Minimum-Perimeter Enclosing k-gon

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Introduction Let $P = p_1, \ldots, p_n$ be a simple polygon (all polygons are assumed convex throughout this paper). A fundamental problem in geometric optimization is to compute a minimum-area or a minimum-perimeter convex k-gon (denoted Q^{A} or Q^{p}, resp.) that encloses P. While efficient algorithms for finding Q^{A} are known for more than 20 years [8, 1, 2], the problem of finding Q^{p} has remained open; the problem is posed as open in [3, 6, 7, 9, 12, 14, 5]. [1] Chang and Yap [8] give a comprehensive classification of the incluson/enclosure problems, but do not mention the minimum-perimeter enclosing k-gon problem (Enc($\mathcal{P}_{all}$, $\mathcal{P}_k$, perimeter), in their terminology) at all.

We give the first polynomial-time algorithms for computing Q^{p}. In order to obtain our solution, we prove a structural result about an optimal polygon: Local optimality implies that it is "flush" with P (Lemma 1). As a by-product we obtain an algorithm for finding the minimum-perimeter "envelope" — a convex k-gon with a specified sequence of interior angles. Our proofs are very simple and are based on elementary geometry.[2]

The exact coordinates of the vertices of Q^{p} are given by the roots of high-degree polynomials. In general, it is impossible to find the coordinates exactly in polynomial time [4]. Thus, given $\varepsilon > 0$ as a part of the input to the problem, we will be satisfied with a $(1+\varepsilon)$-approximate solution.

Finding Q^{p} Our algorithm is based on the following lemma, whose (simple) proof we defer until the next paragraph:

Lemma 1. *Q^{p} is* flush *with P, i.e., one of the edges of Q^{p} contains an edge of P.*

By Lemma 1, we may consider each edge of P as a candidate flush edge with Q^{p} and turn the scene into simple polygon $\overline{P}$ (Fig. 1). This reduces finding Q^{p} to solving n instances of the problem of finding a shortest $(k+1)$-link path in simple polygon $\overline{P}$, a problem which can be solved in polynomial time [12]. Thus we have our main result:

Theorem 2. *Q^{p} can be found in polynomial time.*

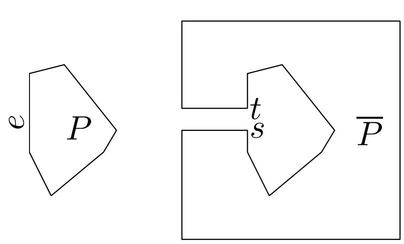

Figure 1: Guess: e is flush with Q^{p}. Reduce to finding shortest $(k+1)$-link s-t path in the simple polygon $\overline{P}$.

[1] A linear-time algorithm exists for the case $k = 3$ [5, 11].

[2] We suggest to solve the minimum-perimeter enclosing k-gon problem by reducing it to the shortest k-link path in simple polygon problem. The algorithms of [12, 14] for the latter are very non-trivial.

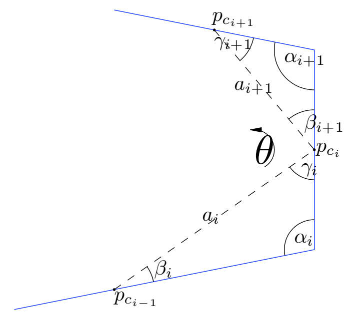

Figure 2: $Q^{\mathsf{p}}(\theta)$ rocks on $\mathsf{C} = (p_{c_1}, \ldots, p_{c_k})$.

The "Flushness" Condition We prove Lemma 1 with the standard method of "rotating calipers" [16].[3] Suppose Q^{p} is not flush with P. Let $\mathsf{C} = \{p_{c_1}, \ldots, p_{c_k}\} \subset \{p_1 \ldots, p_n\}$ be the *rocking* points of the edges of Q^{p}: $\mathsf{C} = bdQ^{\mathsf{p}} \cap P$. Then the perimeter of Q^{p} is (refer to Fig. 2)

$$\mathsf{p}(Q^{\mathsf{p}}) = \sum_{i=1}^{k} \frac{a_i}{\sin \alpha_i}(\sin \beta_i + \sin \gamma_i) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{\beta_i - \gamma_i}{2}$$

Start rotating each edge of Q^{p} around its rocking point counterclockwise by an angle θ; call the polygon formed by such a rotation $Q^{\mathsf{p}}(\theta)$. Let $\theta_{\min}, \theta_{\max}$ ($\theta_{\min} < 0 < \theta_{\max}$) be the angles at which $Q^{\mathsf{p}}(\theta)$ becomes flush with P. For $\theta \in [\theta_{\min}, \theta_{\max}]$, $Q^{\mathsf{p}}(\theta)$ is still a feasible k-gon, enclosing P. The perimeter of $Q^{\mathsf{p}}(\theta)$ as a function f of θ is

$$f(\theta) \equiv \mathsf{p}(Q^{\mathsf{p}}(\theta)) = \sum_{i=1}^{k} \frac{a_i}{\sin \frac{\alpha_i}{2}} \cos \frac{(\beta_i - \theta) - (\gamma_i + \theta)}{2} \quad (1)$$

Since for each i, $\beta_i - \theta$ and $\gamma_i + \theta$ are angles of a triangle, $|(\beta_i - \theta) - (\gamma_i + \theta)| < \pi$. Thus, as $\cos(\cdot)$ is a concave function on $(-\pi/2, \pi/2)$, each summand in (1) is a concave function of θ for $\theta \in (\theta_{\min}, \theta_{\max})$. Hence, $f(\theta)$ is also concave on $(\theta_{\min}, \theta_{\max})$ and attains its minimum at one of the ends of the interval, i.e., when $Q^{\mathsf{p}}(\theta)$ is flush with P. Q.E.D.

Minimum-perimeter envelope DePano and Aggarwal [10] and Mount and Silverman [13] considered the problem of finding the minimum *envelope* — an enclosing convex k-gon with a specified sequence $A = (\alpha_1, \ldots, \alpha_k)$ of angles. The algorithms in [10, 13] for finding the minimum-*area* envelope Q^{A}_A are based (unsurprisingly) on the flushness condition. Note that our "flushness" Lemma 1 actually shows that the minimum-*perimeter* envelope Q^{p}_A is also flush with P and thus, can be

[3] DePano [9] and Chang [7] in their theses proved the lemma for $k = 3$; the proof in [9] uses a complicated trigonometric argument, the proof in [7] is based on the similar result about minimum-*area* enclosing k-gon [10]. Our proof is different from those in [7, 9].

found in polynomial time.[4] By following the algorithm of [13] for finding Q^{A}_A we prove

Theorem 3. *Q^{P}_A can be found in $O(nk \log k)$ time.*

Restricted enclosures In the original statement of the problem, the vertices of the enclosure were allowed to be placed just anywhere in the plane. We propose a generalization, in which two nested polygons P_{out} and $P_{in} \subset P_{out}$ are given, and a minimum convex k-gon restricted to lie in between P_{in} and P_{out} is sought. Of course, the difference between unrestricted and restricted enclosures is that the latter may have some vertices on the boundary of P_{out}; following [12] we say that such vertices are "bash" with P_{out}. As far as we know, this generalization has not been studied before. The problem may be of interest in a classification task where the idea is to build a low-complexity separator between the data points of two types. We make a first small step in solving this type of problems by giving polynomial-time algorithms for finding minimum-area and minimum-perimeter restricted *envelopes* $\overline{Q^{\mathsf{A}}_A}$ and $\overline{Q^{\mathsf{P}}_A}$. Our solution is based on the fact that the optimal restricted polygons are "either flush or bash".

Lemma 4. *$\overline{Q^{\mathsf{P}}_A}$ is either flush with P_{in} or is bash with P_{out}.*

Proof. Otherwise, as in the proof of Lemma 1, start rotating each edge of the envelope around its rocking vertex of P_{in} — the perimeter of $\overline{Q^{\mathsf{P}}_A}$ is a unimodal function of the turn angle and, thus, $\overline{Q^{\mathsf{P}}_A}$ may be rotated in one of the directions, decreasing its perimeter (Fig. 3). □

Next we show that there is only a polynomial number of possible locations for the bash points.

Lemma 5. *Suppose that a bash vertex q_i of $\overline{Q^{\mathsf{P}}_A}$ and the edge e of P_{out} that q_i lies on are given. Suppose that the edges $q_{i-1}q_i$ and q_iq_{i+1} of $\overline{Q^{\mathsf{P}}_A}$ rock on the vertices p_j and p_l of P_{in}. Let $\mathcal{C}$ be the circle through p_j, p_l such that the segment p_jp_l is seen at the angle α_i from the points on $\mathcal{C}$; let a_1, a_2 be the points of intersection (if any) of $\mathcal{C}$ with e. Then either $q_i = a_1$ or $q_i = a_2$ (Fig. 3).*

Theorem 6. *$\overline{Q^{\mathsf{P}}_A}$ may be found in $O(n_{in}^3 n_{out})$ time, where n_{in} and n_{out} are the complexities of P_{in} and P_{out}.*

Proof. If $\overline{Q^{\mathsf{P}}_A}$ is flush with P_{in}, find $\overline{Q^{\mathsf{P}}_A}$ as in Theorem 2. Otherwise, for each triple (p_j, p_l, q_i) (Fig. 3), $\overline{Q^{\mathsf{P}}_A}$ may be found by wrapping the envelope around P_{in}. □

Similarly to Lemma 1, Mount and Silverman showed in [13] that the *area* of the envelope as a function of the turn angle is unimodal. Thus, the above algorithm also works for finding $\overline{Q^{\mathsf{A}}_A}$.

[4] Chang [7, p. 5] and Schwarz et al. [15] mention that [10] solves the *perimeter*-minimization version as well. In fact, perimeter minimization for the general case of arbitrary A (not just $\alpha_i = \frac{k-2}{k}\pi$) is not mentioned in [10] anywhere but in the abstract. We believe (after looking thoroughly into the details of [10]) that [10] did not claim an algorithm for finding Q^{P}_A for arbitrary A.

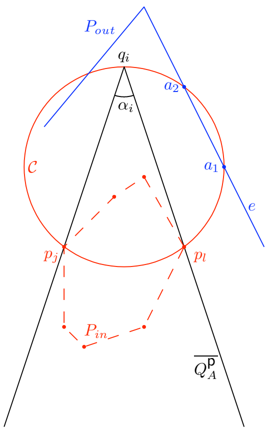

Figure 3: If $q_i \notin \{a_1, a_2\}$, q_i may be moved both clockwise and counterclockwise around $\mathcal{C}$; moving in one of the directions decreases the perimeter of $\overline{Q^{\mathsf{P}}_A}$.

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