^{p+1} + C.$$

For some values of p this is correct, with only a single constant C needed. For other values of p, though, the domain of x^p will consist of more than one piece, and $\frac{1}{p+1} x^{p+1}$ can be modified by a different constant over each piece. For what values of p does this happen?

21. Find $F'(x)$ for the following functions. In parts (a), (b), and (d) do the problems two ways: by finding an antiderivative, and by using the fundamental theorem to get the answer without evaluating an antiderivative. Check that the answers agree.

a) $F(x) = \int_0^x (t^2 + t^3)\, dt.$

b) $F(x) = \int_1^x \frac{1}{u}\,du.$

c) $F(x) = \int_1^x \frac{v}{1+v^3}\,dv.$

d) $F(x) = \int_0^{x^2} \cos t\,dt.$

e) $F(x) = \int_1^{x^2} \frac{v}{1+v^3}\,dv.$ [Hint: let $u = x^2$ and use the chain rule.]

Comment: It may seem that parts (c) and (e) are more difficult than the others. However, there is a way to apply the fundamental theorem of calculus here to get answers to parts (c) and (e) quickly and with little effort.

22. Consider the two functions

$$F(x) = \sqrt{1+x^2} - 1 \quad \text{and} \quad G(x) = \int_0^x \frac{t}{\sqrt{1+t^2}}\,dt.$$

a) Show that F and G both satisfy the initial value problem

$$y' = \frac{x}{\sqrt{1+x^2}}, \qquad y(0) = 0.$$

b) Since an initial value problem typically has a *unique* solution, F and G should be equal. Assuming this, determine the exact value of the following definite integrals.

$$\int_0^1 \frac{t}{\sqrt{1+t^2}}\,dt, \qquad \int_0^2 \frac{t}{\sqrt{1+t^2}}\,dt, \qquad \int_0^5 \frac{t}{\sqrt{1+t^2}}\,dt.$$

23. The connection between integration and differentiation that is provided by the fundamental theorem of calculus makes it possible to determine an integral by solving a differential equation. For example, the accumulation function

$$A(x) = \int_0^x e^{-t^2}\,dt$$

is the solution to the initial value problem

$$y' = e^{-x^2}, \qquad y(0) = 0.$$

Therefore, $A(x)$ can be found by solving the differential equation. As you have seen, Euler's method is a useful way to solve differential equations.

a) Use either a program (e.g., PLOT) or a differential equation solver on a computer to get a graphical solution $A(x)$ to the initial value problem above.

b) Sketch the graph of $y = A(x)$ over the domain $0 \le x \le 4$.

c) Your graph should increase from left to right. How can you tell this even before you see the computer output?

d) Your graph should level off as x increases. Determine $A(5)$, $A(10)$, $A(30)$. (Approximations provided by the computer are adequate here.)

e) Estimate $\lim_{x\to\infty} A(x)$. [The *exact* value is $\sqrt{\pi}/2$.]

f) Determine $\int_0^1 e^{-t^2} dt$ and $\int_0^2 e^{-t^2} dt$.

g) Determine $\int_1^2 e^{-t^2} dt$.

24. Find the area under the curve $y = x^3 + x$ for x between 1 and 4. (See chapter 6.3.)

25. Find the area under the curve $y = e^{3x}$ for x between 0 and $\ln 3$.

26. The **average value** of the function $f(x)$ on the interval $a \le x \le b$ is the integral

$$\frac{1}{b-a}\int_a^b f(x)\,dx.$$

(See the discussion of average value in chapter 6, pages 397–399.)

a) Find the average value of each of the functions $y = x$, x^2, x^3, and x^4 on the interval $0 \le x \le 1$.

b) Explain, using the graphs of $y = x$ and $y = x^2$, why the average value of x^2 is less than the average value of x on the interval $[0, 1]$.

27. a) What are the maximum, minimum, and average values of the function

$$f(x) = x + 2e^{-x}$$

on the interval $[0, 3]$?

b) Sketch the graph of $y = f(x)$ on the interval $[0, 3]$. Draw the line $y = \mu$, where μ is the average value you found in part (a).

c) For which x does the graph of $y = f(x)$ lie above the line $y = \mu$, and for which x does it lie below the line? The region between the graph and the line has two parts—one is above the line (and below the graph) and the other is below the line (and above the graph). Shade these two regions and compare their areas: which is larger? [The two are equal.]

11.2 Integration by Substitution

In the preceding section we converted a couple of general rules for differentiation—the rule for the derivative of a constant times a function and the rule for the derivative of the sum of two functions—into equivalent rules in integral form. In this section we will develop the integral form of the chain rule and see some of the ways this can be used to find antiderivatives.

Suppose we have functions F and G, with corresponding derivatives f and g. Then the chain rule says

$$(F(G(x)))' = F'(G(x))\,G'(x) = f(G(x))\,g(x).$$

If we now take the indefinite integral of these equations, we get

$$F(G(x)) + C = \int (F(G(x)))'\,dx = \int f(G(x))\,g(x)\,dx,$$

where C can be any constant.

The integral form of the chain rule

Reversing these equalities to get a statement about integrals, we obtain:

$$\boxed{\int \boldsymbol{f(G(x))\,g(x)\,dx = F(G(x)) + C.}}$$

This somewhat unpromising expression turns out to be surprisingly useful. Here's how: Suppose we want to find an indefinite integral, and see in the integrand a *pair* of functions G and g, where $G' = g$ and where $g(x)$ can be factored out of the integrand. We then find a function f so that the integrand can be written in the form $f(G(x))g(x)$. Now we only have to find an antiderivative for f. Once we have such an antiderivative, call it F, then the solution to our original problem will be $F(G(x))$. Thus, while our original antiderivative problem is not yet solved, it has been reduced to a

Reduction methods transform problems into equivalent, simpler, problems different, simpler antiderivative problem that, when all goes well, will be easier to evaluate. Such **reduction methods** are typical of many integration techniques. We will see other examples in the remainder of this chapter.

Example 1 Suppose we try to find a formula for the integral

$$\int 3x^2 \left(1 + x^3\right)^7 dx.$$

One way would be to multiply out the expression $\left(1 + x^3\right)^7$, making the integrand a polynomial with many separate terms of different degrees. (The highest degree would be 23; do you see why?) We could then carry out the integration "term-by-term," using the rules for sums and constant multiples of powers of x that were given in the previous section. But this is tedious—even excruciating.

Instead, notice that the expression $3x^2$ is the derivative of $1 + x^3$. If we let $G(x) = 1 + x^3$ and $g(x) = 3x^2$, we can then write the integrand as

$$3x^2 \left(1 + x^3\right)^7 = (G(x))^7 g(x).$$

Now $(G(x))^7$ is clearly just $f(G(x))$ where f is the function which raises its input to the 7th power—$f(x) = x^7$ for any input x. But we recognize f as an elementary function whose antiderivative we can write down immediately as $F(x) = \frac{1}{8}x^8$. Thus the solution to our original problem will be

$$\int 3x^2 \left(1 + x^3\right)^7 dx = F(G(x)) + C = \frac{1}{8}\left(1 + x^3\right)^8 + C.$$

As with any integration problem, we can check our answer by taking the derivative of the right-hand side to see if it agrees with the integrand on the left. You should do this whenever you aren't quite sure of your technique (or your arithmetic!).

Note that the term $3x^2$ that appeared in the integrand above was essential for the procedure to work. The integral

$$\int \left(1 + x^3\right)^7 dx$$

cannot be found by substitution, even though it appears to have a simpler form. (Of course, the integral *can* be found by multiplying out the integrand.)

A compact notation for expressing the integral form of the chain rule

Using differential notation So far the symbol dx (the **differential** of x) under the integral sign has simply been an appendage, tagging along to suggest the Δx portion of the Riemann sums approximating definite integrals. It turns out we can take advantage of this notation to use the integral form of the chain rule more compactly.

Instead of naming the functions G and g as above, we introduce a new variable $u = G(x)$. Then

$$\frac{du}{dx} = G'(x) = g(x),$$

and it is suggestive to multiply out this "quotient" to get

$$du = g(x)\,dx.$$

While this is reminiscent of the microscope equation we met in chapter 3, and 18th century mathematicians took this equation seriously as a relation between two "infinitesimally small" quantities dx and du, we will view it only as a convenient mnemonic device. To see how this simplifies computations, reconsider the previous example. If we let $u = 1 + x^3$, then $du = 3x^2\,dx$, so we can write

$$3x^2(1+x^3)^7\,dx = (\underbrace{1+x^3}_{u})^7\,\underbrace{3x^2\,dx}_{du} = u^7\,du.$$

It follows that

$$\begin{aligned}\int 3x^2\left(1+x^3\right)^7 dx &= \int u^7\,du\\ &= \tfrac{1}{8}u^8 + C\\ &= \tfrac{1}{8}\left(1+x^3\right)^8 + C,\end{aligned}$$

as before. We have arrived at the same answer without having to introduce the cumbersome language of all the auxiliary functions—we simply **substituted** the variable u for a certain expression in x (which we called $G(x)$ before), and replaced $G'(x)dx$ by du. For this reason this technique is called **integration by substitution**. You should always be clear, though, that integration by substitution is just the integral form of the chain rule, a relationship that becomes clear whenever you check the answer substitution gives you.

Example 2 Can we use the method of substitution to find

$$\int \frac{e^{5x}}{6+e^{5x}}\,dx?$$

The numerator is almost the derivative of the denominator. This suggests we let $G(x) = 6 + e^{5x}$, giving $g(x) = G'(x) = 5e^{5x}$. Since we need to be able to factor $g(x)$ out of the integrand, we multiply numerator and denominator by 5 to get

$$\begin{aligned}\int \frac{e^{5x}}{6+e^{5x}}\,dx &= \int \frac{1}{5}\cdot\frac{1}{6+e^{5x}}5e^{5x}\,dx\\ &= \int \frac{1}{5}\cdot\frac{1}{G(x)}g(x)\,dx\\ &= \frac{1}{5}\int f(G(x))g(x)\,dx,\end{aligned}$$

where f is just the reciprocal function—$f(x) = 1/x$. But an antiderivative for f is just $\ln x$, so the desired antiderivative is just

$$\frac{1}{5}F(G(x)) + C = \frac{1}{5}\ln(6+e^{5x}) + C.$$

As usual, you should check this answer by differentiating to see that you really do get the original function.

Now let's see how this works using differential notation. If we set $u = 6 + e^{5x}$, then

$$\frac{du}{dx} = 5e^{5x} \qquad \text{and} \qquad du = 5e^{5x}\,dx.$$

Again we insert a factor of 5 in the numerator and an identical one in the denominator to balance it. Substitutions for u and du then yield the following:

$$\begin{aligned}\int \frac{1}{5}\cdot\frac{5e^{5x}}{6+e^{5x}}\,dx &= \frac{1}{5}\int\frac{5e^{5x}\,dx}{6+e^{5x}}\\ &= \frac{1}{5}\int\frac{du}{u}\\ &= \frac{1}{5}\ln(u) + C\\ &= \frac{1}{5}\ln(6+e^{5x}) + C,\end{aligned}$$

as before.

The basic structure

The two examples above have the same structure. In both, a certain function of x is selected and called u; part of the integrand, namely $u'\,dx$, becomes du, the rest becomes one of the basic functions of u. Specifically:

integrand	u	du	function of u
$3x^2\,(1+x^3)^7\,dx$	$1+x^3$	$3x^2\,dx$	u^7
$\dfrac{e^{5x}}{6+e^{5x}}\,dx$	$6+e^{5x}$	$5e^{5x}\,dx$	$\dfrac{1}{5}\cdot\dfrac{1}{u}$

Note that you may have to do a bit of algebraic reshaping of the integrand to cast it in the proper form. For example, we had to insert a factor of 5 to the numerator of the second example to make the numerator be the derivative of the denominator. There is no set routine to be followed to find an antiderivative most efficiently, or even any way to know whether a particular method will work until you try it. Success comes with experience and a certain amount of intelligent fiddling until something works out.

Example 3 The method of substitution is useful in simple problems, too. Consider

$$\int \cos(3t)\,dt.$$

If we set $u = 3t$, then $du = 3\,dt$ and

$$\begin{aligned}\int \cos(3t)\,dt &= \int \cos(u)\cdot\tfrac{1}{3}\,du\\ &= \tfrac{1}{3}\int \cos(u)\,du\\ &= \tfrac{1}{3}\sin(u) + C\\ &= \tfrac{1}{3}\sin(3t) + C.\end{aligned}$$

Substitution in Definite Integrals

Until now we have been using the technique of substitution to find antiderivatives—that is, to evaluate *indefinite integrals*. Refer back to the integral form of the chain rule given in the box on page 702, and see what happens when we use this equation to evaluate a *definite integral*. Suppose we want to evaluate

$$\int_a^b f(G(x))\,g(x)\,dx.$$

We know that $F(G(x))$ is an antiderivative, so by the fundamental theorem we have

$$\int_a^b f(G(x))\,g(x)\,dx = F(G(x))\big|_a^b = F(G(b)) - F(G(a)).$$

Now suppose we make the substitution $u = G(x)$ and $du = g(x)\,dx$. Then as x goes from a to b, u will go from $G(a)$ to $G(b)$. Moreover, we have

$$\int_{G(a)}^{G(b)} f(u)\,du = F(u)\big|_{G(a)}^{G(b)} = F(G(b)) - F(G(a)),$$

so the two definite integrals have the same value. In other words,

If we make the substitution $u = G(x)$, then

$$\int_a^b f(G(x))\,g(x)\,dx = F(G(b)) - F(G(a)) = \int_{G(a)}^{G(b)} f(u)\,du.$$

This means that to evaluate a definite integral by substitution, we can do everything in terms of u. We don't ever need to find an antiderivative for the original integrand in terms of x or use the original limits of integration.

Example 4 Consider the definite integral

$$\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin x}.$$

We can evaluate this integral by making the substitution $u = 1 + \sin x$, and $du = \cos x\,dx$. Moreover, as x goes from 0 to $\pi/2$, u goes from 1 to 2. Therefore

$$\int_0^{\pi/2} \frac{\cos x\,dx}{1+\sin x} = \int_1^2 \frac{du}{u} = \ln u\big|_1^2 = \ln 2.$$

Check that this is the same answer you would have gotten if you had expressed the antiderivative $\ln u$ in terms of x and evaluated the result at the limits on the original integral.

Example 5 Evaluate

$$\int_0^1 6x^2(1+x^3)^4\,dx.$$

With the substitution $u = 1 + x^3$ and $du = 3x^2\,dx$, as x goes from 0 to 1, u goes from 1 to 2. Our integral thus becomes

$$\int_0^1 6x^2(1+x^3)^4\,dx = \int_0^2 2u^4\,du = \frac{2}{5}u^5\bigg|_1^2 = \frac{64}{5}.$$

Exercises

1. Evaluate the following using substitution. Do parts (a) through (e) in two ways: i. by writing the integrand in the form $f(G(x))g(x)$ (or y or t or whatever the variable is) for appropriate functions f, G, and g, with $G' = g$, and then finding $F = \int f$; and ii. using differential notation. Do the remaining parts in the way you feel most confident.

a) $\displaystyle\int 2y(y^2+1)^{50}\,dy$

b) $\displaystyle\int \sin(5z)\,dz$

c) $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

d) $\displaystyle\int (5t+7)^{50}\,dt$

e) $\displaystyle\int 3u^2\sqrt[3]{u^3+8}\,du$

f) $\displaystyle\int \frac{1}{2v+1}\,dv$

g) $\displaystyle\int \tan x\,dx$

h) $\displaystyle\int \tan^2(x)\sec^2(x)\,dx$

i) $\displaystyle\int \sec(x/2)\tan(x/2)\,dx$

j) $\displaystyle\int \sin(w)\sqrt{\cos(w)}\,dw$

k) $\displaystyle\int \frac{\sin(\sqrt{s})}{\sqrt{s}}\,ds$

l) $\displaystyle\int \sqrt{3-x}\,dx$

m) $\displaystyle\int \frac{dr}{r\ln r}$

n) $\displaystyle\int e^x\sin(1+e^x)\,dx$

o) $\displaystyle\int \frac{y}{1+y^2}\,dy$

p) $\displaystyle\int \frac{w}{\sqrt{1-w^2}}\,dw$

q) $\displaystyle\int \frac{1}{1+4y^2}\,dy$

r) $\displaystyle\int \frac{1}{\sqrt{1-9w^2}}\,dw$

2. Use integration by substitution to find the numerical value of the following. In four of these you should get your answer in two ways: i) by finding an antiderivative for the given integrand, and ii) by using the observation in the box on page 707, comparing the results. You should also check your results for three of the problems by finding numerical estimates for the integrals using RIEMANN.

a) $\displaystyle\int_0^1 \frac{e^s}{e^s+1}\,ds$

b) $\displaystyle\int_0^{\ln e} \frac{e^s}{e^s+1}\,ds$

c) $\displaystyle\int_1^3 \frac{1}{2x+1}\,dx$

d) $\displaystyle\int_{-3}^{-1} \frac{1}{2x+1}\,dx$

e) $\displaystyle\int_0^1 \frac{t}{\sqrt{1+t^2}}\,dt$

f) $\displaystyle\int_0^1 \frac{\sin(\pi\sqrt{t})}{\sqrt{t}}\,dt$

g) $\displaystyle\int_0^2 \frac{1}{1+(x^2/4)}\,dx$

h) $\displaystyle\int_0^{\frac{1}{3}} \frac{1}{\sqrt{1-9y^2}}\,dy$

3. This question concerns the integral $I = \displaystyle\int \sin x \,\cos x\,dx$.

a) Find I by using the substitution $u = \sin x$.

b) Find I by using the substitution $u = \cos x$.

c) Compare your answers to (a) and (b). Are they the same? If not, how do they differ? Since both answers are antiderivatives of $\sin x\,\cos x$, they should differ only by a constant. Is that true here? If so, what is the constant?

d) Now calculate the value of the *definite* integral

$$\int_0^{\pi/2} \sin x\,\cos x\,dx$$

twice, using the two *indefinite* integrals you found in (a) and (b). Do the two values agree, or disagree? Is your result consistent with what you expect?

4. a) Find all functions $y = F(x)$ that satisfy the differential equation

$$\frac{dy}{dx} = x^2\left(1+x^3\right)^{13}.$$

b) From among the functions $F(x)$ you found in part (a), select the one that satisfies $F(0) = 4$.

c) From among the functions $F(x)$ you found in part (a), select the one that satisfies $F(-1) = 4$.

5. Find a function $y = G(t)$ that solves the initial value problem

$$\frac{dy}{dt} = te^{-t^2} \qquad y(0) = 3.$$

6. a) What is the average value of the function $f(x) = x/\sqrt{1+x^2}$ on the interval $[0, 2]$?

b) Show that the average value of the $f(x)$ on the interval $[-2, 2]$ is 0. Sketch a graph of $y = f(x)$ on this interval, and explain how the graph also shows that the average is 0.

7. a) Sketch the graph of the function $y = xe^{-x^2}$ on the interval $[0, 5]$.

b) Find the area between the graph of $y = xe^{-x^2}$ and the x-axis for $0 \leq x \leq 5$.

c) Find the area between the graph of $y = xe^{-x^2}$ and the x-axis for $0 \leq x \leq b$. Express your answer in terms of the quantity b, and denote it $A(b)$. Is $A(5)$ the same number you found in part(b)? What are the values of $A(10)$, $A(100)$, $A(1000)$?

d) It is possible to argue that the area between the graph of $y = xe^{-x^2}$ and the *entire* positive x-axis is $1/2$. Can you develop such an argument?

8. a) Use a computer graphing utility to establish that

$$\sin^2 x = \frac{1 - \cos(2x)}{2}.$$

Sketch these graphs.

b) Find a formula for $\int \sin^2 x \, dx$. (Suggestion: replace $\sin^2 x$ by the expression involving $\cos(2x)$, above, and integrate by substitution.)

c) What is the average value of $\sin^2 x$ on the interval $[0, \pi]$? What is its average value on any interval of the form $[0, k\pi]$, where k is a whole number?

d) Explain your results in part (c) in terms of the graph of $\sin^2 x$ you drew in part (a).

e) Here's a differential equations proof of the identity in part (a). Let $f(x) = \sin^2 x$, and let $g(x) = (1 - \cos(2x))/2$. Show that both of these functions satisfy the initial value problem

$$y'' = 2 - 4y \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 0.$$

Hence conclude the two functions must be the same.

11.3 Integration by Parts

An integral form of the product rule

As in the previous section, suppose we have functions F and G, with corresponding derivatives f and g. If we use the product rule to differentiate $F(x) \cdot G(x)$, we get:

$$(F \cdot G)' = F \cdot G' + F' \cdot G = F \cdot g + f \cdot G.$$

We can turn this into a statement about indefinite integrals:

$$\int (F \cdot g + f \cdot G)\, dx = \int (F \cdot G)'\, dx = F(x) \cdot G(x) + C.$$

Unfortunately, in this form the statement is not especially useful; it applies only when the integrand has two terms of the special form $f \cdot g' + f' \cdot g$. However, if we rewrite the statement in the form

$$\boxed{\int \boldsymbol{F \cdot g\, dx = F \cdot G - \int f \cdot G\, dx}}$$

it becomes very useful.

Example 1 We will use the formula in the box to find

$$\int x \cdot \cos x\, dx.$$

If we label the parts of this integrand as follows:

$$F(x) = x \qquad\qquad g(x) = \cos x,$$

then we have

$$f(x) = 1 \qquad \text{and} \qquad G(x) = \sin x.$$

According to the formula,

$$\begin{aligned}\int x \cdot \cos x\, dx &= x \cdot \sin x - \int \sin x\, dx \\ &= x \cdot \sin x + \cos x + C.\end{aligned}$$

Integrate only part of the integrand

The integrand is first broken into two parts—in this case, x and $\cos x$. One part is differentiated while the other part is integrated. (The part we integrated is $g(x) = \cos x$, and we got $G(x) = \sin x$.) For this reason, the rule described in the box is called **integration by parts**.

As with integration by substitution, integration by parts exchanges one integration task for another: Instead of finding an antiderivative for $F \cdot g$, we must find one for $f \cdot G$. The idea is to "trade-in" one integration problem for a more readily solvable one.

Example 2 Use integration by parts to find

$$\int \ln(x)\,dx.$$

At first glance we can't integrate by parts, because there aren't two parts! But note that we can write

$$\ln(x) = \ln(x) \cdot 1,$$

and then set

$$F(x) = \ln(x), \qquad g(x) = 1.$$

This implies

$$f(x) = \frac{1}{x} \quad \text{and} \quad G(x) = x,$$

and the integration by parts formula now gives us

$$\begin{aligned} \int \ln(x)\,dx &= x \cdot \ln(x) - \int x \cdot \frac{1}{x}\,dx \\ &= x \cdot \ln(x) - \int 1\,dx \\ &= x \cdot \ln(x) - x + C. \end{aligned}$$

The ingredients of a successful integration by parts

We thus see that integration by parts—like integration by substitution—is an art rather than a set routine. If integration by parts is to work, several things must happen. First, you need to see that the method might actually apply. (In Example 2 this wasn't obvious.) Next, you need to identify the parts of the integrand that will be differentiated and integrated, respectively. The wrong choices can lead you away from a solution, rather than towards one. (See example 3 below for a cautionary tale.) Finally, you need to be able to carry out the integration of the new integral $\int f \cdot G\,dx$. As you work you may have to reshape the integrand algebraically. Technique comes with practice, and luck is useful, too.

Example 3 Use integration by parts to find

$$\int t \cdot e^t \, dt.$$

Set

$$F(t) = e^t, \qquad g(t) = t;$$

then

$$f(t) = e^t, \qquad G(t) = \frac{t^2}{2}.$$

The integration by parts formula then gives

$$\int t \cdot e^t \, dt = \frac{t^2}{2} e^t - \int \frac{t^2}{2} e^t \, dt.$$

While this is a true statement, we are not better off—the new integral is *not* simpler than the original. A solution is eluding us here. You will have a chance to do this problem properly in the exercises.

What went wrong?

Exercises

1. Use integration by parts to find a formula for each of the following integrals.

a) $\int x \sin x \, dx$

b) $\int te^t \, dt$

c) $\int we^{-w} \, dw$

d) $\int x \ln x \, dx$

e) $\int \arcsin x \, dx$

f) $\int \arctan x \, dx$

g) $\int x^2 e^{-x} \, dx$

h) $\int u^2 \cos u \, du$

i) $\int x \sec^2 x \, dx$

j) $\int e^{2x}(x + e^x) \, dx$

(Suggestion for part (g): Apply integration by parts twice. After the first application you should have an integral that can itself be evaluated using integration by parts.)

2. Use integration by parts to obtain a formula for

$$\int (\ln x)^2\,dx.$$

Choose $f = \ln x$ and also $g' = \ln x$. To continue you need to find g, the antiderivative of $\ln x$, but this has already been obtained in the text.

3. a) Find $\displaystyle\int x^2 e^x\,dx$.

b) Find $\displaystyle\int x^3 e^x\,dx$. (Reduce this to part (a)).

c) Find $\displaystyle\int x^4 e^x\,dx$.

d) What is the general pattern here? Find a formula for $\displaystyle\int x^n e^x\,dx$, where n is any positive integer.

e) Find $\displaystyle\int e^x \left(5x^2 - 3x + 7\right)\,dx$.

4. a) Draw the graph of $y = \arctan x$ over the interval $0 \le x \le 1$. You could have gotten the same graph by thinking of x as a function of y—write down this relationship and the corresponding y interval.

b) Evaluate

$$\int_0^1 \arctan x\,dx$$

and show on your graph the area this corresponds to.

c) Evaluate

$$\int_0^{\pi/4} \tan y\,dy$$

and show on your graph the area this corresponds to.

d) If we add the results of part (b) and part (c), what do you get? From the geometry of the picture, what should you have gotten?

5. Repeat the analysis of the preceding problem by calculating the value of

$$\int_0^2 x^3\,dx + \int_0^8 y^{1/3}\,dy,$$

and seeing if it agrees with what you would predict by looking at the graphs.

6. Generalize the preceding two problems to the case where f and g are any two functions that are inverses of each other whose graphs pass through the origin.

7. a) What is the average value of the function $\ln x$ on the interval $[1, e]$?

b) What is the average value of $\ln x$ on $[1, b]$? Express this in terms of b. Discuss the following claim: The average value of $\ln x$ on $[1, b]$ is approximately $\ln(b) - 1$ when b is large.

8. a) Sketch the graph of $f(x) = xe^{-x}$ on the interval $[0, 4]$.

b) What is the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le 4$?

c) What is the area between the graph of $y = f(x)$ and the x-axis for $0 \le x \le b$? Express your answer in terms of b, and denote it $A(b)$. What is $A(100)$?

9. Find three solutions $y = f(t)$ to the differential equation

$$\frac{dy}{dt} = 5 - 2\ln t.$$

10. a) Find the solution $y = \varphi(t)$ to the initial value problem

$$\frac{dy}{dt} = te^{-t^2/2}, \qquad y(0) = 2.$$

b) The function $\varphi(t)$ increases as t increases. Show this first by sketching the graph of $y = \varphi(t)$. Show it also by referring to the differential equation that $\varphi(t)$ satisfies. (What is true about the derivative of an increasing function?)

c) Does the value of $\varphi(t)$ increase without bound as $t \to \infty$? If not, what value does $\varphi(t)$ approach?

11. a) **The differential form of Integration by Parts** If u and v are expressions in x, then the product rule can be written as

$$\frac{d}{dx}(u \cdot v) = \frac{du}{dx} \cdot v + u \cdot \frac{dv}{dx}.$$

Explain carefully how this leads to the following statement of integration by parts, and why it is equivalent to the form in the text:

$$\int u\,dv = uv - \int v\,du.$$

b) Solve a couple of the preceding problems using this notation.

Sine and cosine integrals

The purpose of the remaining exercises is to establish integral formulas that we will use to analyze Fourier polynomials and the power spectrum in chapter 12. In the first three, α is a constant:

$$\int \sin^2 \alpha x\,dx = \frac{x}{2} - \frac{1}{4\alpha}\sin 2\alpha x + C,$$
$$\int \cos^2 \alpha x\,dx = \frac{x}{2} + \frac{1}{4\alpha}\sin 2\alpha x + C,$$
$$\int \sin \alpha x \cos \alpha x\,dx = -\frac{1}{4\alpha}\cos 2\alpha x + C.$$

In the remaining four, α and β are *different* constants:

$$\int \sin \alpha x \sin \beta x\,dx = \frac{1}{\beta^2-\alpha^2}\left(\alpha \cos \alpha x \sin \beta x - \beta \sin \alpha x \cos \beta x\right) + C,$$
$$\int \cos \alpha x \cos \beta x\,dx = \frac{1}{\beta^2-\alpha^2}\left(\beta \cos \alpha x \sin \beta x - \alpha \sin \alpha x \cos \beta x\right) + C,$$
$$\int \sin \alpha x \cos \beta x\,dx = \frac{1}{\beta^2-\alpha^2}\left(\beta \sin \alpha x \sin \beta x + \alpha \cos \alpha x \cos \beta x\right) + C,$$
$$\int \cos \alpha x \sin \beta x\,dx = \frac{1}{\beta^2-\alpha^2}\left(-\alpha \sin \alpha x \sin \beta x - \beta \cos \alpha x \cos \beta x\right) + C.$$

12. a) In the later exercises we shall make frequent use of the following "trigonometric identities":

$$2 \sin \alpha x \cos \alpha x = \sin 2\alpha x,$$
$$\cos^2 \alpha x - \sin^2 \alpha x = \cos 2\alpha x,$$
$$\sin^2 \alpha x + \cos^2 \alpha x = 1.$$

Using a graphing package on a computer, graph together the functions

$$2 \sin \alpha x \cos \alpha x \qquad \text{and} \qquad \sin 2\alpha x$$

to show that they seem to be identical. (That is, show that they "share phosphor.") Then do the same for the pairs of functions in the other two identities.

b) We can give a different argument for the identities above using the ideas we have developed in studying initial value problems. To prove the first identity, for instance, let $f(x) = 2 \sin \alpha x \cos \alpha x$, and let $g(x) = \sin 2\alpha x$. Show that both functions satisfy

$$y'' = -4\alpha^2 y, \quad \text{with} \quad y(0) = 0 \quad \text{and} \quad y'(0) = 2\alpha.$$

Hence conclude the two functions must be the same.

c) Find an initial value problem that is satisfied by both $f(x) = \cos^2 \alpha x - \sin^2 \alpha x$ and by $g(x) = \cos 2\alpha x$.

13. **Evaluating** $\int \sin^2 x \, dx$

a) Using integration by parts, show that

$$\int \sin^2 x \, dx = -\sin x \cos x + \int \cos^2 x \, dx.$$

b) Using the identity $\sin^2 \alpha x + \cos^2 \alpha x = 1$, show that the new integral can be written as

$$x - \int \sin^2 x \, dx.$$

c) Combining (a) and (b) algebraically, show that

$$2 \int \sin^2 x \, dx = -\sin x \cos x + x + C.$$

d) Using algebra and a trigonometric identity, conclude that

$$\int \sin^2 x \, dx = \frac{x}{2} - \frac{1}{4} \sin 2x + C.$$

14. Modify the argument of the preceding exercise to show

$$\int \sin^2 \alpha x \, dx = \frac{x}{2} - \frac{1}{4\alpha} \sin 2\alpha x + C.$$

and

$$\int \cos^2 \alpha x \, dx = \frac{x}{2} + \frac{1}{4\alpha} \sin 2\alpha x + C.$$

15. **Evaluating** $\int \sin^2 \alpha x \, dx$

Determine this integral anew, without using integration by parts, by carrying out the following steps.

a) From the trigonometric identities on page 716, deduce that

$$2 \sin^2 \alpha x = 1 - \cos 2\alpha x.$$

b) Using the formula in (a), conclude that

$$\int \sin^2 \alpha x \, dx = \frac{x}{2} - \frac{1}{4\alpha} \sin 2\alpha x + C.$$

16. **Evaluating** $\int \cos^2 \alpha x \, dx$

Using only algebra and the identity $\sin^2 \alpha x + \cos^2 \alpha x = 1$, show that the previous exercise implies

$$\int \cos^2 \alpha x \, dx = \frac{x}{2} + \frac{1}{4\alpha} \sin 2\alpha x + C.$$

17. **Evaluating** $\int \sin \alpha x \cos \alpha x \, dx$

a) Using the identity $\sin 2\alpha x = 2 \sin \alpha x \cos \alpha x$, deduce the following formula

$$\int \sin \alpha x \cos \alpha x \, dx = -\frac{1}{4\alpha} \cos 2\alpha x + C.$$

b) Using integration by substitution, obtain the alternative formula

$$\int \sin \alpha x \cos \alpha x \, dx = \frac{1}{2\alpha} \sin^2 \alpha x + C.$$

c) Show that your results in (a) and (b) are compatible. (For example, use exercise 15 (a).)

18. **Evaluating** $\int \sin \alpha x \sin \beta x \, dx$

a) Use integration by parts to show that

$$\int \sin \alpha x \sin \beta x \, dx = -\frac{1}{\alpha} \cos \alpha x \sin \beta x + \frac{\beta}{\alpha} \int \cos \alpha x \cos \beta x \, dx.$$

b) Using integration by parts again, show that the new integral in part (a) can be written as

$$\int \cos \alpha x \cos \beta x \, dx = \frac{1}{\alpha} \sin \alpha x \cos \beta x + \frac{\beta}{\alpha} \int \sin \alpha x \sin \beta x \, dx.$$

c) Let $J = \int \sin \alpha x \sin \beta x \, dx$; show that combining (a) and (b) gives

$$J = -\frac{1}{\alpha} \cos \alpha x \sin \beta x + \frac{\beta}{\alpha^2} \sin \alpha x \cos \beta x + \frac{\beta^2}{\alpha^2} J.$$

d) Solve (c) for J to find

$$\int \sin \alpha x \sin \beta x \, dx = \frac{1}{\beta^2 - \alpha^2} (\alpha \cos \alpha x \sin \beta x - \beta \sin \alpha x \cos \beta x) + C.$$

19. Imitate the methods of the preceding exercise to deduce

$$\int \cos \alpha x \cos \beta x \, dx = \frac{1}{\beta^2 - \alpha^2} (\beta \cos \alpha x \sin \beta x - \alpha \sin \alpha x \cos \beta x) + C$$

and

$$\int \sin \alpha x \cos \beta x \, dx = \frac{1}{\beta^2 - \alpha^2} (\beta \sin \alpha x \sin \beta x + \alpha \cos \alpha x \cos \beta x) + C.$$

20. **Evaluating** $\int \cos \alpha x \sin \beta x \, dx$

This integral is the same as one in the preceding exercise, if you exchange the factors α and β. Do that, and obtain the formula

$$\int \cos \alpha x \sin \beta x \, dx = 1\alpha^2 - \beta^2(\alpha \sin \alpha x \sin \beta x + \beta \cos \alpha x \cos \beta x) + C.$$

21. Determine the following.

a) $\displaystyle\int_0^{2\pi} \sin^2 x \, dx$.

b) $\displaystyle\int_0^{n\pi} \cos^2 x \, dx$, n a positive integer.

c) $\displaystyle\int_0^{2\pi} \sin x \sin \beta x \, dx$, $\beta \neq 1$.

d) $\displaystyle\int_0^{\pi} \sin x \sin \beta x \, dx$, $\beta \neq 1$.

11.4 Separation of Variables and Partial Fractions

One of the principal uses of integration techniques is to find closed form solutions to differential equations. If you look back at the methods we have developed so far in this chapter, they are all applicable to differential equations of the form $y' = f(t)$ for some function f—that is, the rate at which y changes is a function of the independent variable only. In such cases we only need to find an antiderivative F for f, choose the constant C to satisfy the initial value, and we have our solution. As we saw in the early chapters, though, the behavior of y' often depends on the values of y rather than on t—think of the S-I-R model or the various predator-prey problems. In this section we will see how our earlier techniques can be adapted to apply to problems of this sort as well.

The Differential Equation $y' = y$

A new method for solving $y' = y$

As you know, the exponential functions $y = Ce^t$ are the solutions to the differential equation $dy/dt = y$. Let's put aside this knowledge for a moment and rediscover these solutions using a new method. The method involves the connection between inverse functions and their derivatives. With it we will be able to explore a variety of problems that had been beyond our reach.

Find the inverse function instead

The idea behind the new method is quite simple: Instead of thinking of y as a function of t, convert to thinking of t as a function of y, thereby looking for the **inverse function**. We know that the derivative of the inverse function is the reciprocal of the derivative of the original function, so we can rewrite the given differential equation by using its reciprocal:

The new differential equation...

$$\frac{dy}{dt} = y \qquad \text{becomes} \qquad \frac{dt}{dy} = \frac{1}{y}.$$

Then solve the new differential equation $dt/dy = 1/y$. While this may not look very different, it has the property that the rate of change of the *dependent variable*—now t—is expressed as a function of the *independent* variable—now y. But this is just the form we have been considering in the earlier sections of this chapter. A solution to the new equation is a function $t = g(y)$ whose derivative is $1/y$. This is one of the basic antiderivatives listed in the table on page 689:

...and its solution

$$t = g(y) = \ln y + k,$$

where k is an arbitrary constant.

The inverse ...

The solution to the original differential equation $dy/dt = y$ is the inverse of $t = \ln y + k$. We find it by solving this equation for y:

$$\begin{aligned} t - k &= \ln y \\ e^{t-k} &= y \\ e^t \cdot e^{-k} &= y \end{aligned}$$

...solves the original problem

Thus $y(t) = Ce^t$, where we have replaced the constant e^{-k} by C.

Indefinite integrals

The differential equation expressed using differentials

The language of indefinite integrals and differentials again provides a convenient mnemonic for this new method. First, we use the original differential equation to relate the differentials dy and dt:

$$dy = \frac{dy}{dt}\, dt = y\, dt.$$

This use of differentials is introduced on page 704. At this point the equation makes sense for either t or y being the independent variable. Now if we try to integrate this equation with respect to t, we get

$$y = \int dy = \int y(t)\, dt.$$

We can't find the last integral, because we don't know what y is as a function of t. Remember, an indefinite integral is an antiderivative, so an expression of the form

$$\int f\,dt$$

represents a function $F(t)$ whose derivative is $f(t)$. If f is *not* given by a formula in t, there is no way to get a formula for $F(t)$.

Suppose, though, that we divide both sides of the differential equation $dy = y\,dt$ by y, and integrate with respect to y. The equation takes the form

$$\frac{dy}{y} = dt.$$

Now, if we introduce indefinite integrals, we have

$$\int \frac{dy}{y} = \int dt.$$

The variables are now *separated*

This time the variables y and t have been separated from each other, and we *can* find the integrals. In fact,

$$\ln y = \int \frac{dy}{y} = \int dt = t + b,$$

where b is an arbitrary constant. (We could just as easily have added the constant to the left side instead—do you see why we don't have to add a constant to both sides?) To complete the work we solve for y:

The solution once again

$$y = e^{t+b} = e^t \cdot e^b = Ce^t,$$

where $C = e^b$.

Summary

The first time we went through the method, we replaced

$$\frac{dy}{dt} = y \qquad \text{by} \qquad \frac{dt}{dy} = \frac{1}{y}.$$

These differential equations express the same relation between y and t. Each is just the reciprocal of the other. In the first, y depends on t; in the second, though, t depends on y. The second time we went through the method, using indefinite integrals, we replaced

$$dy = y\,dt \qquad \text{by} \qquad dt = \frac{dy}{y}.$$

This change was also algebraic, and it had the same effect: the dependent variable changed from y to t. More important, in the new differential equation (using the differentials themselves!) the variables are separated. That allows us to do the integration. We get a solution in the form $t = g(y)$. The solution to the original problem is the *inverse* $y = f(t)$ of $t = g(y)$.

To integrate, separate the variables

Separation of Variables

With the method of **separation of variables**, introduced in the previous pages, we can obtain formulas for solutions to a number of differential equations that were previously accessible only by Euler's method. Recall that one of the clear advantages of a *formula* is that it allows us to see how the parameters in the problem affect the solution. We'll look at two problems. First we'll show how the method can explain the rather baffling formula for supergrowth that we gave in chapter 4. Then, using the method of **partial fractions**, to be discussed next, we'll give a formula for logistic growth.

Supergrowth

In chapter 4.2 we modelled the growth of a population Q by the initial value problem

$$\frac{dQ}{dt} = kQ^{1.2}, \qquad Q(0) = A.$$

To get a formula for the solution, transform the differential equation in the following way:

Separate the variables

$$\frac{dQ}{dt} = k\,Q^{1.2} \quad \rightsquigarrow \quad dQ = k\,Q^{1.2}\,dt \quad \rightsquigarrow \quad \frac{dQ}{Q^{1.2}} = k\,dt.$$

Now integrate:

$$\int \frac{dQ}{Q^{1.2}} = \int k\,dt.$$

Because the variables have been separated, the integrals can be found:

$$\int \frac{dQ}{Q^{1.2}} = \int Q^{-1.2}\, dQ = \frac{1}{-.2} Q^{-.2}$$

$$\int k\, dt = kt + C$$

Solve for Q

Therefore, $\frac{1}{-.2} Q^{-.2} = kt + C$. Now we must solve this equation for Q. Here is one possible approach. First, we can write

$$Q^{-.2} = -.2(kt + C) \;=\; C_1 - .2kt.$$

To simplify the expression, we have replaced $-.2C$ by a *new* constant C_1. Since

$$(Q^{-.2})^{-5} = Q^{-.2\times -5} = Q^1 = Q,$$

we'll raise both sides of the previous equation to the power -5:

$$Q(t) = (Q^{-.2})^{-5} = (C_1 - .2kt)^{-5}.$$

Bring in the initial condition

The last step is to incorporate the initial condition $Q(0) = A$. According to the new formula for $Q(t)$,

$$Q(0) = (C_1 - .2k \cdot 0)^{-5} = C_1^{-5} = A.$$

Solving $A = C_1^{-5}$ for C_1, we get

$$C_1 = A^{-1/5} = \frac{1}{\sqrt[5]{A}}.$$

We are now done:

$$Q(t) = \left(\frac{1}{\sqrt[5]{A}} - .2kt \right)^{-5}.$$

Interpreting the formula

This is the formula that appears on page 214. It shows how the parameters k and A affect the solution. In particular, we called this *supergrowth* because the model predicts that the population Q becomes infinite when

$$\frac{1}{\sqrt[5]{A}} - .2kt = 0; \quad \text{that is, when} \quad t = \frac{1}{.2k\sqrt[5]{A}}.$$

Partial Fractions

Using separation of variables with a partial fractions decomposition (to be described below), we will obtain a formula for the solution to the logistic equation (chapter 4.1). The method of **partial fractions** is a useful tool for solving many integration problems.

Logistic growth

Consider this initial value problem associated with the logistic differential equation:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{C}\right), \qquad P(0) = A.$$

We will find a formula for the solution that incorporates the growth parameter k and the carrying capacity C.

Step 1: separate the variables

The first step is to transform the equation into one where the variables are separated:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{C}\right) \quad \rightsquigarrow \quad \frac{dP}{P(1 - P/C)} = k\,dt.$$

Integrating the new equation we get

$$\int \frac{dP}{P(1 - P/C)} = \int k\,dt.$$

The denominator has an unfamiliar form

We are stuck now, because the integral on the left doesn't appear in our table of integrals (page 689). If the denominator had *only* P or *only* $1 - P/C$, we could use the natural logarithm. The difficulty is that the denominator is the product of both terms.

There is a way out of the difficulty. We will use algebra to transform the integrand into a form we can work with. The first step is to simplify the denominator a bit:

$$\frac{1}{P(1 - P/C)} = \frac{1}{P(C/C - P/C)} = \frac{1}{P(C - P)/C} = \frac{C}{P(C - P)}.$$

(This wasn't essential; it just makes later steps easier to write.) The next step will be the crucial one. To understand why we take it, consider the rule for adding two fractions:

$$\frac{\alpha}{(x + a)} + \frac{\beta}{(x + b)} = \frac{\alpha(x + b) + \beta(x + a)}{(x + a)(x + b)}$$

Step 2: write the integrand as a sum of simple fractions

The denominator is a product—very much like the product $P(C-P)$ in our integrand! Perhaps we can write *that* as a sum of two simpler fractions:

$$\frac{C}{P(C-P)} = \frac{\alpha}{P} + \frac{\beta}{C-P}.$$

What values should α and β have? According to the rule for adding fractions,

$$\frac{\alpha}{P} + \frac{\beta}{C-P} = \frac{\alpha(C-P)+\beta P}{P(C-P)},$$

and this should equal the original integrand:

$$\frac{\alpha(C-P)+\beta P}{P(C-P)} = \frac{C}{P(C-P)}.$$

Determining α and β

Since the denominators are equal, the numerators must also be equal:

$$\alpha(C-P)+\beta P = C.$$

In fact, they must be equal *as polynomials in the variable* P. If we rewrite the last equation, collecting terms that involve the same power of P, we get

$$(\beta-\alpha)P + \alpha C = 0\cdot P + 1\cdot C.$$

Since two polynomials are equal precisely when their coefficients are equal, it follows that

$$\beta-\alpha = 0 \qquad \alpha = 1.$$

Thus $\alpha = \beta = 1$, and we have

The partial fractions decomposition

$$\frac{C}{P(C-P)} = \frac{1}{P} + \frac{1}{C-P}.$$

The simpler expressions on the right are called **partial fractions**. Their denominators are the different *parts* of the denominator of the integrand. The equation that expresses the integrand as a sum of partial fractions is called a **partial fractions decomposition**.

We can now return to the integral equation we are trying to solve:

$$\int \frac{C}{P(C-P)}\,dP = \int k\,dt.$$

The right hand side equals $kt+b$, where b is the usual constant of integration. Thanks to the partial fractions decomposition, the left hand side can be written

Step 3: evaluate the integrals

$$\int \frac{C}{P(C-P)}\,dP = \int \frac{1}{P}\,dP + \int \frac{1}{C-P}\,dP.$$

The first integral on the right is straightforward:

$$\int \frac{1}{P}\,dP = \ln P.$$

The second can be solved by using the substitution $C - P = u$, with $dP = -du$:

$$\int \frac{1}{C-P}\,dP = \int \frac{-du}{u} = -\ln u = -\ln(C-P).$$

Putting everything together we find

$$\ln P - \ln(C-P) = \ln\left(\frac{P}{C-P}\right) = kt + b.$$

As we have seen, separation of variables usually leaves us with an inverse function to find. This problem is no different. We must solve the last equation for P. The first step is to exponentiate both sides:

Step 4: solve for P

$$\frac{P}{C-P} = e^{kt+b} = e^b \cdot e^{kt} = Be^{kt}.$$

To simplify the expression a bit, we have replaced e^b by $B = e^b$. Multiplying both sides by $C - P$ gives

$$P = Be^{kt}(C-P) = CBe^{kt} - PBe^{kt}.$$

Now bring the last term over to the left, and then factor out P:

$$P + PBe^{kt} = \left(1 + Be^{kt}\right) P = CBe^{kt}.$$

The final step is to divide by the coefficient $1 + Be^{kt}$:

The formula for $P(t)$

$$P(t) = \frac{CBe^{kt}}{1 + Be^{kt}}.$$

Lastly, we must see how the initial condition $P(0) = A$ affects the solution. We could substitute $t = 0$, $P = A$ into the last formula, but that produces an algebraic mess. We want to know how A affects the constant B,

Step 5: incorporate the initial condition and we can see that directly by making our substitutions into the equation above:

$$\frac{P}{C-P} = Be^{kt} \quad \rightsquigarrow \quad \frac{A}{C-A} = Be^0 = B.$$

Now replace B by $A/(C-A)$ in our formula for $P(t)$. This yields

$$P(t) = \frac{CAe^{kt}/(C-A)}{1 + Ae^{kt}/(C-A)} = \frac{CAe^{kt}}{C-A+Ae^{kt}}$$

If we write $-A + Ae^{kt} = A(e^{kt}-1)$, we get one of the standard forms of the solution to the logistic equation:

The complete solution

$$P(t) = \frac{CAe^{kt}}{C + A(e^{kt}-1)}.$$

Remark. The method of partial fractions can be used to evaluate integrals of the form

$$\int \frac{dx}{(x+a_1)(x+a_2)\cdots(x+a_n)} \qquad \text{or} \qquad \int \frac{P(x)}{Q(x)}\,dx,$$

where $P(x)$ and $Q(x)$ are arbitrary polynomials. Tables of integrals and calculus references describe how the method works in these cases. As one example, though, let's compute the antiderivative of the cosecant function, since we will need it in the next section.

Example—The antiderivative of the cosecant We first use a trigonometric identity to transform the integral slightly:

$$\begin{aligned}\int \csc x\,dx &= \int \frac{1}{\sin x}\,dx \\ &= \int \frac{\sin x}{\sin^2 x}\,dx \\ &= \int \frac{\sin x\,dx}{1-\cos^2 x}.\end{aligned}$$

If we now make the substitution $u = \cos x$, with $du = -\sin x\,dx$, this becomes

$$\begin{aligned}\int \csc x\,dx &= \int \frac{-du}{1-u^2}\\ &= \int \frac{-1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right) du\\ &= -\frac{1}{2}\int \frac{du}{1+u} - \frac{1}{2}\int \frac{du}{1-u}\\ &= -\frac{1}{2}\ln(1+u) + \frac{1}{2}\ln(1-u) + C\\ &= \frac{1}{2}\ln\frac{1-u}{1+u} + C\\ &= \frac{1}{2}\ln\frac{1-\cos x}{1+\cos x} + C.\end{aligned}$$

We can simplify this slightly by multiplying both numerator and denominator by $1 + \cos x$ to get

The final form of the antiderivative of the cosecant

$$\begin{aligned}\int \csc x\,dx = \frac{1}{2}\ln\frac{1-\cos^2 x}{(1+\cos x)^2} + C &= \ln\left|\frac{\sin x}{1+\cos x}\right| + C\\ &= -\ln|\csc x + \cot x| + C.\end{aligned}$$

Note that in the next to last line we used the general fact about logarithms that $n \ln A = \ln(A^n)$ for any value of n and any $A > 0$. Note also that since the domain of the secant function consists of infinitely many separate intervals. the "$+\,C$" at the end of the antiderivative needs to be interpreted as potentially a different value of C over each interval.

In the same fashion we can obtain the antiderivative for the secant function:

The antiderivative of the secant

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C.$$

Exercises

Separation of variables

1. Use the method of separation of variables to find a formula for the solution of the differential equation $dy/dt = y + 5$. Your formula should contain an arbitrary constant to reflect the fact that many functions solve the differential equation.

2. Use the method of separation of variables to find formulas for the solutions to the following differential equations. In each case your formula should be expressed in terms of the input variable that is indicated (e.g., in part (a) it is t).

a) $dy/dt = 1/y$.

b) $dz/dx = 3/(z-2)$.

c) $dy/dx = x/y$

d) $dy/dx = y/x$

e) $du/dv = u/(u-1)$

f) $dv/dt = -\sqrt{v}$

3. **A cooling liquid**. According to Newton's law of cooling (see chapter 4.1), in a room where the ambient temperature is A, the temperature Q of a hot object will change according to the differential equation

$$\frac{dQ}{dt} = -k(Q - A).$$

The constant k gives the rate at which the object cools.

a) Find a formula for the solution to this equation using the method of separation of variables. Your formula should contain an arbitrary constant.

b) Suppose A is 20°C and k is .1° per minute per °C. If time t is measured in minutes, and $Q(0) = 90$°C, what will Q be after 20 minutes?

c) How long does it take for the temperature to drop to 30°C?

4. a) Suppose a cold drink at 36°F is sitting in the open air on a summer day when the temperature is 90°F. If the drink warms up at a rate of .2°F per minute per °F of temperature difference, write a differential equation to model what will happen to the temperature of the drink over time.

b) Obtain a formula for the temperature of the drink as a function of the number of minutes t that have passed since its temperature was 36°F.

c) What will the temperature of the drink be after 5 minutes; after 10 minutes?

d) How long will it take for the drink to reach 55°F?

5. **A leaking tank**. In chapter 4.2 we used the differential equation

$$\frac{dV}{dt} = -k\sqrt{V}$$

to model the volume $V(t)$ of water in a leaking tank after t hours (see page 221).

a) Use the method of separation of variables to show that

$$V(t) = \frac{k^2}{4}\,(C-t)^2$$

is a solution to the differential equation, for any value of the constant C.

b) Explain why the function

$$V(t) = \begin{cases} \dfrac{k^2}{4}(C-t)^2 & \text{if } 0 \le t \le C, \\ 0 & \text{if } C < t. \end{cases}$$

is *also* a solution to the differential equation. Why is *this* solution more relevant to the leaking tank problem than the solution in part (a)?

6. **A falling body with air resistance**. We have used the differential equation

$$\frac{dv}{dt} = -g - bv$$

to model the motion of a body falling under the influence of gravity (g) and air resistance (bv). Here v is the velocity of the body at time t. (See pages 223–224.)

a) Solve the differential equation by separating variables, and obtain

$$v(t) = \frac{1}{b}\,\left(Ce^{-bt} - g\right),$$

where C is an arbitrary constant.

b) Now impose the initial condition $v(0) = 0$ (so the body starts it fall from rest) to determine the value of C. What is the formula for $v(t)$ now?

c) Exercise 21 on page 223 gives the solution to the initial value problem as

$$v(t) = \frac{g}{b}\,\left(2^{-bt/.69} - 1\right).$$

Reconcile this expression with the one you obtained in part (b) of this exercise.

d) The distance $x(t)$ that the body has fallen by time t is given by the integral

$$x(t) = \int_0^t v(t)\,dt, \quad \text{because} \quad \frac{dx}{dt} = v \quad \text{and} \quad x(0) = 0.$$

Use your formula for $v(t)$ from part (b) to find $x(t)$.

7. a) **Supergrowth**. We have analyzed the differential equation

$$\frac{dQ}{dt} = kQ^p$$

when $p = 1.2$ (and, of course, when $p = 1$). Find a formula for the solution $Q(t)$ when $p = 2$. Your formula should contain an arbitrary constant C.

b) Add the initial condition $Q(0) = A$. This will fix the value of the constant C. What is the formula for $Q(t)$ when the initial condition is incorporated?

c) Your formula in part (b) should demonstrate that Q becomes infinite at some finite time $t = \tau$. When is τ? Your answer should be expressed in terms of the growth constant k and the initial population size A.

d) Suppose the values of k and A are known only imprecisely, and they could be in error by as much as 5%. That makes the value of τ uncertain. Which error causes the greater uncertainty: the error in k or the error in A? (See the discussion of error analysis for the supergrowth model on pages 215–217.)

8. **General supergrowth**. Find the solution to the initial value problem

$$\frac{dQ}{dt} = kQ^p, \qquad Q(0) = A$$

for *any* value of the power p. For which values of p does Q blow up to ∞ at a finite time $t = \tau$? What is τ?

Partial fractions

9. Use the method of partial fractions to determine the values of α, β, and γ in the following equations.

a) $\dfrac{1}{(x-1)(x+2)} = \dfrac{\alpha}{x-1} + \dfrac{\beta}{x+2}$

b) $\dfrac{x}{(x-1)(x+2)} = \dfrac{\alpha}{x-1} + \dfrac{\beta}{x+2}$

c) $\dfrac{1}{x(x^2-1)} = \dfrac{\alpha}{x} + \dfrac{\beta}{x-1} + \dfrac{\gamma}{x+1}$

d) $\dfrac{x}{2x^2+3x+1} = \dfrac{\alpha}{2x+1} + \dfrac{\beta}{x+1}$

e) $\dfrac{1}{x(x^2+1)} = \dfrac{\alpha}{x} + \dfrac{\beta x + \gamma}{x^2+1}$

[Note that x^2+1 can't be factored.]

10. Find a formula for each of these indefinite integrals.

a) $\displaystyle\int \frac{3\,dx}{(x-1)(x+2)}$

b) $\displaystyle\int \frac{5x+3}{(x-1)(x+2)}\,dx$

c) $\displaystyle\int \frac{dt}{t(t^2-1)}$

d) $\displaystyle\int \frac{x\,dx}{1-x^2}$

e) $\displaystyle\int \frac{1-u}{u^2-4}\,du$

f) $\displaystyle\int \frac{x^2+2x+1}{x(x^2+1)}\,dx$

11. Determine

a) $\displaystyle\int_2^3 \frac{3\,dx}{(x-1)(x+2)}$

b) $\displaystyle\int_2^4 \frac{dt}{t(t^2-1)}$

c) $\displaystyle\int_0^{\pi/4} \frac{x\,dx}{1-x^2}$

d) $\displaystyle\int_1^{\sqrt{3}} \frac{x^2+2x+1}{x(x^2+1)}\,dx$

12. Mirror the derivation of $\displaystyle\int \csc x\,dx$ to find $\displaystyle\int \sec x\,dx$.

13. Consider the particular logistic growth model defined by

$$\frac{dP}{dt} = .2P\left(1-\frac{P}{10}\right)\ \text{lbs/hr}, \qquad P(0) = .5\ \text{lbs}$$

(Compare this with the fermentation problems, pages 195–197.)

a) Obtain the formula for the solution to this initial value problem.

b) How large will P be after 3 hours; after 10 hours?

c) When will P reach one-half the carrying capacity–that is, for which t is $P = 5$ lbs?

14. Derive the formula for $\displaystyle\int \sec x\,dx$ given on page pagerefsecant, using methods similar to those used to find an antiderivative for the cosecant function.

11.5 Trigonometric Integrals

The preceding sections have covered the main integration techniques and concepts likely to be needed by most users of calculus. These techniques, together with the numerical methods discussed in chapter 11.6, should be part of the basic tool kit of every practitioner of calculus. For those going on in physics or mathematics, there are additional methods, largely involving trigonometric functions in various ways, that are sometimes useful. The purpose of this section is to develop the most commonly used of these techniques.

Recall that there are only a few simple antiderivatives we can write down immediately by inspection. All non-numerical integration techniques consist of finding transformations that will reduce some new class of integration problems to a class we already know how to solve. Once we have a new class of solvable problems, then we look for other classes of problems that can be reduced to this new class, and so on. The techniques we will be developing in this section involve ways of making such transformations through the use of basic trigonometric identities, typically in conjunction with integration by parts or by substitution. Before we proceed with the integration techniques, it will be helpful to list the trigonometric identities used.

Review of trigonometric identities

The most frequently used identity is

$$\sin^2 x + \cos^2 x = 1,$$

and the equivalent form obtained by dividing through by $\cos^2 x$:

$$\tan^2 x + 1 = \sec^2 x,$$

and by $\sin^2 x$:

$$1 + \cot^2 x = \csc^2 x.$$

The only other identities you will need have already been encountered:

$$\sin 2x = 2 \sin x \cos x \quad \text{and} \quad \cos 2x = \cos^2 x - \sin^2 x,$$

plus the two other forms of the second of these identities,

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x) \quad \text{and} \quad \sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

Inverse Substitution

The method of substitution outlined in chapter 11.2 worked by taking a complicated integrand and breaking it down into simpler components, reducing the problem of finding an antiderivative for something in the form $f(G(x))g(x)$ to the problem of finding an antiderivative for f. In some cases, though, we go in the opposite direction: we have an integral $\int f(x)\,dx$ we want to find but can't evaluate directly. Instead, we can find a function $G(u)$ with derivative $g(u)$ such that we can find an antiderivative for $f(G(u))g(u)$. Since we know this integral is $F(G(u))$, we can now figure out what the desired function F must be. As with the earlier substitution techniques, this **inverse substitution** is conveniently expressed using differential notation.

Success sometimes comes by making things more complicated

Example 1 Suppose we want to evaluate

$$\int \sqrt{4-x^2}\,dx.$$

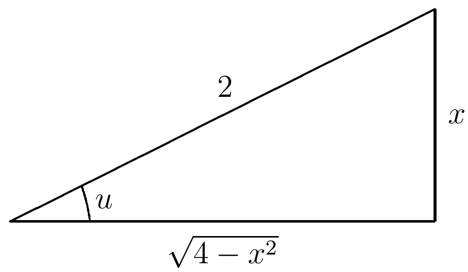
If we substitute $x = 2\sin u$, so that $dx = 2\cos u\,du$, look what happens:

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= \int \sqrt{4-(2\sin u)^2}\,2\cos u\,du \\ &= \int \sqrt{4-4\sin^2 u}\,2\cos u\,du \\ &= \int 2\sqrt{1-\sin^2 u}\,2\cos u\,du \\ &= \int 2\cos u\,2\cos u\,du \\ &= 4\int \cos^2 u\,du.\end{aligned}$$

But this is just an antiderivative we have already found in the exercises in chapter 11.3, namely

$$\begin{aligned}\int \cos^2 u\,du &= \frac{u}{2} + \frac{1}{4}\sin 2u + C \\ &= \frac{u}{2} + \frac{1}{4}\cdot 2\sin u\cos u + C \\ &= \frac{1}{2}(u + \sin u\cos u) + C.\end{aligned}$$

To find the desired antiderivative for the original function of x, we now replace u by its expression in terms of x by inverting the relationship: If $x = 2\sin u$, then $\sin u = x/2$, and $u = \arcsin(x/2)$. As we found in chapter 11.1, drawing a picture expressing the relationship between x and u makes it easy to visualize the other trigonometric functions:



From the picture we see that

$$\cos u = \frac{\sqrt{4-x^2}}{2} \qquad \text{and} \qquad \tan u = \frac{x}{\sqrt{4-x^2}}.$$

We can now find an expression for the desired antiderivative in terms of x:

$$\begin{aligned}\int \sqrt{4-x^2}\,dx &= 2(u + \sin u \cos u) + C \\ &= 2\left(\arcsin\frac{x}{2} + \frac{x}{2}\frac{\sqrt{4-x^2}}{2}\right) + C \\ &= 2\arcsin\frac{x}{2} + \frac{x\sqrt{4-x^2}}{2} + C.\end{aligned}$$

As usual, you should check this result by differentiating the right-hand side to see that you do obtain the integrand on the left.

Some useful substitutions

Similar substitutions allow us to evaluate other integrals involving square roots of quadratic expressions. Here is a summary of useful substitutions. In each case, a is a positive real number.

To transform	$a^2 - x^2$	let	$x = a\sin u$;
To transform	$a^2 + x^2$	let	$x = a\tan u$;
To transform	$x^2 - a^2$	let	$x = a\sec u$.

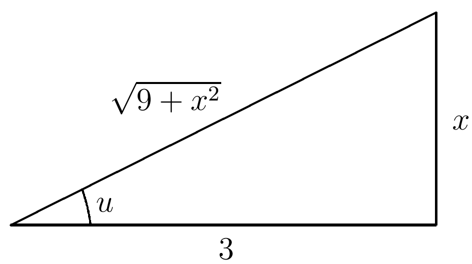
Example 2 Integrate

$$\int \frac{dx}{\sqrt{x^2+9}}.$$

If we set $x = 3\tan u$, then $dx = 3\,\sec^2 u\,du$, and the integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+9}} &= \int \frac{3\,\sec^2 u\,du}{\sqrt{9\,\sec^2 u}}\\ &= \int \sec u\,du\\ &= \ln|\sec u + \tan u| + C,\end{aligned}$$

(as we saw in chapter 11.4). To express this in terms of x, we again draw a picture showing the relation between u and x:



From this picture we see that $\sec u = \sqrt{9+x^2}/3$. Therefore

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+9}} &= \ln\left|\frac{\sqrt{9+x^2}}{3} + \frac{x}{3}\right| + C\\ &= \ln\left|\frac{\sqrt{9+x^2}+x}{3}\right| + C = \ln\left|\sqrt{9+x^2}+x\right| + C',\end{aligned}$$

where $C' = C - \ln 3$ is a new constant. As usual, you should differentiate to check that this really is the claimed antiderivative

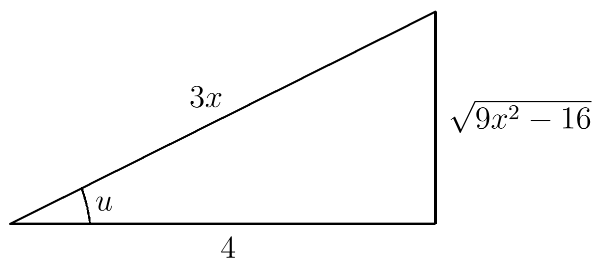
Example 3 Evaluate

$$\int \frac{dx}{\sqrt{9x^2-16}}.$$

We first write $\sqrt{9x^2-16}$ as $\sqrt{9}\sqrt{x^2-(16/9)} = 3\sqrt{x^2-(16/9)}$. Using the substitution $x = (4/3)\sec u$, with $dx = (4/3)\sec u\tan u\,du$ gives

$$\int \frac{dx}{\sqrt{9x^2-16}} = \int \frac{(4/3)\sec u\ \tan u\, du}{3\sqrt{(16/9)\sec^2 u - (16/9)}}$$
$$= \int \frac{(4/3)\sec u\ \tan u\, du}{3\cdot(4/3)\tan u} = \frac{1}{3}\int \sec u\, du$$
$$= \frac{1}{3}\ln|\sec u + \tan u| + C.$$

Again we need a picture to relate x and u:



Thus $\tan u = \sqrt{9x^2-16}/4$, which gives

$$\int \frac{dx}{\sqrt{9x^2-16}} = \frac{1}{3}\ln|\sec u + \tan u| + C$$
$$= \frac{1}{3}\ln\left|\frac{3x}{4} + \frac{\sqrt{9x^2-16}}{4}\right| + C$$
$$= \frac{1}{3}\ln\left|3x + \sqrt{9x^2-16}\right| + C',$$

where $C' = C - (\ln 4)/3$.

As usual, you should differentiate this final expression to confirm that it really is the desired antiderivative.

Inverse Substitution and Definite Integrals

We saw on page 707 in chapter 11.2 how to use substitution to evaluate a definite integral. When we transformed an integral originally expressed in terms of a variable x into one expressed in terms of a variable u, the two integrals had the same numerical value. The same can be done with the inverse substitution technique we have just been considering. Let's see how this works. Suppose we start with a function $f(x)$ to be integrated over an interval $[a, b]$. If we only knew an antiderivative F for f, we could easily write

$$\int_a^b f(x)\,dx = F(b) - F(a),$$

as usual. In the examples we've just been considering, we found the antiderivative for f by making a substitution $x = G(u)$ for some function G and then finding an antiderivative for $f(G(u))g(u)$, where $G' = g$. This antiderivative we know is $F(G(u))$, where F is the function we are trying to find. We were able to obtain F by replacing u by its expression in x. To do this we needed to find the inverse function G^{-1} for G, so that $x = G(u)$ was equivalent to $u = G^{-1}(x)$, and $F(x) = F(G(G^{-1}(x)))$. It is this last step we can eliminate in calculating definite integrals.

If we want x to go from a to b, what must u do? What interval of u values will get transformed to this interval of x values by G. The value of u such that $G(u) = a$ is $A = G^{-1}(a)$. Similarly the u value that gets transformed to b is $B = G^{-1}(b)$. Thus under the substitution $x = G(u)$, as u goes from A to B, x will go from a to b. Now look at the corresponding definite integral:

$$\begin{aligned}\int_A^B f(G(u))\,g(u)\,du &= F(G(u))\Big|_A^B \\ &= F(G(B)) - F(G(A)) \\ &= F(G(G^{-1}(b))) - F(G(G^{-1}(a))) \\ &= F(b) - F(a),\end{aligned}$$

which is just the desired value of the original definite integral. To summarize,

If we make the substitution $x = G(u)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a) = \int_A^B f(G(u))\,g(u)\,du,$$

where $A = G^{-1}(a)$ and $B = G^{-1}(b)$.

Let's look back at a couple of the preceding examples to see how this works.

Example 4 Evaluate

$$\int_{-2}^{2} \sqrt{4 - x^2}\,dx.$$

In Example 1 we found an antiderivative for this function by making the substitution $x = 2\sin u = G(u)$. The inverse function is then $G^{-1}(x) = \arcsin(x/2)$. To get x to go from -2 to 2, u must go from $G^{-1}(-2) = \arcsin(-1) = -\pi/2 = A$ to $G^{-1}(2) = \arcsin 1 = \pi/2 = B$. Then to evaluate the integral from $x = -2$ to $x = 2$, we only need to evaluate the antiderivative we found for the u integral between $u = -\pi/2$ and $u = \pi/2$.

$$\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 2(u + \sin u\, \cos u)\Big|_{-\pi/2}^{\pi/2} = \pi - (-\pi) = 2\pi.$$

(Note that this is just half the area of a circle of radius 2. How could we have foreseen this result from the form of the problem?)

Example 5 Suppose we wanted the integral

$$\int_0^3 \frac{dx}{\sqrt{x^2 + 9}}.$$

In Example 2 we found an antiderivative by letting $x = 3\tan u$. Here $G^{-1}(x) = \arctan(x/3)$. For x to go from 0 to 3, u must go from 0 to $\arctan 1 = \pi/4$. Using the u-antiderivative we found in Example 2, we have

$$\begin{aligned}\int_0^3 \frac{dx}{\sqrt{x^2 + 9}} &= \ln|\sec u + \tan u|\Big|_0^{\pi/4} \\ &= \ln|\sqrt{2} + 1| - \ln|1 + 0| = \ln(\sqrt{2} + 1).\end{aligned}$$

Completing The Square

Integrands involving terms of the form $Ax^2 + Bx + C$ can always be put in the form $A(u^2 \pm b^2)$ for a suitable variable u and constant b. The technique for doing this is the standard method of **completing the square**:

$$\begin{aligned}Ax^2 + Bx + C &= A\left(x^2 + \frac{B}{A}x\right) + C \\ &= A\left(x^2 + \frac{B}{A}x + \frac{B^2}{4A^2}\right) + C - \frac{B^2}{4A} \\ &= A\left(x + \frac{B}{2A}\right)^2 + \frac{4AC - B^2}{4A}.\end{aligned}$$

The substitutions

$$u = x + \frac{B}{2A} \qquad \text{and} \qquad b = \frac{\sqrt{|4AC - B^2|}}{2A}$$

then transform the problem to a form where we can use the techniques already developed. The following examples should make this clear.

Example 6 Consider the integral

$$\int \frac{dx}{x^2 + 4x + 5}.$$

This may not immediately remind us of anything we've seen before. But if we rewrite it in the form

$$\int \frac{dx}{(x^2 + 4x + 4) + 1} = \int \frac{dx}{(x+2)^2 + 1},$$

it now begins to resemble something involving an arctangent. In fact, if we make the substitution $u = x + 2$, so $du = dx$, we can write

$$\begin{aligned} \int \frac{dx}{x^2 + 4x + 5} &= \int \frac{du}{u^2 + 1} \\ &= \arctan u + C \\ &= \arctan(x + 2) + C. \end{aligned}$$

Example 7 The technique of completing the square even works for expressions we could have factored directly, if we had noticed:

$$\begin{aligned} \int \frac{dx}{x^2 + 4x + 3} &= \int \frac{dx}{(x+2)^2 - 1} \\ &= \int \frac{dx}{(x + 2 - 1)(x + 2 + 1)} \\ &= \int \frac{dx}{(x+1)(x+3)} \\ &= \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{2} \int \frac{dx}{x+3} \\ &= \frac{1}{2} \ln \left| \frac{x+1}{x+3} \right| + C. \end{aligned}$$

Example 8 Evaluate

$$\int \frac{dx}{\sqrt{6x - x^2}}.$$

Note that $6x - x^2 = -(x^2 - 6x) = -(x-3)^2 + 9 = 9 - (x-3)^2$. If we now substitute $x - 3 = 3u$, with $dx = 3du$, we get

$$\begin{aligned}
\int \frac{dx}{\sqrt{6x - x^2}} &= \int \frac{dx}{\sqrt{9 - (x-3)^2}} \\
&= \int \frac{3\,du}{\sqrt{9 - 9u^2}} = \int \frac{du}{\sqrt{1 - u^2}} \\
&= \arcsin u + C \\
&= \arcsin \frac{x-3}{3} + C.
\end{aligned}$$

Trigonometric Polynomials

A **trigonometric polynomial** is any sum of constant multiples of products of trigonometric functions. The preceding techniques have shown some cases where such trigonometric polynomials can arise, even though the original problem had no apparent reference to trigonometric functions. There are many different ways of breaking trigonometric polynomials down into special cases which can then be integrated. We will develop one way which has the virtue of using few special cases, so that it can be used fairly automatically. It also introduces a powerful tool—that of **reduction formula**—which can be used to generate mathematical results interesting in their own right. One example is the striking representation of π derived in chapter 12.1. Other examples are developed in the exercises at the end of this section.

Since every trigonometric function is expressible in terms of sines and cosines, any trigonometric polynomial can be written as a sum of terms of the form $c \sin^m x \cos^n x$ where c is a constant and m and n are integers—positive, negative, or 0. For instance, $5 \sec^2 x \tan^5 x$ can be rewritten as $5 \sin^5 x \cos^{-7} x$. To find antiderivatives for trigonometric polynomials, it therefore suffices to be able to evaluate integrals of the form

$$\int \sin^m x \cos^n x \, dx.$$

We will see how to find antiderivatives for functions of this sort by breaking the problem into a series of special cases:

Category I	either $m \geq 0$ or $n \geq 0$ (or both)
Case 1	$m = 1$ or $n = 1$
Case 2	$m = 0$ or $n = 0$
Category II	m and n both negative

Category I: Either $m \geq 0$ or $n \geq 0$ (or both)

Assume for the sake of explicitness that $m \geq 0$. We can then use the identity $\sin^2 x = 1 - \cos^2 x$ to replace $\sin^m x$ entirely by cosine terms if m is even, or to replace all but one of the sine terms by cosines if m is odd. A similar replacement can be made if $n \geq 0$.

Example 9

$$\begin{aligned}\sin^4 x \cos^6 x &= (1 - \cos^2 x)^2 \cdot \cos^6 x \\ &= (1 - 2\cos^2 x + \cos^4 x) \cdot \cos^6 x \\ &= \cos^6 x - 2\cos^8 x + \cos^{10} x.\end{aligned}$$

(Note that in this example we could just as well have expressed $\cos^6 x$ entirely in terms of $\sin x$.)

Example 10

$$\begin{aligned}\sin^3 x \cos^{-8} x &= \sin x \cdot (1 - \cos^2 x) \cdot \cos^{-8} x \\ &= \sin x \cos^{-8} x - \sin x \cos^{-6} x.\end{aligned}$$

Example 11

$$\begin{aligned}\sin^{-7} x \cos^7 x &= \sin^{-7} x \cdot (1 - \sin^2 x)^3 \cdot \cos x \\ &= \sin^{-7} x \cos x - 3\sin^{-5} x \cos x + 3\sin^{-3} x \cos x \\ &\quad - \sin^{-1} x \cos x.\end{aligned}$$

We can thus reduce any problem in Category I to one of two special cases:

Case 1	$m = 1$ or $n = 1$
Case 2	$m = 0$ or $n = 0$

We will now see how to find antiderivatives for these cases.

Case 1: $m = 1$ or $n = 1$ Since the two possibilities are analogous, we will consider the case with $n = 1$. Then m can be any real number at all, not necessarily an integer. We make the substitution $u = \sin x$, so that $du = \cos x\,dx$, and

$$\int \sin^m x \cos x\,dx = \int u^m\,du = \begin{cases} \dfrac{1}{m+1}\,u^{m+1} + C & \text{if } m \neq -1, \\ \ln|u| + C & \text{if } m = -1. \end{cases}$$

Replacing u by its expression in x we have the antiderivative:

$$\boxed{\int \sin^m x \cos x\,dx = \begin{cases} \dfrac{1}{m+1}\sin^{m+1} x + C & \text{if } m \neq -1, \\ \ln|\sin x| + C & \text{if } m = -1. \end{cases}}$$

The antiderivative of the cotangent

Remark: The instance $m = -1$ in this case is worth singling out, as it gives us an antiderivative for $\cot x$:

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C.$$

Integrals where $m = 1$ are handled in a completely analogous fashion. You should check that

$$\boxed{\int \cos^n x \sin x\,dx = \begin{cases} \dfrac{-1}{n+1}\cos^{n+1} x + C & \text{if } n \neq -1, \\ -\ln|\cos x| + C & \text{if } n = -1. \end{cases}}$$

The antiderivative of the tangent

Remark: Notice that $n = -1$ gives us an antiderivative for $\tan x$:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C.$$

Case II: $m = 0$ or $n = 0$ Again the two possibilities are analogous, so we will look at instances where $n = 0$. There are a number of clever ways for dealing with antiderivatives of functions of this form, many of them depending on special subcases according to whether m is even or odd, positive or negative, etc. We will develop a single method which deals with all cases in the same way.

Think of $\sin^n x$ as $\sin^{n-1} x \cdot \sin x$ and use integration by parts with

$$F(x) = \sin^{n-1} x \qquad \text{and} \qquad g(x) = \sin x;$$

then

$$f(x) = (n-1)\sin^{n-2} x\, \cos x \qquad \text{and} \qquad G(x) = -\cos x.$$

Therefore

$$\int \sin^n x\, dx = -\sin^{n-1} x\, \cos x + (n-1)\int \sin^{n-2} x\, \cos^2 x\, dx.$$

Now since $\cos^2 x = 1 - \sin^2 x$, we can rewrite the integral on the right-hand side as

$$\int \sin^{n-2} x\, \cos^2 x\, dx. = \int \sin^{n-2} x\, dx - \int \sin^n x\, dx$$

—an expression involving the original integral we are trying to evaluate! If we now substitute this expression in our original equation and bring all the terms involving $\sin^n x$ over to the left-hand side, we have

$$n\int \sin^n x\, dx = -\sin^{n-1} x\, \cos x + (n-1)\int \sin^{n-2} x\, dx,$$

so that

$$\boxed{\int \sin^n x\, dx = \frac{-1}{n}\sin^{n-1} x\, \cos x + \frac{n-1}{n}\int \sin^{n-2} x\, dx.}$$

We thus have a **reduction formula** which reduces the problem of finding an antiderivative for $\sin^n x$ to the problem of finding an antiderivative for $\sin^{n-2} x$. This in turn can be reduced to finding an antiderivative for $\sin^{n-4} x$, and so on, until we get down to having to find an antiderivative for $\sin x$ (if n is odd), or for 1 (if n is even). A reduction formula

Example 12

$$\begin{aligned}\int \sin^5 x\, dx &= \frac{-1}{5}\sin^4 x \cos x + \frac{4}{5}\int \sin^3 x\, dx \\ &= \frac{-1}{5}\sin^4 x \cos x + \frac{4}{5}\left(\frac{-1}{3}\sin^2 x \cos x + \frac{2}{3}\int \sin x\, dx\right) \\ &= \frac{-1}{5}\sin^4 x \cos x - \frac{4}{15}\sin^2 x \cos x - \frac{8}{15}\cos x + C.\end{aligned}$$

Check this answer by taking the derivative of the right-hand side. To show that this derivative really is equal to the integrand on the left, you will need to express all the cosines in terms of sines.

Example 13 If we let $n = 2$, we quickly get the antiderivative for $\sin^2 x$ that we've needed at several points already:

$$\int \sin^2 x\,dx = \frac{-1}{2}\sin x\cos x + \frac{1}{2}\int 1\,dx$$
$$= \frac{x}{2} - \frac{1}{2}\sin x\cos x.$$

In its current form, the reduction formula works best for $n > 0$

The reduction formula as stated is most convenient for $n > 0$, although it is true for any number $n \neq 0$. For if $n < 0$, though, we want to *increase* the exponent, replacing a problem of finding an antiderivative for $\sin^n x$ by a problem where the exponent is less negative. We can do this by rearranging the formula as

$$\int \sin^{n-2} x\,dx = \frac{n}{n-1}\int \sin^n x\,dx + \frac{1}{n-1}\sin^{n-1} x\cos x.$$

The reduction formula for negative exponents

Since we are interested in negative exponents, call $n-2$ by a new name, $-k$. But if $n - 2 = -k$, then $n = -k + 2$, and we can rewrite our formula as

$$\boxed{\int \sin^{-k} x\,dx = -\frac{1}{k-1}\sin^{-(k-1)} x\cos x + \frac{k-2}{k-1}\int \sin^{-(k-2)} x\,dx.}$$

With this formula we can reduce the problem of finding an antiderivative for $\sin^{-k} x$ to the problem of finding an antiderivative for $\sin^{-k+2} x$. This in turn can be reduced to finding an antiderivative for $\sin^{-k+4} x$, and so on, until we get up to having to find an antiderivative for $\sin^{-1} x$ (if k is odd), or for 1 (if k is even). All we need, then, is an antiderivative for $\sin^{-1} x$. But $\sin^{-1} x = \csc x$, and in chapter 11.4 (page 728) we found that

$$\int \csc x\,dx = \int \sin^{-1} x\,dx = -\ln|\csc x + \cot x| + C.$$

We can now handle antiderivatives for any negative integer exponent of the sine function.

Example 14 We can check this formula by trying $k = 2$, which will give us the antiderivative of $\csc^2 x$:

$$\begin{aligned}\int \csc^2 x = \int \sin^{-2} x\,dx \\ &= -\frac{1}{1}\sin^{-1} x\ \cos x + \frac{0}{1}\int \sin^0 x\,dx \\ &= -\sin^{-1} x\ \cos x + C = -\cot x + C,\end{aligned}$$

as it should.

Example 15

$$\begin{aligned}\int \sin^{-3} x\,dx &= -\frac{1}{2}\sin^{-2} x\ \cos x + \frac{1}{2}\int \sin^{-1} x\,dx \\ &= \frac{1}{2}\left(-\sin^{-2} x \cos x - \ln|\csc x + \cot x|\right) + C.\end{aligned}$$

In the exercises you are asked to derive the following reduction formulas for the cosine function:

$$\int \cos^m x\,dx = \frac{1}{m}\cos^{m-1} x\ \sin x + \frac{m-1}{m}\int \cos^{m-2} x\,dx,$$

$$\int \cos^{-m} x\,dx = \frac{1}{m-1}\cos^{-m+1} x\ \sin x + \frac{m-2}{m-1}\int \cos^{-m+2} x\,dx.$$

Category II: Both $m < 0$ and $n < 0$

If we divide the identity $\cos^2 x + \sin^2 x = 1$ by $\sin^2 x\ \cos^2 x$, we get the identity

$$\sin^{-2} x + \cos^{-2} x = \sin^{-2} x\ \cos^{-2} x.$$

We will now use this identity to express anything of the form $\cos^{-r} x\ \sin^{-s} x$ (where $r > 0$ and $s > 0$) as a sum of terms of the form $\sin^{-h} x$, or $\cos^{-i} x$, or $\sin x\ \cos^{-j} x$, or $\sin^{-k} x\ \cos x$. Since we learned how to find antiderivatives for expressions like these in the previous cases, we will then be done.

The trick in transforming $cos^{-r} x\ \sin^{-s} x$ to the desired form is to multiply by $(\cos x\ \cos^{-1} x)$ or $(\sin x\ \sin^{-1} x)$ as needed so that both the sine and the cosine terms appear to *even* negative exponents. Then simply keep using the identity above until there's nothing left to use it on. The following three examples should make clear how the reduction then works.

Example 16 (r and s both even already)

$$\begin{aligned}
\sin^{-4} x \, \cos^{-6} x &= \left(\sin^{-2} x \, \cos^{-2} x\right)^2 \, \cos^{-2} x \\
&= \left(\sin^{-2} x + \cos^{-2} x\right)^2 \, \cos^{-2} x \\
&= (\sin^{-4} x + 2\sin^{-2} x \, \cos^{-2} x + \cos^{-4} x) \, \cos^{-2} x \\
&= (\sin^{-4} x + 2(\sin^{-2} x + \cos^{-2} x) + \cos^{-4} x) \, \cos^{-2} x \\
&= \sin^{-4} x \, \cos^{-2} x + 2\sin^{-2} x \, \cos^{-2} x + 2\cos^{-4} x \\
&\qquad + \cos^{-6} x \\
&= \sin^{-2} x \, (\sin^{-2} x + \cos^{-2} x) + 2(\sin^{-2} x + \cos^{-2} x) \\
&\qquad + 2\cos^{-4} x + \cos^{-6} x \\
&= \sin^{-4} x + \sin^{-2} x \, \cos^{-2} x + 2\sin^{-2} x + 2\cos^{-2} x \\
&\qquad + 2\cos^{-4} x + \cos^{-6} x \\
&= \sin^{-4} x + (\sin^{-2} x + \cos^{-2} x) + 2\sin^{-2} x \\
&\qquad + 2\cos^{-2} x + 2\cos^{-4} x + \cos^{-6} x \\
&= \sin^{-4} x + 3\sin^{-2} x + 3\cos^{-2} x + 2\cos^{-4} x + \cos^{-6} x.
\end{aligned}$$

While this process is tedious, it requires little thought—you simply replace $\sin^{-2} x \, \cos^{-2} x$ with $\sin^{-2} x + \cos^{-2} x$ at every opportunity until there is no negative-exponent sine term multiplying any negative-exponent cosine term. We will use this result to demonstrate how to deal with cases where either r or s (or both) is odd.

Example 17 (r even and s odd)

$$\begin{aligned}
\sin^{-4} x \, \cos^{-5} x &= \sin^{-4} x \, \cos^{-6} x \, \cos x \\
&= \sin^{-4} x \, \cos x + 3\sin^{-2} x \, \cos x + 3\cos^{-1} x \\
&\qquad + 2\cos^{-3} x + \cos^{-5} x
\end{aligned}$$

Example 18 (both r and s odd)

$$\begin{aligned}
\sin^{-3} x \, \cos^{-5} x &= \sin x \, \sin^{-4} x \, \cos^{-6} x \, \cos x \\
&= \sin^{-3} x \, \cos x + 3\sin^{-1} x \, \cos x + 3\sin x \, \cos^{-1} x \\
&\qquad + 2\sin x \, \cos^{-3} x + \sin x \, \cos^{-5} x
\end{aligned}$$

Exercises

1. Find the following antiderivatives (a is a positive constant):

a) $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

b) $\displaystyle\int \frac{dx}{\sqrt{1-4x^2}}$

c) $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

d) $\displaystyle\int \frac{x\,dx}{\sqrt{4+x^2}}$

e) $\displaystyle\int \frac{dx}{(a^2-x^2)^{3/2}}$

f) $\displaystyle\int \frac{dx}{4+x^2}$

g) $\displaystyle\int \frac{x\,dx}{4+x^2}$

h) $\displaystyle\int \frac{dx}{x\sqrt{4+x^2}}$

i) $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-a^2}}$

j) $\displaystyle\int \frac{dx}{(a^2+x^2)^2}$

2. Evaluate the following integrals:

a) $\displaystyle\int_1^{-1} \frac{dx}{4-x^2}$

b) $\displaystyle\int_1^2 \sqrt{x^2-1}\,dx$

c) $\displaystyle\int_0^{\pi/3} x\sec^2 x\,dx$

d) $\displaystyle\int_0^1 \frac{dx}{(2-x^2)^{3/2}}$

e) $\displaystyle\int_0^{\infty} \frac{dx}{9+x^2}$

f) $\displaystyle\int_a^{2a} x^3\sqrt{x^2-a^2}\,dx$

3. Sketch the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, labelling the coordinates of the points where it crosses the x-axis and the y-axis. Prove that the area of this ellipse is πab.

4. Find the following antiderivatives:

a) $\displaystyle\int \frac{dx}{\sqrt{x^2-2x-8}}$

b) $\displaystyle\int \frac{dx}{x^2+6x+10}$

c) $\displaystyle\int \frac{dx}{\sqrt{x^2+6x+8}}$

d) $\displaystyle\int \frac{x\,dx}{x^2+4x+5}$

e) $\displaystyle\int \frac{x\,dx}{\sqrt{5+4x-x^2}}$

f) $\displaystyle\int \frac{(2x+7)\,dx}{4x^2+4x+5}$

g) $\displaystyle\int \frac{(4x-3)\,dx}{\sqrt{-x^2-2x}}$

h) $\displaystyle\int \frac{dx}{(a^2-x^2-2x)^2}$

5. a) If $x = a \sec u$, where a is a constant, draw a right triangle containing an angle u with lengths of sides specified to reflect this relation between x, a, and u.

b) In terms of x and a, what is $\sin u$? What is $\cos u$? What is $\tan u$?

6. Evaluate the following:

a) $\displaystyle\int \frac{dx}{\sin x \, \cos x}$

b) $\displaystyle\int \cos^3 x \, \sin^{-4} x \, dx$

c) $\displaystyle\int \csc^4 x \, \cot^2 x \, dx$

d) $\displaystyle\int \sin 3x \, \cot 3x \, dx$

e) $\displaystyle\int_0^{\pi/2} \sin^n x \, \cos^3 x \, dx$

f) $\displaystyle\int \tan^5 x \, dx$

g) $\displaystyle\int \frac{\sin^3 5x \, dx}{\sqrt[3]{\cos 5x}}$

h) $\displaystyle\int \frac{\cos^3(\ln x) \, dx}{x}$

i) $\displaystyle\int \sec^4 x \, \ln(\tan x) \, dx$

j) $\displaystyle\int_0^{a/2} \frac{dx}{(a^2 - x^2)^{3/2}}$

7. Use the analysis of Example 17 (page 748) to find an antiderivative for $\sin^{-4} x \, \cos^{-5} x$.

Reduction formulas

8. Derive the reduction formulas for the cosine function given on page 747.

9. a) By writing $\tan^n x = \tan^{n-2} x(\sec^2 x - 1)$, get a reduction formula which expresses $\displaystyle\int \tan^n x \, dx$ in terms of $\displaystyle\int \tan^{n-2} x \, dx$.

b) Use this evaluation formula to find $\displaystyle\int \tan^6 x \, dx$.

c) Show that

$$\int_0^{\pi/4} \tan^n x \, dx = \begin{cases} \dfrac{1}{n-1} - \dfrac{1}{n-3} + \cdots \pm \dfrac{1}{3} \mp 1 \pm \pi/4 & \text{if } n \text{ is even,} \\ \dfrac{1}{n-1} - \dfrac{1}{n-3} + \cdots \pm \dfrac{1}{4} \mp \dfrac{1}{2} \pm \dfrac{1}{2} \ln 2 & \text{if } n \text{ is odd.} \end{cases}$$

d) Give a clear argument why $\lim_{n\to\infty} \int_0^{\pi/4} \tan^n x\,dx = 0.$

e) Prove that

$$\lim_{k\to\infty} \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots \pm \frac{1}{2k+1}\right) = \frac{\pi}{4},$$

and

$$\lim_{k\to\infty} \left(1 - \frac{1}{2} + \frac{1}{3} - \cdots \pm \frac{1}{k}\right) = \ln 2.$$

10. a) By writing $\sec^n x$ as $\sec^{n-2} x \sec^2 x$ and using integration by parts, get a reduction formula which expresses $\int \sec^n x\,dx$ in terms of $\int \sec^{n-2} x\,dx$.

b) Since $\sec x = \cos^{-1} x$, the formula you got in part (a) could also have been obtained from the reduction formula for cosines on page 747. Try it and see if the formulas are in fact the same.

11. a) Find a reduction formula that expresses

$$\int x^n e^x\,dx \quad \text{in terms of} \quad \int x^{n-1} e^x\,dx.$$

b) Using the results of part (a), show that

$$\frac{1}{n!}\int_0^t x^n e^x\,dx = e^t\left(\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!} - \cdots \pm \frac{t^2}{2!} \mp t \pm 1\right) \mp 1.$$

c) Explain why, for a fixed value of t,

$$\lim_{n\to\infty} \frac{1}{n!}\int_0^t x^n e^x\,dx = 0.$$

d) Prove that

$$\lim_{n\to\infty} \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \cdots \pm \frac{t^n}{n!}\right) = e^{-t}.$$

12. a) Find a reduction formula expressing

$$\int \frac{dx}{(1+x^2)^n} \quad \text{in terms of} \quad \int \frac{dx}{(1+x^2)^{n-1}};$$

you can do this using integration by parts, or you can use a trigonometric substitution.

b) What is the exact value of $\displaystyle\int_0^1 \frac{dx}{(1+x^2)^5}$?

13. Our approach to integrating trigonometric polynomials was to express everything in the form $\sin^m x \cos^n x$. We can just as readily express everything in the form $\sec^j x \tan^k x$ (j and k integers—positive, negative, or 0), and develop our technique by dealing with various cases of this. See if you can work out the details, trying to parallel the approach developed in the text using sines and cosines as our basic functions.

11.6 Simpson's Rule

A return to numerical methods

This chapter has concentrated on formulas for antiderivatives, because a formula conveys compactly a lot of information. However, you must not lose sight of the fact that most antiderivatives cannot be found by such analytic methods. The integrand may be a data function, for instance, and thus have no formula. And even when the integrand is given by a formula, there may be no formula for the antiderivative itself. One possibility in such cases is to **approximate** such a function by a function—such as a polynomial—for which we can readily find an antiderivative. In chapter 10 we saw some methods for doing this. In chapter 12 we introduce Fourier series, providing another family of approximating functions for which antiderivatives can be readily obtained. Another approach is to find a desired definite integral using approximating rectangles, as we did in chapter 6.

In any case, numerical methods are inescapable, but accurate results require many calculations. This takes time—even on a modern high-speed computer. A numerical method is said to be **efficient** if it gets accurate results quickly, that is, with relatively few calculations. In chapter 6 we saw that *midpoint* Riemann sums are much more efficient than left or right *endpoint* Riemann sums. We will look at these and other methods in detail in this section. The most efficient method we will develop is called Simpson's rule.

Efficient numerical integration

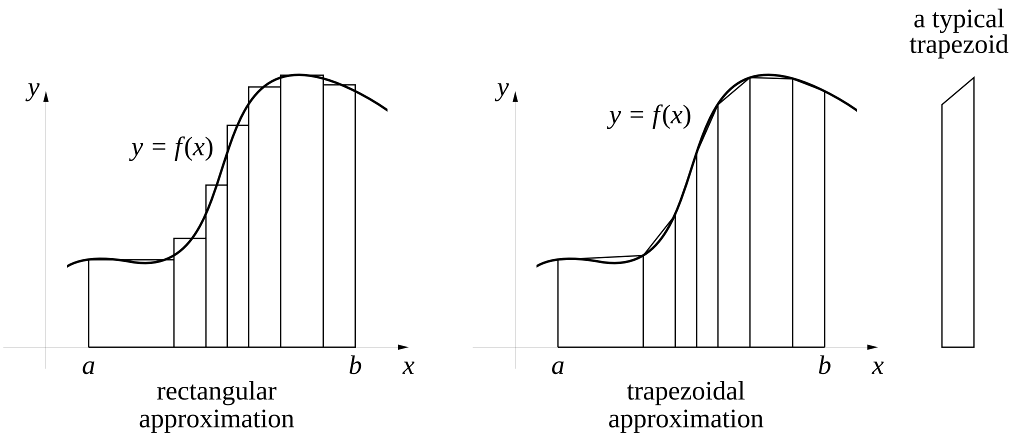
The Trapezoid Rule

We interpret the integral

$$\int_a^b f(x)\,dx$$

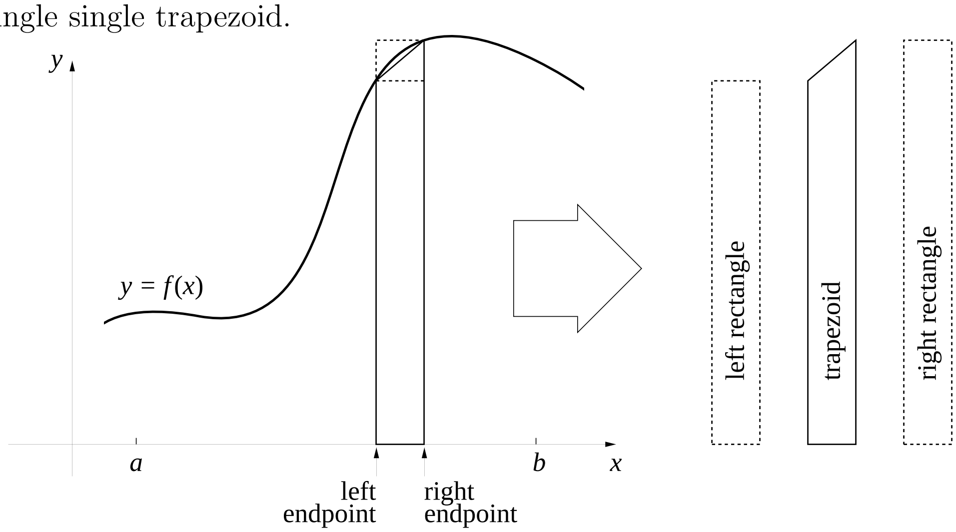
as the area under the graph $y = f(x)$ between $x = a$ and $x = b$. We interpret a Riemann sum as the total area of a collection of rectangles that approximate the area under the graph. The tops of the rectangles are level, and they represent the graph of step function. Clearly, we get a better approximation to the graph by using slanted lines. They form the tops of a sequence of **trapezoids** that approximate the area under the graph.

Replace rectangles by trapezoids



Let's figure out the areas of these trapezoids. They are related in a simple way to the rectangles that we would construct at the right and left endpoints to calculate Riemann sums. To see the relation, let's take a closer look at a single single trapezoid.

Each trapezoid is sandwiched between two rectangles. . .



. . . whose average area equals the area of the trapezoid

It is sandwiched between two rectangles, one taller and one shorter. In our picture the height of the taller rectangle is f(right endpoint). We will call it the *right rectangle*. The height of the shorter rectangle is f(left endpoint). We will call it the *left rectangle*. For other trapezoids the left rectangle may be the taller one. In any case, the trapezoid is exactly half-way between the two rectangles in size, and thus its area is the *average* of the areas of the rectangles:

$$\text{area trapezoid} = \tfrac{1}{2}\left(\text{area left rectangle} + \text{area right rectangle}\right).$$

The trapezoidal approximation is the average of left and right Riemann sums

If we sum over the areas of all of the trapezoids, the areas of the right rectangles sum to the right Riemann sum, and similarly for the left rectangles. In follows that

$$\text{trapezoidal approximation} = \tfrac{1}{2}(\text{left Riemann sum} + \text{right Riemann sum}).$$

The pictures make it clear that the trapezoidal approximation should be significantly better than either a left or a right Riemann sum. To test this numerically, let's get numerical estimates for the integral

$$\int_1^3 \frac{1}{x}\,dx = \ln 3 = 1.098612288668\ldots.$$

Comparing approximations

(The relation between the trapezoid approximation and the left and right Riemann sums holds for any choice Δx_k of subintervals. However, we will use equal subintervals to make the calculations simpler.) Here is how our four main estimates compare when we use 100 subintervals.

$n = 100$	approximation	error
right	1.09197525	6.63×10^{-3}
left	1.10530858	-6.69×10^{-3}
midpoint	1.09859747	1.48×10^{-5}
trapezoidal	1.09864191	-2.90×10^{-5}

The figures in this table are calculated to 8 decimal places, and the column marked *error* is the difference

$$1.09861229 - \text{approximation},$$

so that the error is negative if the approximation is too large. The left Riemann sum is too large, for instance.

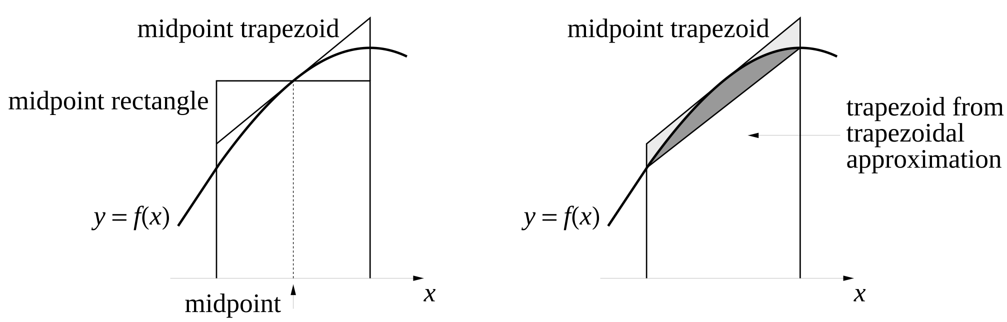
Before we comment on the differences between the estimates, let's gather more data. Here are the calculations for 1000 subintervals. Note that the midpoint and trapezoidal approximations are more than 1000 times better than the left or right Riemann sums!

$n = 1000$	approximation	error
right	1.09794591	6.663×10^{-4}
left	1.09927925	-6.669×10^{-4}
midpoint	1.09861214	1.4×10^{-7}
trapezoidal	1.09861258	-2.9×10^{-7}

A surprise: the midpoint approximation is even better than the trapezoidal

We expected the trapezoidal approximation to be better than either the right or left Riemann sum. The surprising observation is that the midpoint Riemann sum is even better! In fact, it appears that the midpoint Riemann sum has only *half* the error of the trapezoidal approximation.

The figures below explain geometrically why the midpoint approximation is better than the trapezoidal. The first step is shown on the left. Take a midpoint rectangle (whose height is f(midpoint)), and rotate the top edge around the midpoint until it is tangent to the graph of $y = f(x)$. Call this a **midpoint trapezoid**. Notice that the trapezoid has the same area as the rectangle.



Shading depicts the errors

The second step is to compare the midpoint trapezoid to the one used in the trapezoidal approximation. This is done on the right. The error coming from the midpoint trapezoid is shaded light gray, while the error from the trapezoidal approximation is dark gray. The midpoint trapezoid is the better approximation. Since the midpoint rectangle has the same area as the midpoint trapezoid, we now see why the midpoint Riemann sum is more accurate than the trapezoidal approximation. This picture also explains why the errors of the two approximations have different signs, which we noticed first in the tables.

Simpson's Rule

Combine good approximations...

Our goal is a calculation scheme for integrals that gives an error is as small as possible. The trapezoidal and the midpoint approximations are both good—but we can combine them to get something even better. Here is why. The tables and the figure above indicate that the *errors* in the two approximations have opposite signs, and the midpoint error is only about half the size of the trapezoidal error (in absolute value). Thus, if we form the sum

$$2 \times \text{midpoint approximation} + \text{trapezoidal approximation},$$

...so that most of the error cancels

then most of the error will cancel. Now this sum is approximately three times the value of the integral, because each term in it approximates the integral itself. Therefore, if we divide by three, then

$$\tfrac{2}{3} \times \text{midpoint approximation} + \tfrac{1}{3} \times \text{trapezoidal approximation}$$

should be a superb approximation to the integral.

Let's try this approximation on our test integral

$$\int_1^3 \frac{1}{x}\,dx$$

with $n = 100$ subintervals. Using the numbers from the table on page 754, we obtain

$$\tfrac{2}{3} \times 1.09859747 + \tfrac{1}{3} \times 1.09864191 = 1.098612283,$$

which gives the value of the integral accurate to 8 decimal places. This method of approximating integrals is called **Simpson's rule**.

Using Riemann sums to carry out Simpson's rule

We can use the program RIEMANN to do the calculation. First calculate left and right Riemann sums, and take their average. That is the trapezoidal approximation. Then calculate the midpoint Riemann sum. Since

$$\text{trapezoid} = \tfrac{1}{2} \times \text{left} + \tfrac{1}{2} \times \text{right},$$

Simpson's rule reduces to this combination of left, right, and midpoint sums:

$$\begin{aligned}
&\tfrac{2}{3} \times \text{midpoint} + \tfrac{1}{3} \times \text{trapezoidal} \\
&\quad = \tfrac{2}{3}\text{midpoint} + \tfrac{1}{3}\left(\tfrac{1}{2} \times \text{left} + \tfrac{1}{2} \times \text{right}\right) \\
&\quad = \tfrac{2}{3} \times \text{midpoint} + \tfrac{1}{6} \times \text{left} + \tfrac{1}{6} \times \text{right} \\
&\quad = \tfrac{1}{6}\left(4 \times \text{midpoint} + \text{left} + \text{right}\right)
\end{aligned}$$

Simpson's rule:

$$\int f(x)\,dx \approx \frac{1}{6}\left(\text{left sum} + \text{right sum} + 4 \times \text{midpoint sum}\right)$$

The accuracy of Simpson's rule

You can get even more accuracy if you keep track of more digits in the left, right, and midpoint Riemann sums. For example, if you estimate

$$\int_1^3 \frac{1}{x}\,dx = \ln 3 = 1.098\,612\,288\,668\ldots$$

to 14 decimal places, you will get the following.

left:	1.105 308 583 647 79
right:	1.091 975 250 314 45
midpoint:	1.098 597 475 005 31

When combined these give the estimate 1.098 612 288 997 3, which differs from the true value by less than 3.3×10^{-10}. In other words, the calculation is actually correct to 9 decimal places.

An error bound for Simpson's rule

It is possible to get a bound on the error produced by using Simpson's rule to estimate the value of

$$\int_a^b f(x)\,dx.$$

(See the discussion of error bounds in chapter 6.3.) Specifically,

$$\left| \int_a^b f(x)\,dx - \text{Simpson's rule} \right| \leq \frac{M(b-a)^5}{2880\,n^4},$$

where n is the number of subintervals used in the Riemann sums and M is a bound on the size of the fourth derivative of f:

$$|f^{(4)}(x)| \leq M \qquad \text{for all } a \leq x \leq b.$$

The crucial factor n^{-4}

The most important factor in the error bound is the n^4 that appears in the denominator. In our example $n = 100 = 10^2$, and this leads to the factor $1/n^4 = 10^{-8}$ in the error bound. As we saw, the actual error was less than 10^{-9}. Essentially, n is the number of computations we do, and error bound tells how many decimal places of accuracy we can count on. According to the error bound, a ten-fold increase in the number of computations produces four more decimal places of accuracy. *That* is why Simpson's method is efficient.

Exercises

Because Simpson's rule is so efficient, we can use it to get accurate values of some of the fundamental constants of mathematics. For example, since

$$4 \cdot \int_0^1 \frac{dx}{1+x^2} = 4\arctan(x)\,\Big|_0^1 = 4\arctan(1) = 4 \cdot \frac{\pi}{4} = \pi,$$

we can estimate the value of π by using Simpson's rule to approximate this integral.

1. Evaluate the expression above (including the factor of 4) use Simpson's rule with $n = 2$, 4, 8, and 16. How accurate is each of these estimates of π; that is, how many decimal places of each estimate agree with the true value of π?

2. a) Over the interval $0 \le x \le 1$ it is true that

$$|f^{(4)}(x)| \le 96 \qquad \text{when} \qquad f(x) = \frac{4}{1+x^2}.$$

(You don't need to show this, but how might you do it?) Use this bound to show that $n = 256 = 2^8$ will guarantee that you can find the first 10 decimals of π by using the method of the previous question.

b) Show that if $n = 128 = 2^7$ then the error bound for Simpson's rule does *not* guarantee that you can find the first 10 decimals of π by the same method.

c) Run Simpson's rule with $n = 2^7$ to estimate π. How many decimal places *are* correct? Does this surprise you? In fact the error bound is too timid: it says that the error is no larger than the bound it gives, but the actual error may be much smaller. From your work in part (a), which power of 2 is sufficient to get 10 decimal places accuracy?

3. a) In chapter 6.3 (page 391) a left Riemann sum for

$$\int_0^1 e^{-x^2}\,dx$$

with 1000 equal subdivisions gave 3 decimal places accuracy. How many subdivisions n are needed to get that much accuracy using Simpson's rule? Let n be a power of 2. Start with $n = 1$ and increase n until three digits stabilize.

b) If you use Simpson's rule with $n = 1000$ to estimate this integral, how many digits stabilize?

4. On page 374 a midpoint Riemann sum with $n = 10000$ shows that

$$\int_1^3 \sqrt{1 + x^3}\,dx = 6.229959\ldots.$$

How many subdivisions n are needed to get this much accuracy using Simpson's rule? Start with $n = 1$ and keep doubling it until seven digits stabilize.

11.7 Improper Integrals

The Lifetime of Light Bulbs

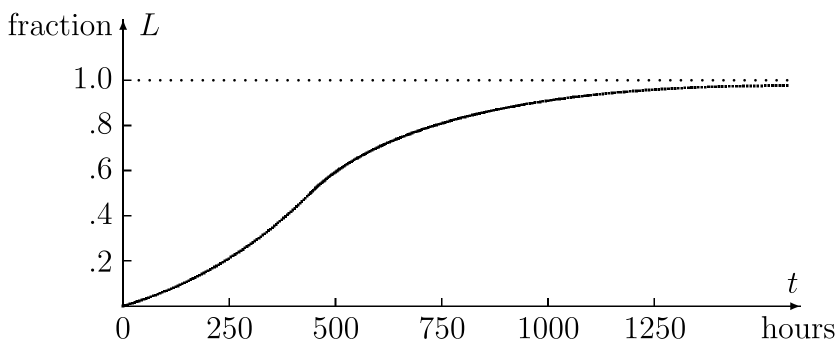
The lifetime of a light bulb is unpredictable

Ordinary light bulbs are supposed to burn about 700 hours, but of course some last longer while others burn out more quickly. It is impossible to know, in advance, the lifetime of a particular bulb you might buy, but it is possible to describe what happens to a large batch of bulbs.

Suppose we take a batch of 1000 light bulbs, start them burning at the same time, and note how long it takes each one to burn out. Let

$$L(t) = \text{fraction of bulbs that burn out before } t \text{ hours}$$

Then $L(t)$ might have a graph that looks like this:



In this example, $L(400) \approx .5$, so about half the bulbs burned out before 400 hours. Furthermore, all but a few have burned out by 1250 hours.

The burnout rate

Manufacturers are very concerned about the way the lifetime of light bulbs varies. They study the output of their factories on a regular basis. It is more common, though, for them to talk about the *rate* r at which bulbs burn out. The rate varies over time, too. In fact, in terms of L, r is just the derivative

$$r(t) = L'(t) \quad \text{bulbs per hour.}$$

However, if we *start* with the rate r, then we get L as the integral

$$L(t) = \int_0^t r(s)\,ds.$$

Lifetime is the integral of burnout rate...

This is yet another consequence of the fundamental theorem of calculus. The integral expression is quite handy. For example, the fraction of bulbs that burn out between $t = a$ hours and $t = b$ hours is

$$L(b) - L(a) = \int_a^b r(s)\,ds.$$

We can even use the integral to say that all the bulbs burn out eventually:

$$L(t) = \int_0^t r(s)\,ds = 1 \qquad \text{when } t \text{ is sufficiently large.}$$

...but there is no upper limit to the lifetime

In practice r is the average burnout rate for many batches of light bulbs, so we can't identify the precise moment when L becomes 1. All we can really say is

$$L(\infty) = \int_0^\infty r(s)\,ds.$$

This is called an **improper integral**, because it cannot be calculated directly: its "domain of integration" is infinite. By definition, its value is obtained as a limit of ordinary integrals:

An integral is *improper* if the domain of integration is infinite

$$\int_0^\infty r(s)\,ds = \lim_{b\to\infty}\int_0^b r(s)\,ds.$$

The **normal density function** of probability theory provides us with another example of an improper integral. In a simple form, the function itself is

$$f(x) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{x^2}{2}\right).$$

(Recall, $\exp(x) = e^x$.) If x is any *normally distributed quantity* whose average value is 0, then the probability that a randomly chosen value of x lies between the numbers a and b is

$$\int_a^b f(x)\,dx.$$

Since the probability that x lies *somewhere* on the x-axis is 1, we have

The normal probability distribution involves an improper integral

$$\int_{-\infty}^\infty f(x)\,dx = 1.$$

This is an improper integral, and its value is defined by the limit

$$\int_{-\infty}^\infty f(x)\,dx = \lim_{b\to\infty}\int_{-b}^b f(x)\,dx.$$

In the exercises you will have a chance to evaluate this integral.

Evaluating Improper Integrals

An integral with an infinite domain of integration is only one kind of improper integral. A second kind has a finite domain of integration, but the integrand becomes infinite on that domain. For example,

An integral is also improper if its integrand becomes infinite

$$\int_0^1 \frac{dx}{x} \qquad \text{and} \qquad \int_0^1 \ln x\,dx$$

are both improper in this sense. In both cases, the integrand becomes infinite as $x \to 0$. Because the difficulty lies at the endpoint 0, we define

$$\int_0^1 \frac{dx}{x} = \lim_{a \to 0} \int_a^1 \frac{dx}{x}.$$

More generally,

$$\int_a^b f(x)\,dx$$

is an improper integral if $f(x)$ becomes infinite at some point c in the interval $[a, b]$. In that case we define

$$\int_a^b f(x)\,dx = \lim_{q \to 0} \left(\int_a^{c-q} f(x)\,dx + \int_{c+q}^b f(x)\,dx \right).$$

In effect, we avoid the bad spot but "creep up" on it in the limit.

Antiderivatives help find improper integrals

Indefinite integrals—that is, antiderivatives—can be a great help in evaluating improper integrals. Here are some examples.

Example 1. We can evaluate

$$\int_0^\infty e^{-x}\,dx$$

by noting first that $\int e^{-x}\,dx = -e^{-x}$. Therefore

$$\int_0^b e^{-x}\,dx = -e^{-x}\Big|_0^b = -e^{-b} - \left(-e^{-0}\right) = 1 - e^{-b}$$

and

$$\int_0^\infty e^{-x}\,dx = \lim_{b \to \infty} \int_0^b e^{-x}\,dx = \lim_{b \to \infty} \left(1 - e^{-b}\right) = 1.$$

Example 2. To evaluate $\int_0^1 \ln x\,dx$, we use the indefinite integral

$$\int \ln x\,dx = x\,\ln x - x.$$

Thus

$$\int_a^1 \ln x\,dx = x\ \ln x - x\Big|_a^1 = -1 - (a\ln a - a) = a - 1 - a\ln a.$$

By direct calculation (using a graphing package, for instance) we can find

$$\lim_{a\to 0}\ a\ln a = 0;$$

therefore

$$\int_0^1 \ln x\,dx = \lim_{a\to 0}\int_a^1 \ln x\,dx = \lim_{a\to 0}\ a - 1 - a\ln a = -1.$$

You should not assume that an improper integral always has a finite value, though. Consider the next example.

Example 3. $\displaystyle\int_0^1 \frac{dx}{x} = \lim_{a\to 0}\int_a^1 \frac{dx}{x} = \lim_{a\to 0}\ln(x)\Big|_a^1 = \lim_{a\to 0}(\ln(1) - \ln(a)) = \infty.$

This is forced because $\displaystyle\lim_{a\to 0}\ \ln(a) = -\infty$, which you can see from the graph of the logarithm function.

Exercises

1. Find the value of each of the following improper integrals. (The value may be ∞.)

a) $\displaystyle\int_{-\infty}^0 e^x\,dx$

b) $\displaystyle\int_1^\infty \frac{du}{u}$

c) $\displaystyle\int_0^1 \frac{dy}{y^2}$

d) $\displaystyle\int_0^{\pi/2} \tan x\,dx$

e) $\displaystyle\int_0^\infty xe^{-x}\,dx$

f) $\displaystyle\int_1^\infty \frac{du}{u^2}$

g) $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx$

h) $\displaystyle\int_1^3 \frac{x}{x^2-1}\,dx$

2. Use the reduction formula for $\displaystyle\int \frac{dx}{(1+x^2)^n}$ you found on page 752 to find the exact value of $\displaystyle\int_0^\infty \frac{dx}{(1+x^2)^{10}}$

The normal density function

The next two questions concern the improper integral

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx$$

of the normal density function defined on page 761. The goal is to determine the value of this integral.

3. First, use RIEMANN to estimate the value of

$$\frac{1}{\sqrt{2\pi}}\int_{-b}^{b} e^{-x^2/2}\,dx$$

when b has the different values 1, 10, 100, and 1000. On the basis of these results, estimate

$$\lim_{b\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-b}^{b} e^{-x^2/2}\,dx.$$

This gives one estimate of the value of the improper integral.

4. a) To construct a second estimate, begin by sketching the graph of the normal density function

$$f(x) = \frac{1}{\sqrt{2\pi}}e^{-x^2/2}$$

on an interval centered at the origin. Use the graph to argue that

$$\frac{1}{\sqrt{2\pi}}\int_{-b}^{b} e^{-x^2/2}\,dx = 2\left(\frac{1}{\sqrt{2\pi}}\int_{0}^{b} e^{-x^2/2}\,dx\right) = \sqrt{2/\pi}\int_{0}^{b} e^{-x^2/2}\,dx$$

and therefore

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2}\,dx = \sqrt{2/\pi}\int_{0}^{\infty} e^{-x^2/2}\,dx.$$

b) Now consider the accumulation function

$$F(t) = \sqrt{2/\pi}\int_{0}^{t} e^{-x^2/2}\,dx.$$

We want to find $F(\infty) = \lim_{t\to\infty} F(t)$. According to the fundamental theorem of calculus, $y = F(t)$ satisfies the initial value problem

$$\frac{dy}{dt} = \sqrt{2/\pi} \cdot e^{-t^2/2}, \qquad y(0) = 0.$$

Use a differential equation solver (e.g., PLOT) to graph the solution $y = F(t)$ to this problem. From the graph determine

$$F(\infty) = \lim_{t\to\infty} F(t).$$

c) Does your results in part (b) and question 2 agree? Do they agree with the value the text claims for the improper integral. (Remember, the value is the probability that a randomly chosen number will lie *somewhere* on the number line between $-\infty$ and $+\infty$.)

The gamma function

The **factorial function** is defined for a positive integer n by the formula

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1.$$

For example, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$, and $10! = 3628800$. The factorial function is used often in diverse mathematical contexts, but its use is sometimes limited by the fact that it is defined only for positive integers. How might the function be defined on an expanded domain, so that we could deal with expressions like $tfrac12!$, for example? The *gamma function* answers this question.

The **gamma function** $\Gamma(x)$ is defined by the improper integral

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt.$$

Notice that t is the active variable in this integral. While the integration is being performed, x is treated as a constant; for the integral to converge, we need $x > 0$.

5. Show that $\Gamma(1) = 1$.

6. Using integration by parts, show that $\Gamma(x+1) = x \cdot \Gamma(x)$. You may use the fact that

$$\frac{t^p}{e^t} \to 0 \text{ as } t \to +\infty.$$

The property $\Gamma(x+1) = x \cdot \Gamma(x)$ makes the gamma function like the factorial function, because

$$(n+1)! = (n+1) \cdot \underbrace{n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 3 \cdot 2 \cdot 1}_{= n!} = (n+1) \cdot n!.$$

Notice there is a slight difference, though. We explore this now.

7. Using the property $\Gamma(x+1) = x \cdot \Gamma(x)$, calculate $\Gamma(2)$, $\Gamma(3)$, $\Gamma(4)$, $\Gamma(5)$, and $\Gamma(6)$. On the basis of this evidence, fill in the blank:

$$\text{For a positive integer } n, \qquad \Gamma(n) = \underline{\qquad}!$$

Using this relation, give a computable meaning to the expression $\frac{1}{2}!$.

8. Estimate the value of $\Gamma(1/2)$. (Exercises 2 and 3 above offer two ways to estimate the value of an improper integral.)

9. In fact, $\Gamma(1/2) = \sqrt{\pi}$ exactly. You can show this by employing several of the techniques developed in this chapter. Start with

$$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-\frac{1}{2}}\, dt.$$

a) Make the substitution $u = (2t)^{\frac{1}{2}}$ and show that the integral becomes

$$\Gamma(1/2) = \sqrt{2} \int_0^\infty e^{-u^2/2} du.$$

b) From exercise 4 you know

$$\sqrt{2/\pi} \int_0^\infty e^{-u^2/2}\, du = 1.$$

(Check this.) Now, using some algebra, show $\Gamma(1/2) = \sqrt{\pi}$.

c) Compare your estimate for $\Gamma(1/2)$ from exercise 7 with the exact value $\sqrt{\pi}$.

10. a) Determine the exact values of $\Gamma(3/2)$ and $\Gamma(5/2)$.

b) In exercise 6 you gave a meaning to the expression $\frac{1}{2}!$; can you now give it an exact value?

11.8 Chapter Summary

The Main Ideas

- A function F is an **antiderivative** of f if $F' = f$. *Every* antiderivative of f is equal to $F + C$ for some appropriately chosen constant C. We write $\int f(x)\,dx = F(x) + C$.

- Differentiation rules for combinations of functions yield corresponding anti-differentiation rules. Among these are the **constant multiple** and **addition** rules. The chain rule for differentiation corresponds to **integration by substitution**. The product rule for differentiation corresponds to **integration by parts**.

- The derivative of a function and of its inverse are reciprocals. When $y = f(t)$ and $t = g(y)$ are inverses:

$$\frac{dt}{dy} = \frac{1}{dy/dt}.$$

- In some cases, the method of **separation of variables** can be used to find a *formula* for the solution of a differential equation.

- A numerical method for estimating an integral is **efficient** if it gets accurate results with relatively few calculations. The **trapezoidal approximation** is the average of a left and a right *endpoint* Riemann sum and is more efficient than either. *Midpoint* Riemann sums are even more efficient than trapezoidal approximations.

- The most efficient method developed in this chapter is **Simpson's rule**. Simpson's rule approximates an integral by

$$\int_a^b f(x)\,dx \approx \frac{1}{6}\left(\text{left sum} + \text{right sum} + 4 \times \text{midpoint sum}\right).$$

- An **improper integral** is one that cannot be calculated directly. The problem may be that its "domain of integration" is infinite or that the integrand becomes infinite on that domain. Its value is obtained as a limit of ordinary integrals.

Expectations

- You should be able to find antiderivatives of basic functions.
- You should be able to find antiderivatives of combinations of functions using the **constant multiple** and **addition** rules, as well as the **method of substitution** and **integration by parts**.
- You should be able to rewrite an integrand given as a quotient using the method of **partial fractions**.
- You should be able to express the derivative of an invertible function in terms of the derivative of its inverse. In particular, you should be able to differentiate the **arctangent**, **arcsine** and **arccosine** functions.
- You should be able to solve a differential equation using the method of **separation of variables**.
- You should be able to adapt the program RIEMANN to approximate integrals using the **trapezoid rule** and **Simpson's rule**.
- You should be able to find the value of an **improper integral** as the limit of ordinary integrals.