Steady-State Geotherm

Problem: Calculate the setady-state geotherm for a 30 km thick crust with a uniform distribution of heat producing elements. Assume that the average heat production (A) is 2.0×10^{-6} W/m^3, that the steady mantle heat flux into the base of the crust is 1.0×10^{-2} W/m^2, that the thermal conductivity (k) of the crust is 2.5 W/mK, that the volumetric heat capacity (ρC_P) of the crust is 2.5×10^6 J/m^{3}K, and that the temperature (T) at the surface is 0°C. Let the depth z=0 at the surface and z=-3.0 x 10^4 m at the base of the crust. The required heat conduction equations are:

$$\text{heat flux} = -k\left(\frac{\partial T}{\partial z}\right)_t \qquad \text{and} \qquad \left(\frac{\partial T}{\partial t}\right)_z = \frac{k}{\rho C_P}\left(\frac{\partial^2 T}{\partial z^2}\right)_t + \frac{A}{\rho C_P}$$

where t (s) is the time, ρ (Kg/m^3) is the density of the crust, and C_P (J/KgK) is the specific heat capacity of the crust. The second equation assumes (1) that the thermal parameters for the crust are uniform throughout the crust and (2) that the symmetry of the problem permits a one-dimensional solution.

In the steady-state, $\partial T/\partial t = 0$. Therefore, the heat conduction equation reduces to

$$\left(\frac{d^2T}{dz^2}\right) = -\frac{A}{k},$$

which is a comparatively simple differential equation. The solution is of the form

$$T = \left(-\frac{A}{2k}\right)z^2 + \alpha z + \beta \qquad \text{with} \qquad \left(\frac{dT}{dz}\right) = \left(-\frac{A}{k}\right)z + \alpha$$

where α and β are constants. At the surface, z=0 and T=0; therefore, β=0. At z=-30,000 m,

$$\text{heat flux} = -k\left(\frac{dT}{dz}\right) = -k\left(-\frac{A}{k}\right)z - k\alpha = 0.01 \text{ W/m}^2,$$

which may be solved for α to yield

$$\alpha = \frac{Az}{k} - \frac{0.01}{k} = \frac{\left(2.0 \times 10^{-6}\right)\left(-3.0 \times 10^{4}\right)}{2.5} - \frac{0.01}{2.5} = 0.028 \text{ K/m}.$$

The solution is then

$$T = -\left(4.0 \times 10^{-7}\right)z^2 - (0.028)\,z .$$

The heat flow at the surface for this model is given by

$$\text{heat flux} = -k\left(\frac{dT}{dz}\right)_{z=0} = -k\alpha = (2.5)\,(0.028) = 0.07 \text{ W/m}^2 .$$