5

HOLES

5.1. INTRODUCTION

One of the major open problems in the field of art gallery theorems is to establish a theorem for polygons with holes. A *polygon with holes* is a polygon P enclosing several other polygons $H_1, \ldots, H_h$, the holes. None of the boundaries of $P, H_1, \ldots, H_h$ may intersect, and each of the holes is empty. P is said to bound a *multiply-connected* region with h holes: the region of the plane interior to or on the boundary of P, but exterior to or on the boundary of $H_1, \ldots, H_h$. (A polygon without holes is said, in contrast, to be *simply-connected*.) Similarly we define an *orthogonal polygon with holes* to be an orthogonal polygon with orthogonal holes, with all edges aligned with the same pair of orthogonal axes. For both general polygons with holes and orthogonal polygons with holes, a gap remains between the available necessity and sufficiency proofs. In this chapter we discuss these problems, and present partial results obtained by Aggarwal and Shermer.

Recall that the proof of Theorem 2.1 established that orthogonal polygons with holes may be convexly quadrilateralized. But we have yet to prove that arbitrary polygons with holes may be triangulated.

LEMMA 5.1. A polygon P with holes may be triangulated.

Proof. Let P have h holes and n vertices in total. The proof is by induction on h primarily, and n secondarily. Theorem 1.2 establishes the basis of the induction for $h = 0$. For the general case, let d be a completely internal diagonal, whose existence can be guaranteed by the same argument as used in Theorem 1.2: choose an arbitrary convex vertex v_2, with neighbors v_1 and v_3, on the outer boundary of P, and let $d = v_1v_3$ if this is internal, and otherwise let $d = v_2x$, where x is the closest vertex to v_2 measured perpendicular to v_1v_3. If d has one endpoint on a hole, then it increases n by 2, but decreases h by 1. If d has both endpoints on the outer boundary of P, then it partitions P into two polygons P_i with $n_i < n$ vertices and $h_i \leq h$ holes, $i = 1, 2$. In either case, the induction hypothesis applies and establishes the theorem. □

The number of triangles and quadrilaterals that result from triangulation and quadrilateralization are dependent on the number of holes:

LEMMA 5.2. Let a polygon P with h holes have n vertices total, counting vertices on the holes as well as on the outer boundary. Then a triangulation of P has $t = n + 2h - 2$ triangles, and a quadrilateralization has $q = n/2 + h - 1$ quadrilaterals.

Proof. Let the outer boundary of P have n_0 vertices, and let the ith hole have n_i vertices; thus $n = n_0 + n_1 + \cdots + n_h$. The sum of the interior angles of the outer boundary is $(n_0 - 2)180$ degrees; the sum of the exterior angles of the ith hole is $(n_i + 2)180$. Thus

$$180[(n_0 - 2) + (n_1 + 2) + \cdots + (n_h + 2)] = 180t$$

or $t = n + 2h - 2$. Since $q = t/2$, $q = n/2 + h - 1$.

The same result may be obtained with Euler's Theorem. There are $V = n$ vertices, $F = t + h + 1$ faces, one for each triangle and hole, plus the exterior face, and $E = (3t + n)/2$ edges, where three per triangle plus the boundary counts each edge twice. Then $V - E + F = 2$ yields $t = n + 2h - 2$ as above. □

Throughout the remainder of the chapter, we will use n, h, t, and q to designate the quantities defined in this lemma and P to represent a polygon with holes (including the holes).

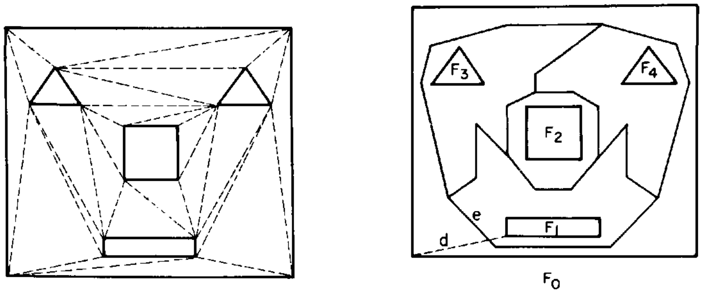
The best sufficiency result for both the general and the orthogonal problems is the following theorem.

THEOREM 5.1 [O'Rourke 1982]. For a polygon of n vertices with h holes, $\lfloor (n + 2h)/3 \rfloor = \lceil t/3 \rceil$ combinatorial guards suffice to dominate any triangulation, and for an orthogonal polygon, $\lfloor (n + 2h)/4 \rfloor = \lceil q/2 \rceil$ combinatorial guards suffice to dominate any quadrilateralization.

Proof.[1] First we note that the equivalences of $\lceil t/3 \rceil$ and $\lfloor (n + 2h)/3 \rfloor$, and $\lceil q/2 \rfloor$ and $\lfloor (n + 2h)/4 \rfloor$, follow directly from Lemma 5.2 by substitution. Thus this theorem is a direct extension of the sufficiency halves of Theorems 1.1 and 2.2, which established respectively that $\lfloor n/3 \rfloor = \lceil t/3 \rceil$ and $\lfloor n/4 \rfloor = \lceil q/2 \rceil$ guards suffice.

Given a polygon P with holes, triangulate it into t triangles; call the triangulation T. The plan of the proof is to "cut" the polygon along diagonals of the triangulation in order to remove each hole by connecting it to the exterior of P. It is clear that every hole must have diagonals in T from some of its vertices to either other holes or the outer boundary of P. Cutting along any such diagonal either merges the hole with another, or connects it to the outside. In either case, each cut reduces the number of holes by one. We are not quite finished, however, because we need to choose the cuts so that the result is a single polygon: it may be that a choice of cuts results in several disconnected pieces.

1. I have incorporated several ideas from Aggarwal (1984).



a b

Fig. 5.1. A triangulation graph of a polygon with holes (a) and its dual (b): each hole in (a) is surrounded by a cycle in (b).

Let $\bar{T}$ be the (non-weak) dual of the triangulation. $\bar{T}$ is a planar graph of maximum degree three, which, in its natural embedding, has h bounded faces $F_1, \ldots, F_h$, one per hole of P. Let F_0 be the exterior unbounded face. Choose any face F_i that shares at least one edge e with F_0. There must be such a face because there must be a diagonal of T from the outer boundary to some hole, and the dual of this diagonal in $\bar{T}$, is e. Removal of e from $\bar{T}$ merges F_i with F_0 without disconnecting the graph. See Fig. 5.1 for an example. Note that removal of an edge in $\bar{T}$ is equivalent to cutting P along the corresponding diagonal of T. Continuing to remove edges of $\bar{T}$ shared with the exterior face in this manner guarantees that a single connected graph results.

Let P' be the polygon that results after all holes are cut in the above manner. Then P' has $n + 2h$ vertices, since two vertices are introduced per cut, but because cuts do not create new triangles, it still has t triangles. Applying Theorem 1.1 to P' yields coverage by $\lfloor (n + 2h)/3 \rfloor = \lceil t/3 \rceil$ guards.

The proof for orthogonal polygons is exactly the same, except that Theorem 2.2 is invoked to obtain the result. □

Although this easily obtained theorem has a pleasing form when expressed in terms of t and q, it appears to be weak: no one has found examples of polygons that require this many guards. In fact, it is difficult to find an example that requires more than $\lfloor n/3 \rfloor$ guards independent of the number of holes. But we show in the next section that there are such polygons.

5.2. GENERAL POLYGONS WITH HOLES

Sidarto discovered the one-hole polygon shown in Fig. 5.2a. It has $n = 8$ vertices, $h = 1$ hole, and requires three guards. Note that $3 > \lfloor 8/3 \rfloor$. Shermer discovered the polygons in Figs. 5.2b and 5.2c, which also have

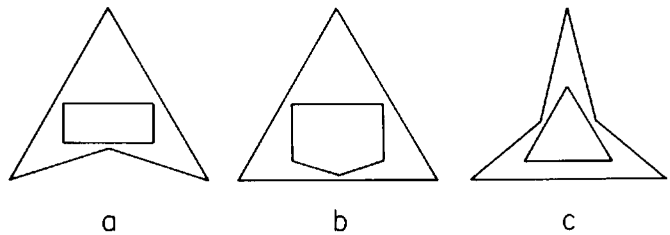


Fig. 5.2. One-hole polygons of 8 vertices that require 3 guards.

eight vertices and require three guards. These one-hole examples can be extended to establish $\lfloor (n+1)/3 \rfloor$ necessity for one hole: Figs. 5.3a and 5.3b show two examples for $n = 11$, due, respectively, to Shermer and Delcher.[2] Finally, the examples can be extended to more than one hole: Fig. 5.4 shows Shermer's method of stitching together copies of the basic one-hole example. The polygon shown has $n = 24$ vertices, $h = 3$ holes, and requires nine guards. This example establishes $\lfloor (n+h)/3 \rfloor$ necessity for h holes. We will not attempt to prove that the claimed number of guards is necessary in the examples just mentioned, as it should be obvious from the figures. The following theorem summarizes the implications of these examples.

THEOREM 5.2 [Shermer 1982]. $\lfloor (n+h)/3 \rfloor$ guards are sometimes necessary for a polygon of n vertices and h holes.

Note that Fig. 5.4 also establishes that $\lfloor 3n/8 \rfloor$ guards are sometimes necessary if we express the result solely as a function of n.

The gap between the necessity of $\lfloor (n+h)/3 \rfloor$ and the sufficiency of $\lfloor (n+2h)/3 \rfloor$ has proved very difficult to close. Since the gap widens as h increases, it is not as insignificant as it might first appear. The strongest result available is that $\lfloor (n+h)/3 \rfloor$ guards suffice for $h = 1$, a theorem proved independently by Aggarwal and Shermer (Shermer 1984). We will follow Shermer's proof technique here.

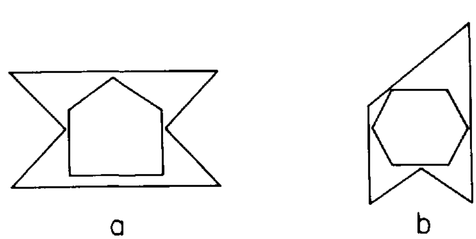


Fig. 5.3. One-hole polygons of 11 vertices that require 4 guards.

2. I assigned this as a homework problem in my computational geometry class. Julian Sidarto and Thomas Shermer, and Arthur Delcher, were students in that class in 1982 and 1985, respectively.

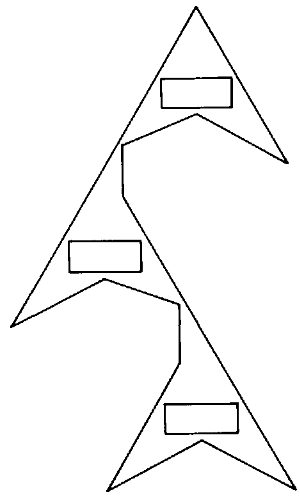


Fig. 5.4. A polygon of 24 vertices with 3 holes that requires 9 guards.

5.2.1. Reduced Triangulations

Before outlining the proof, we first perform a reduction that eliminates irrelevancies. The dual of a triangulation of a polygon with one hole has one cycle surrounding the hole, with (perhaps) several trees attached to the cycle. The next lemma shows that we can clip all the trees down to at most one node. Define a *reduced triangulation* as one such that every subgraph of the triangulation dual G that may be disconnected from G by the removal of a single arc, has exactly one node. Note that this definition is independent of the number of holes in the polygon from which the triangulation derives. We restrict the next lemma to one-hole polygons although it does extend to the general case.

LEMMA 5.3. If $\lfloor (n+1)/3 \rfloor$ combinatorial guards suffice to dominate every reduced triangulation of a polygon of n vertices and one hole, then $\lfloor (n+1)/3 \rfloor$ guards suffice to dominate every triangulation of n vertices and one hole.

Proof. The proof is by induction on the number of trees of more than one node attached to cycles of the triangulation dual G. The basis is established by the antecedent of the lemma: $\lfloor (n+1)/3 \rfloor$ guards suffice for a reduced triangulation, which by definition has no attached trees of more than one node. For the general step, assume $\lfloor (n+1)/3 \rfloor$ guards suffice for any triangulation with $s' < s$ trees of at least two nodes, and let G be a non-reduced triangulation with s such trees. Let T be one of these trees, detachable from G by the removal of one arc r. The situation is as illustrated in Fig. 5.5. Let a and b be the endpoints of the diagonal whose dual is r. Let m be the number of vertices in the polygon Q composed of the triangles of T, *not including* a and b. We show that all but at most the root

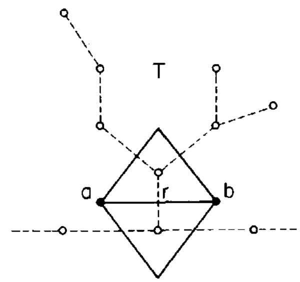


Fig. 5.5. A tree T attached at diagonal ab to a cycle, which extends to the left and right.

triangle of T can be covered "efficiently," that is, with one guard per three vertices. The proof proceeds in three cases, depending on the value of $m \bmod 3$. The easiest cases are considered first.

Case 0 ($m = 3k$). The polygon Q has $m+2$ vertices, and it may therefore be covered by $\lfloor (m+2)/3 \rfloor = k$ guards by Theorem 1.1. Let $P - Q$ be the polygon remaining after removal of Q—that is, the deletion of all vertices in T except a and b, and all incident edges. Since $P - Q$ has $s - 1$ attached trees of one node or more, the induction hypothesis guarantees coverage with $\lfloor (n - m + 1)/3 \rfloor$ guards. Thus P may be covered with

$$\lfloor (n-m+1)/3 \rfloor + k = \lfloor [(n-m+1)+m]/3 \rfloor = \lfloor (n+1)/3 \rfloor.$$

guards.

Case 2 ($m = 3k + 2$). The strategy used in Case 0 will lead to $k+1$ guards here, which is insufficient for our purposes, so another approach must be taken. Augment Q to Q' by adding the triangle on the other side of ab, whose apex is x. Q' is a polygon of $m + 3 = 3k + 5$ vertices and may therefore be covered with $\lfloor (3k+5)/3 \rfloor = k+1$ guards by Theorem 1.1. Fisk's proof of that theorem (Section 1.2.1) assigns one vertex of triangle abx a guard. If x is assigned a guard, it may be moved to a or b while maintaining complete coverage of Q'. Thus we may assume that a or b is assigned a guard. Suppose without loss of generality that a is assigned a guard. Let P' be the result of removing all of Q', all triangles incident on a, and splitting vertex x into two vertices. See Figs. 5.6a and 5.6b. P' has $n - m - 1 + 1$ vertices, since it is missing the m vertices of Q and vertex a,

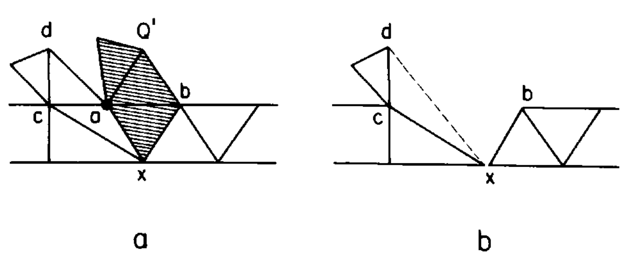


Fig. 5.6. When $3k+2$ vertices comprise T (a), the hole is removed by splitting x (b).

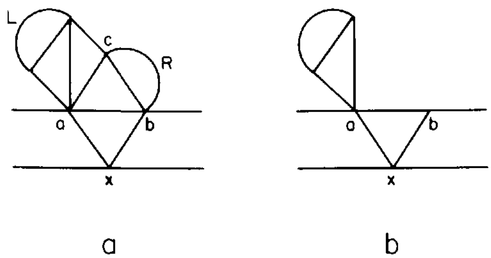


Fig. 5.7. When $3k+1$ vertices comprise T (a), one case is handled by covering R and abc together, and L separately (b).

but gains a vertex from the split of x. Splitting x removes the hole, but P' is not necessarily a polygon, as pieces may be attached at vertices only. But now connect each vertex of P' that was adjacent to a in P, to x. In Fig. 5.6b, vertex d is so connected. These connections are not always geometrically possible, but for this case we are only concerned with the combinatorial structure of the graph. The reconnections do not increase the number of vertices, but they restore P' to be a triangulation graph of a polygon without holes. There is now no need to use the induction hypothesis; rather apply Theorem 1.1 to P', resulting in coverage by

$$\lfloor (n-m)/3 \rfloor = \lfloor [(n-3(k+1))+1]/3 \rfloor = \lfloor (n+1)/3 \rfloor - (k+1)$$

guards. Together with the $k+1$ guards used to cover Q', the lemma is established in this case.

Case 1 ($m=3k+1$). Let the triangle forming the root of T be abc. Let l be the number of vertices in the left subtree L, not including a and c, and let r be the number of vertices in the right subtree R, not including b and c. Thus $m=l+r+1$; see Fig. 5.7a. We consider two subcases dependent on the values of l and $r \bmod 3$.

Subcase 1a ($l=3k_1$ and $r=3k_2$; $m=3(k_1+k_2)+1$). As in Case 0, cover L and R with k_1 and k_2 guards. By the induction hypothesis, $P-L-R$ can be covered with

$$\lfloor [n-(l+r)+1]/3 \rfloor = \left\lfloor [n-3(k_1+k_2)+1]/3 \right\rfloor = \lfloor (n+1)/3 \rfloor - (k_1+k_2)$$

guards, establishing the theorem. This is the only case in which T cannot be entirely removed, but is instead reduced to a single triangle abc.

Subcase 1b ($l=3k_1+1$ and $r=3k_2+2$; $m=3(k_1+k_2+1)+1$). Let R' be the polygon obtained by adding abc to R. R' has $3k_2+5$ vertices. Cover R' with k_2+1 guards by Theorem 1.1. Fisk's coloring procedure guarantees that one vertex of abc is assigned a guard. If either a or b (say a) is guarded, then proceed exactly as in Case 2: delete R' and a, and split x. The calculations are just as in Case 2, establishing the lemma. If on the other hand c is guarded, then delete R' and all triangles of L incident on c, as in

Fig. 5.7b. This leaves at most $l + 1 = 3k_1 + 2$ vertices either disconnected from P or attached at a. Addition of graph edges if necessary restores this piece to a triangulation graph of a polygon L' without increasing the number of vertices. Cover L' with k_1 guards by Theorem 1.1. Now the remainder of P has $n - m$ vertices and $s - 1$ attached trees. By the induction hypothesis it may be covered with

$$\lfloor (n - m + 1)/3 \rfloor = \left\lfloor [n - 3(k_1 + k_2 + 1)]/3 \right\rfloor = \lfloor n/3 \rfloor - (k_1 + k_2 + 1)$$

guards. Together with the $k_2 + 1$ guards used to cover R', and the k_1 guards used for L', complete coverage has been achieved with fewer than $\lfloor (n + 1)/3 \rfloor$ guards.

All cases have now been covered, and the lemma established. □

The idea of reconnecting "broken" pieces into a polygon triangulation graph is from Shermer (1985). Note that this technique was used only when induction was unnecessary, or was applied only to an attached tree. This is crucial, as the geometry of the reduced triangulation is important in the proof of Theorem 5.3 below, and cannot be warped by curved reconnections that need to be straightened in the manner used in Lemma 3.1.

5.2.2. Tough Triangulations

We may now proceed with Shermer's proof of $\lfloor (n + 1)/3 \rfloor$ sufficiency for a polygon with one hole. The first step of the proof reveals why the problem is hard: there exist triangulations of polygons with h holes that require $\lfloor (n + 2h)/3 \rfloor$ *combinatorial* guards for domination. Thus the problem cannot be reduced to pure combinatorics by an arbitrary triangulation. Before proving this we introduce some notation.[3] Lemma 5.3 permits us to restrict attention to reduced triangulations. Let T be a reduced triangulation of a polygon with one hole. Then T consists of a single cycle of triangles, each with perhaps one attached triangle that is not part of the cycle. A cycle triangle is *based* on the inner boundary if it has exactly one vertex, its apex, on the outer boundary of the polygon, and based on the outer boundary if just its apex is on the inner boundary. Note that the base edge of a cycle triangle based on the inner boundary may not itself be on the inner boundary because of a tree attached to the base; this is why the definition is phrased in terms of the apex. Label a cycle triangle "1" if it has no attached non-cycle triangle, and "2" if it does. Then T is represented as a string of characters over the alphabet {"1", "2", "/"}, formed by concatenating all the labels of the cycle triangles, and inserting a "/" between labels λ_1 and λ_2 if the λ_1 triangle is based on the inner boundary and the λ_2 triangle is based on the outer boundary, or vice versa. Thus each "/" records a switch in basing. This string of characters will be called the *string* associated with T.

Figure 5.8 shows an example. Starting at the indicated lowest triangle and

3. The notation is due to Shermer (1984), but slightly modified here.

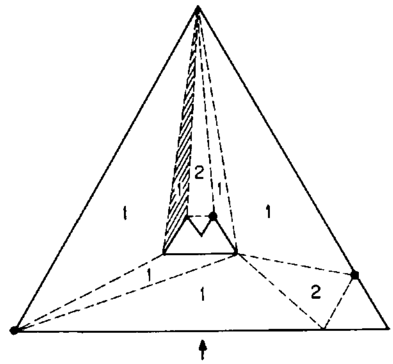


Fig. 5.8. A triangulation of 10 vertices with string 121/121/1/1/ that requires 4 guards: the 3 shown (dots) do not cover the shaded triangle.

proceeding counterclockwise, we obtain the string 121/121/1/1/. Note that the sum of the integers in the string is equal to the total number of triangles in T, and because $t = n$ when $h = 1$ by Lemma 5.2, this is the same as the number of vertices of the polygon. We will employ standard regular expression notation to condense the strings: "+" for "or," s^k for k repetitions of string s, and s^* for zero or more repetitions of s. Thus the above string is equivalent to $(121/)^2(1/)^2$ and is an instance of $(1(21)^*/)^4$. We consider two strings equivalent if one is a cyclic shift of the other, or a cyclic shift of the reverse of another. Finally note that the strings make no distinction between the inner and outer boundaries, and in fact this distinction is irrelevant for combinatorial guards.

A complete characterization of those triangulations that require $\lfloor(n+2h)/3\rfloor$ combinatorial guards for $h = 1$ is provided by the following theorem (Shermer 1984).

THEOREM 5.3 [Shermer 1984]. A reduced triangulation T of a polygon with one hole requires $\lfloor(n+2)/3\rfloor$ combinatorial guards for complete domination iff the string for T has the form $(1(21)^*/)^{6k-2}$.

We will call a string that is an instance of $(1(21)^*/)^{6k-2}$ *tough*. Figure 5.8 satisfies the theorem: $n = 10$ and it requires $\lfloor 12/3\rfloor = 4$ combinatorial guards; an attempted cover with three guards is shown in the figure. Figure 5.9 shows a polygon with the string $(121/)^{10}$; here $n = 40$ and $\lfloor 42/3\rfloor = 14$ guards are required. Even triangulations whose strings are tough but do not correspond to any non-degenerate polygon require $\lfloor(n+2)/3\rfloor$ combinatorial guards. Figure 5.10 shows the smallest possible instance, $(1/)^4$, where $n = 4$ and $g = \lfloor 6/2\rfloor = 2$. Figure 5.8 is the smallest instance realizable as a polygon. All these examples are from Shermer (1984).

Proof of Theorem 5.3. We first prove that a triangulation graph T with a tough string requires $\lfloor(n+2)/3\rfloor$ combinatorial guards. The proof is by induction, in two parts. First it is shown that the claim holds for strings of

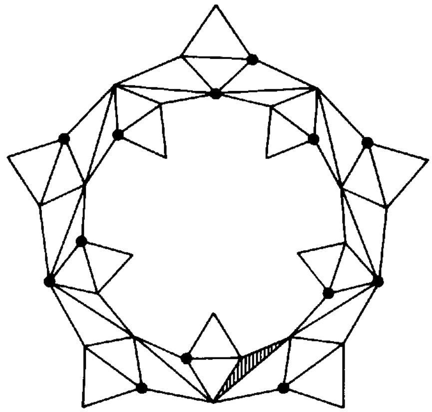


Fig. 5.9. A triangulation of 40 vertices with string $(121/)^{10}$ that requires 14 guards: the 13 shown (dots) do not cover the shaded triangle.

the form $(1/)^{6k-2}$. Then it is shown that each addition of a (21) section requires another guard.

Triangulations of the form $(1/)^{6k-2}$ have a particularly simple structure, illustrated for $k=2$ in Fig. 5.11. Each vertex is adjacent to exactly three triangles. Thus $g=\lceil t/3\rceil$ guards are necessary. But since $t=n=6k-2$, $g=\lceil (6k-2)/3\rceil = 2k = \lfloor (n+2)/3\rfloor$, establishing the first claim.

Now assume that all triangulations T of n vertices with tough strings require $g=\lfloor (n+2)/3\rfloor$ guards, and consider adding three vertices to such a T by insertion of a (21) section S after a "1" triangle and before a "/" switch. Clearly any triangulation of $n+3$ vertices that is an instance of the tough form $(1(21)^*/)^{6k-2}$ can be obtained by such an insertion. Assume, in contradiction to our goal, that the insertion does not increase the number of guards required beyond g. We claim then that T could have been covered by $g-1$ guards. S must be covered in one of the three ways illustrated in Figs. 5.12a, 5.12b, or 5.12c. If S is covered as in Fig. 5.12a, then removal of the section and the guard results in domination of T by $g-1$ guards. If S is covered as in Fig. 5.12b, then deleting S merges two guards, again resulting in coverage by $g-1$ guards. Finally, if S is covered as in Fig. 5.12c, then removal of S leaves two guards, one of which (the bottom one in the figure) is superfluous because every triangle to which it is adjacent is already covered. So again T can be dominated by $g-1$ guards. This contradicts the assumption that g are necessary, establishing that the form $(1(21)^*/)^{6k-2}$ always requires $\lfloor (n+2)/3\rfloor$ combinatorial guards.

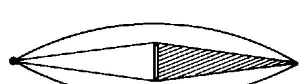


Fig. 5.10. A triangulation of 4 vertices with string $(1/)^4$ that requires 2 guards: the 1 shown (dot) does not cover the shaded triangle.

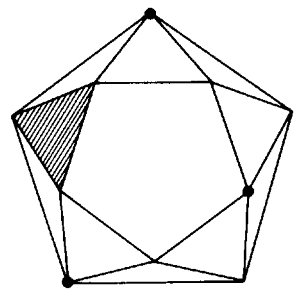


Fig. 5.11. A triangulation of 10 vertices with string $(1/)^{10}$ that requires 4 guards: the 3 shown (dots) do not cover the shaded triangle.

Now we prove the theorem in the other direction, in the contrapositive form: if a triangulation T does *not* have a tough string, then fewer than $\lfloor (n+2)/3 \rfloor$ combinatorial guards suffice for domination. Each 1 in a tough string must be followed by $(2+/)$, and each 2 by 1. Thus, any non-tough triangulation must contain a fragment of the form 11, 22, or 2/. Each of these cases is treated separately.

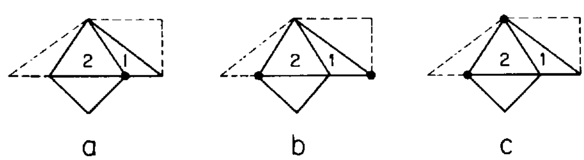


Fig. 5.12. Three ways to guard a (21) section.

Case 1 (11). Let ab be the diagonal shared between the two "1" triangles, with b an apex of both, as shown in Fig. 5.13a. Place a guard at b, delete all covered triangles, and add in extra edges as needed to restore to a polygon with no holes, as shown in Fig. 5.13b. The result is a triangulation of a polygon of no more than $n-2$ vertices, and so T may be dominated with $1 + \lfloor (n-2)/3 \rfloor = \lfloor (n+1)/3 \rfloor$ guards.

Case 2 (22). Again let ab be the shared diagonal, with a incident to all four triangles, as shown in Fig. 5.14a. Place a guard at a, delete the four adjacent triangles, and split node b into two nodes, as shown in Fig. 5.14b. The result is a polygon of $n-2$ vertices, so again T can be dominated with $1 + \lfloor (n-2)/3 \rfloor = \lfloor (n+1)/3 \rfloor$ guards.

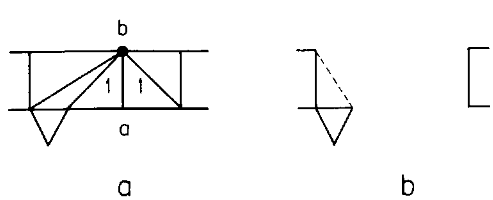


Fig. 5.13. Guarding a 11 fragment (a) removes the hole (b).

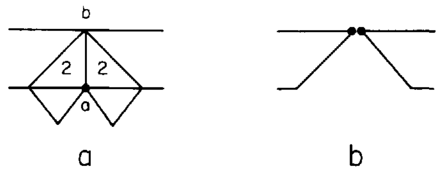


Fig. 5.14. Guarding a 22 fragment (a) removes the hole after splitting a vertex (b).

Case 3 (2/). Here we must consider several subcases, depending on the triangles adjacent to the 2/ fragment.

Case 3a (2/1). See Fig. 5.15a. Place a guard at a and delete adjacent triangles. The result is a polygon of $n-2$ vertices, and we proceed as in Case 1.

Case 3b (2/2). Again we consider subcases.

Subcase (2/21). See Fig. 5.15b. Place a guard at a as in Case 3a.

Subcase (2/22). This was already handled in Case 2.

Subcase (2/2/1). This was already handled in Case 3a.

Subcase (2/2/2). See Fig. 5.15c. Place a guard at a, delete all adjacent triangles, and split vertex b. Now proceed as in Case 2.

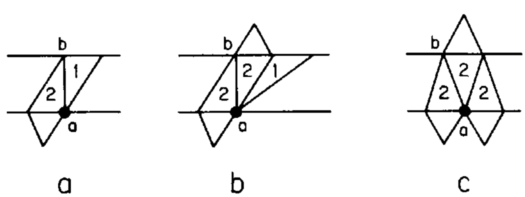


Fig. 5.15. The three cases for the fragment 2/ are all handled by guarding vertex a.

We have thus shown that $\lfloor (n+1)/3 \rfloor$ combinatorial guards suffice whenever one of the fragments 11, 22, or 2/ are present in T's string, establishing the theorem. □

5.2.3. Convex Pairs and Triplets

The second step of Shermer's proof of Theorem 5.3 is to further characterize those one-hole polygon triangulations that might require $\lfloor (n+2)/3 \rfloor$ guards, this time involving the geometry of the triangulation and using geometric guards. In particular, if a tough triangulation contains either a "c-pair" or a "c-triplet," then $\lfloor (n+1)/3 \rfloor$ guards suffice. The third and final step is to show that every tough triangulation must contain one of these two structures.

A *c-pair* is a pair of adjacent cycle triangles that together form a convex quadrilateral.

Fig. 5.16. Flipping a 1/1 c-pair leads to 11/1 (a) or 2/1/1 (b).

LEMMA 5.4. A polygon with a tough triangulation containing a c-pair may be covered with $\lfloor (n+1)/3 \rfloor$ vertex guards.

Proof. The strategy is to flip the diagonal of the c-pair, changing the structure of the triangulation to non-tough. Since the triangulation has the string $(1(21)^*/)^{6k-2}$, the c-pair has either the form 1/1 or 21 (or equivalently 12). Each case is considered separately.

Case 1 (1/1). Flip the diagonal of the c-pair. Since the quadrilateral is convex, this is possible. The resulting triangulation is not tough, as can be seen in Fig. 5.16. If the triangle preceding the c-pair is of type 1, then the fragment 1/(1/1) is changed to 11/1 (Fig. 5.16a). If the preceding triangle is of type 2, then the fragment 2(1/1) is changed to 2/1/1. Neither of these new fragments are substrings of any tough string. By Theorem 5.3, then, $\lfloor (n+1)/3 \rfloor$ guards suffice.

Case 2 (21). Again flip the diagonal. The resulting triangulation, shown in Fig. 5.17, is not reduced. But this is just Case 2 of the proof of Lemma 5.3. Place a guard at *a*, delete all adjacent triangles, split vertex *c* in two, and restore to a polygon triangulation by adding diagonals as necessary. Three vertices are deleted, and one added. Since the result is a polygon, coverage by $1 + \lfloor (n-2)/2 \rfloor = \lfloor (n+1)/3 \rfloor$ guards has been achieved. □

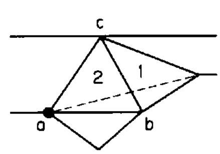


Fig. 5.17. Flipping a 21 c-pair leads to the case considered in Fig. 5.6.

A *c-triplet* is a triple (A, B, C) of consecutive cycle triangles such that first, B is of string type 1, and second, the union of the three triangles may be partitioned into two convex pieces.

LEMMA 5.5. A polygon with a tough triangulation containing a c-triplet may be covered with $\lfloor (n+1)/3 \rfloor$ vertex guards.

Proof. Let a be the vertex common to the c-triplet triangles A, B, and C, as shown in Fig. 5.18a. Delete B and split vertex a. The result is a polygon of no holes with $n+1$ vertices, which may therefore be covered with $\lfloor (n+1)/3 \rfloor$ vertex guards by Theorem 1.1. In particular, perform the

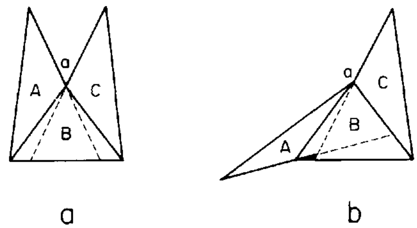


Fig. 5.18. A c-triplet is covered if A and C are covered (a), but B is not covered if the triangles do not form a c-triplet.

coverage with Fisk's coloring procedure; then both A and C must have a guard in one of their corners. Now put back B. Because the three triangles form a c-triplet, B is also covered by the guards covering A and C. □

Note that if the triangles did not form a c-triplet, as in Fig. 5.18b, B would not necessarily be covered. Similarly, if B were of string type 2, the triangle attached to B would not necessarily be covered.

We finally come to the last step of the proof. For a triangle t_i, define the open cone delimited by the two edges of t_i passing through the apex as $\alpha(i)$, and define the similar region off the right base vertex as $\beta(i)$; see Fig. 5.19.

LEMMA 5.6. Any tough triangulation of a polygon contains either a c-pair or a c-triplet.

Proof. The proof is by contradiction. Assume a tough triangulation contains no c-pair or c-triplet. Then we will show that it cannot close into a cycle, and so is not the triangulation of a polygon with one hole.

Identify two adjacent cycle triangles of the form 1/1; such a fragment must exist because the general form is $(1(21)^*/)^{6k-2}$. We will identify triangles by subscripts on their type. The selected 1/1 fragment is labeled $1_0/1_1$. We expand this string to the right in all possible ways compatible with the general tough form, and show that a particular geometric structure always results. Let a string S end at the right with 1_i, and let v_i be the vertex at the tip of the ear 1_i. Then define an embedding of S to be *nesting* if v_i is

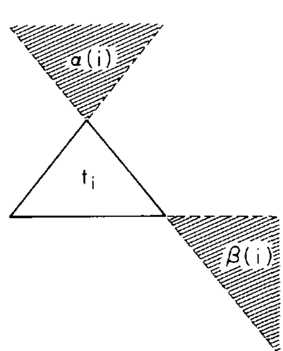


Fig. 5.19. The apex cone α and base cone β for a triangle.

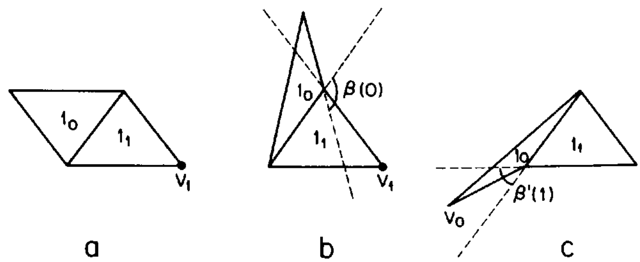


Fig. 5.20. $1_0/1_1$ is nesting.

in the base cone $\beta(i-1)$ of the triangle adjacent to 1_i. We now show that $1_0/1_1$ is nesting.

The general form of this fragment is as shown in Fig. 5.20a. In order to avoid a c-pair, either the configuration shown in Fig. 5.20b or 5.20c must hold. In Fig. 5.20b, $v_1 \in \beta(0)$, and so the nesting definition is satisfied. Figure 5.20c is just Fig. 5.20b reflected in a horizontal line, and we assume without loss of generality that 5.20b obtains.

The string $1_0/1_1$ may be extended only with /1 or 21 while remaining compatible with the tough form. We consider each case separately.

Case 1 ($1_0/1_1/1_2$). The general form is shown in Fig. 5.21a. In order to avoid a c-pair in $1_1/1_2$, either $v_2 \in \alpha(1)$ or $v_2 \in \beta(1)$. The former choice (Fig. 5.21b) leads to a c-triplet, and the latter choice (Fig. 5.21c) is a nesting configuration.

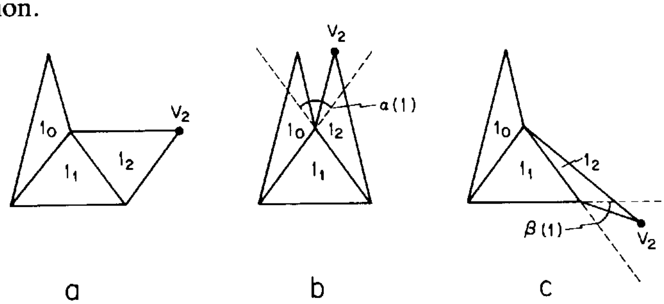


Fig. 5.21. $1_0/1_1/1_2$ is nesting.

Case 2 ($1_0/1_1/2_21_3$). As in Case 1, we must have $v_2 \in \beta(1)$. To avoid a c-pair in 2_21_3, we must have $v_3 \in \beta(2)$, as illustrated in Fig. 5.22. Again the configuration is nesting.

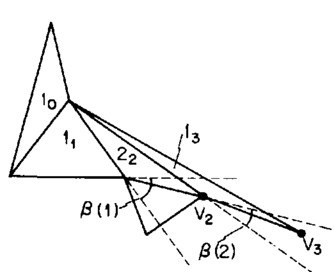


Fig. 5.22. $1_0/1_1/2_21_3$ is nesting.

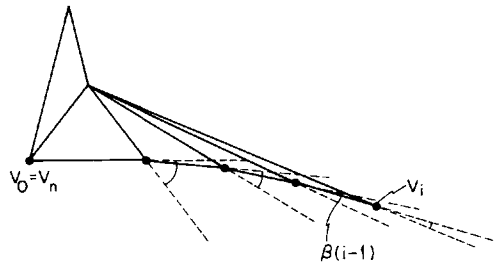


Fig. 5.23. Repeated nesting prevents v_n from coinciding with v_0.

Both Case 1 and 2 may be extended only with /1 or with 21. Extension of Case 1 results in the same two cases again, although the possibility that $v_3 \in \alpha(2)$ is blocked by 1_0, so this choice does not have to be ruled out by showing that it leads to a c-triple. Similarly extension of Case 2 brings us back to the same two cases. We conclude that every embedding of the string compatible with the tough form is nesting.

But now the contradiction is immediate. The repeated nesting forces $v_i \in \beta(i-1)$, and since these base cones are clearly nested inside one another (see Fig. 5.23), the embedding cannot wrap back around to permit $v_n = v_0$. □

THEOREM 5.4 [Aggarwal, Shermer 1984]. $\lfloor (n+1)/3 \rfloor$ vertex guards suffice to cover any n vertex polygon with one hole.

Proof. Lemma 5.3 established that if the theorem holds for reduced triangulations, then it holds for all triangulations. So we restrict our attention to reduced triangulations. Theorem 5.3 shows that if the reduced triangulation is not tough, then $\lfloor (n+1)/3 \rfloor$ vertex guards suffice. So we need only consider tough triangulations. Lemmas 5.4 and 5.5 show that if a tough triangulation contains a c-pair or a c-triplet, then $\lfloor (n+1)/3 \rfloor$ guards suffice. And Lemma 5.6 shows that every tough triangulation contains one of these structures, so there are no further possibilities. □

It does not seem easy to extend this proof to more than one hole. Nevertheless, there is considerable evidence for the following conjecture.

CONJECTURE 5.1 $\lfloor (n+h)/3 \rfloor$ vertex guards are sufficient to cover any polygon of n vertices and h holes.

5.3. ORTHOGONAL POLYGONS WITH HOLES

The status of the art gallery problem for orthogonal polygons is similar to that for general polygons in that it is unsolved in its most general form. There are, however, four interesting differences: the number of guards does not seem to be dependent on h, there is a simple proof of the one-hole theorem, there is a two-hole theorem, and vertex guards do not suffice for more than one hole.

Recall that the quadrilateralization theorem (2.1) holds for orthogonal polygons with holes. However, the coloring argument used to obtain $\lfloor n/4 \rfloor$ sufficiency does not work if there are cycles in the dual of the quadrilateralization. Nevertheless, no examples of orthogonal polygons with holes are known to require more than $\lfloor n/4 \rfloor$ guards. This leads to the following conjecture.

CONJECTURE 5.2. $\lfloor n/4 \rfloor$ point guards suffice to cover any orthogonal polygon of n vertices, independent of the number of holes.

The gap between this conjecture and the best general result, $\lfloor (n+2h)/4 \rfloor$ (Theorem 5.1), is substantial.

Aggarwal established the truth of the conjecture for $h = 1$ and $h = 2$. His proof for one hole is long and complicated (Aggarwal 1984). The two-hole theorem is by no means a simple extension of the one-hole theorem; further complications arise.[4] Recently Shermer found a simple proof of the one-hole theorem. This is the only proof we will present in this section.

His proof is "simple," however, only if we accept a non-trivial lemma proved by Aggarwal to the effect that only reduced quadrilateralizations need be studied. A *reduced quadrilateralization* is one for whose dual G the following conditions hold:

(1) Every subgraph that may be disconnected from G by the removal of a single arc of G has exactly one node, called a *leaf*;
(2) the quadrilaterals of no two such leaf nodes share a vertex.

For a polygon with one hole, the dual of a reduced quadrilateralization is a single cycle with attached leaf nodes satisfying condition (2). Note that the definition of a reduced quadrilateralization parallels that of a reduced triangulation used in the previous section, with the additional restriction of discarding neighboring non-cycle quadrilaterals.

Aggarwal established the following analog of Lemma 5.3.

LEMMA 5.7. If $\lfloor n/4 \rfloor$ guards suffice to dominate every reduced quadrilateralization of n vertices and one hole, then $\lfloor n/4 \rfloor$ guards suffice to cover every quadrilateralization of n vertices and one hole.

The proof of this lemma is at least as complex as that of Lemma 5.3, but it is very similar in spirit, and we will not detail it here (Aggarwal 1984, Prop. 3.10). This lemma permits us to concern ourselves solely with reduced quadrilateralizations.

We need a simple characterization of the cycle quadrilaterals of one-hole orthogonal polygons before proceeding. Each cycle quadrilateral has all four of its vertices on the boundary of the polygon. If a quadrilateral has

4. This proof is only sketched in Aggarwal (1984), but he has rather detailed notes.

two vertices on the exterior boundary and two on the hole boundary it is called *balanced*; otherwise it is called *skewed*.

LEMMA 5.8 Any quadrilateralization of an orthogonal polygon with one hole has an even number (at least four) of balanced quadrilaterals.

Proof. We first establish that the number of polygon edges bounding the cycle quadrilaterals towards the exterior is even. Of course this is trivial if all the quadrilateral edges are polygon edges, because an orthogonal polygon has an even number of edges. Let $e_1, \ldots, e_k$ be the cycle quadrilateral edges towards the exterior, and let n_i be the number of polygon edges in the portion of the polygon P_i bound by e_i that does not include the hole. If e_i is a polygon edge, then $n_i = 1$; otherwise n_i is odd, since P_i is quadrilateralizable and therefore has an even number of boundary edges including e_i. Since each n_i is odd, and $\sum_{i=1}^{k} n_i$, the total number of polygon edges, is even, the number of terms k must be even. This establishes the claim.

Since each balanced quadrilateral contributes 1 to k, and each skewed quadrilateral contributes 0 or 2 to k, the number of balanced quadrilaterals must be even. To establish that there must be at least four balanced quadrilaterals, note that the four extreme edges of the hole (top, bottom, left, right), cannot be part of a skewed quadrilateral. □

We may now proceed with Shermer's proof of the one-hole theorem.

THEOREM 5.5 [Aggarwal 1984]. $\lfloor n/4 \rfloor$ vertex guards suffice to cover any n vertex orthogonal polygon with one hole.

Proof [Shermer 1985]. Let Q be a reduced quadrilateralization of an orthogonal polygon with one hole. Associate a graph H with Q as follows. The nodes of H correspond to the quadrilaterals of Q, and two nodes are connected by an arc iff their quadrilaterals share a *vertex*. An example is

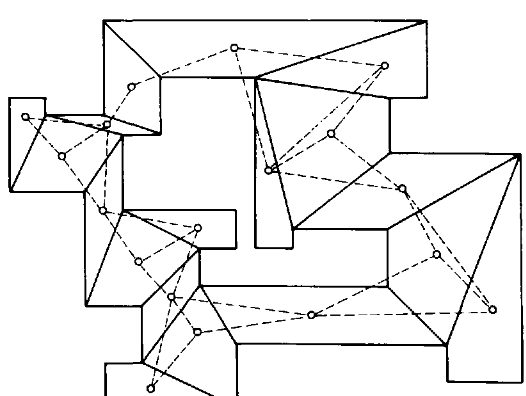


Fig. 5.24. Two nodes are adjacent in H if their quadrilaterals share a vertex.

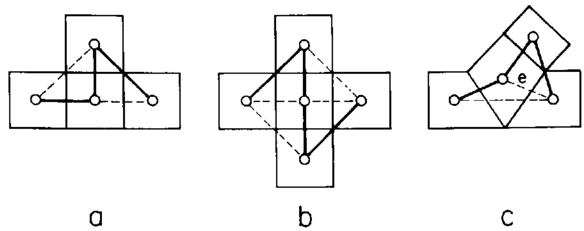


Fig. 5.25. Hamiltonian paths through balanced (a and b) and skewed (c) cycle quadrilaterals.

shown in Fig. 5.24. We claim that H is Hamiltonian, that is, it contains a cycle that touches each node exactly once.

First it is easy to see that the quadrilaterals that form a cycle in the dual of Q form a cycle in H as well, since quadrilaterals that share a diagonal share vertices. We now "stitch" the leaf nodes into this cycle. Let A, B, and C be three consecutive cycle quadrilaterals. If B is balanced it may have either one or two attached leaf quadrilaterals (Figs. 5.25a and 5.25b); if B is skewed, it may have one attached leaf quadrilateral (Fig. 5.25c). In all three cases it is possible to form a Hamiltonian path from A to C including B and any attached leaf nodes, as illustrated in the figures. Concatenation of these Hamiltonian paths for all cycle quadrilaterals playing the role of B can be seen to result in a Hamiltonian cycle γ for H by the following argument.

A leaf attached to a skew quadrilateral may be brought into the cycle in only one way, as shown in Fig. 5.25c. To reduce the graph to situations where choice remains, contract the edge e shown in Fig. 5.25c for every such skew quadrilateral, and delete the attached leaf node. Perform this contraction of e even if the skew quadrilateral has no attached leaf node. After contraction (or "squashing") of e, every edge of H incident to either endpoint of e is made incident to one node that "represents" both endpoints. Applying this transformation to the complicated section of H shown in Fig. 5.26a, for example, reduces it to the simpler fragment shown in Fig. 5.26b. After contraction of all such skew quadrilaterals, the resulting graph H' is a mixture of the two cases in Figs. 5.25a and 5.25b. Because no two leaf quadrilaterals share a common vertex by definition of a reduced quadrilateralization, and because the contraction process does not destroy this property, H' is a simple pasting together of the patterns in Figs. 5.25a and 5.25b. Note that the Hamiltonian path in those figures always use the edge(s) between a balanced cycle quadrilateral and its attached leaf node(s) (the vertical edges in Fig. 5.25). Contracting these edges produces a further reduced graph H'' which is always a simple cycle; see Fig. 5.26c. Now start with the obvious Hamiltonian cycle for H'', and "reverse" the transformations above. From H'' to H' there are choices available, but it is clear that a Hamiltonian cycle can always be achieved: because every leaf node is adjacent to three consecutive cycle nodes, an exit in the required direction is always available to a path traveling the vertical edge from cycle node to

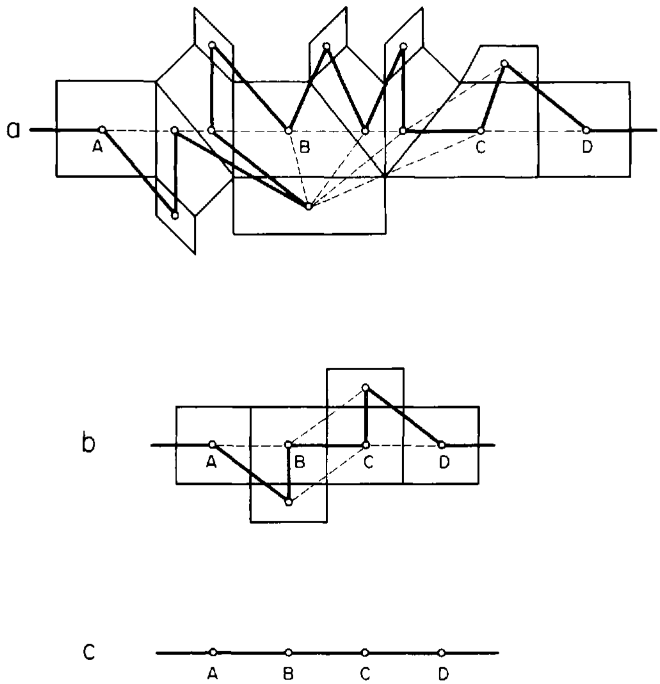


Fig. 5.26. A section of H with a Hamiltonian path (a), after contraction of skewed quadrilaterals (b), and after contraction of vertical edges (c).

leaf node. Reversal from H' to H is straightforward, as there is no choice. The result is the claimed Hamiltonian cycle γ of H.

Let q be the number of quadrilaterals in Q. Recall that by Lemma 5.2, $q = n/2$. If q is even, then every other edge of γ forms a perfect matching in H: a set of edges that is incident on each node exactly once. Each edge of the matching corresponds to a vertex of the polygon. Placing guards at the vertices associated with the edges of the matching covers the two quadrilaterals whose nodes are endpoints of the edge. Thus this guard placement covers the entire polygon with $q/2 = n/4 = \lfloor n/4 \rfloor$ guards.

Now suppose that q is odd. If the number of cycle quadrilaterals is even, then there must be at least one leaf quadrilateral. If the number of cycle quadrilaterals is odd, then by Lemma 5.8 there must be at least one skewed cycle quadrilateral. Both of these cases guarantee the existence of three quadrilaterals consecutive in γ that can be covered by one vertex guard. Place a guard at this vertex and delete the three nodes from γ, forming γ'. γ' has an even number of nodes and forms a Hamiltonian path. Again every other edge of γ' represents a perfect matching, and placing guards at the corresponding vertices results in complete coverage. The number of guards used is $1 + (q-3)/2 = (q-1)/2 = (n-2)/4 = \lfloor n/4 \rfloor$. □

As mentioned earlier, Aggarwal has also proven that $\lfloor n/4 \rfloor$ guards suffice

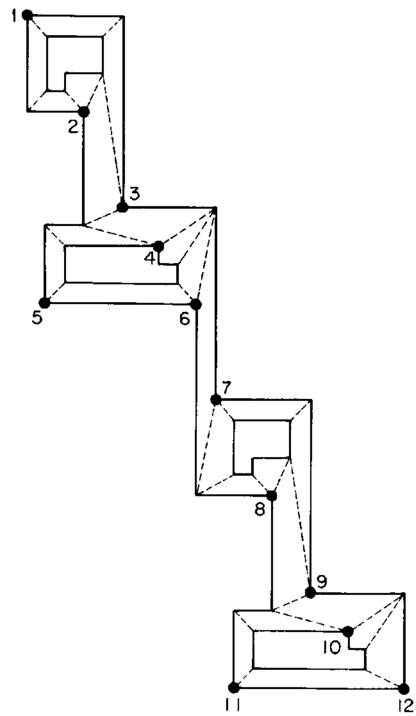


Fig. 5.27. A four-hole polygon of 44 vertices that requires 12 vertex guards.

to cover any orthogonal polygon with two holes. The most interesting aspect of the two-hole theorem is that $\lfloor n/4 \rfloor$ vertex guards do *not* suffice for polygons of two or more holes: Fig. 5.27 shows a four-hole polygon with $n = 44$ that require 12 vertex guards, 3 surrounding each hole.[5] However, $10 < \lfloor 44/4 \rfloor$ guards suffice if they are not restricted to vertices: movement of guards 2 and 8 horizontally to the right to the polygon boundary permits the elimination of guards 3 and 9. Extension of this example to multiple holes has led Aggarwal and Shermer to make the following conjectures (respectively).

CONJECTURE 5.3. $\lfloor 3n/11 \rfloor$ vertex guards are sufficient to cover any orthogonal polygon with any number of holes.

CONJECTURE 5.4. $\lfloor (n+h)/4 \rfloor$ vertex guards are sufficient to cover any orthogonal polygon with any number of holes.

5. Aggarwal (1984, p. 137), as modified by Shermer.