Zero-Parity Stabbing Information

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Abstract

Everett et al. [EHN96, EHN97] introduced several varieties of stabbing information for the lines determined by pairs of vertices of a simple polygon P, and established their relationships to vertex visibility and other combinatorial data. In the same spirit, we define the "zero-parity (ZP) stabbing information" to be a natural weakening of their "weak stabbing information," retaining only the distinction among {zero, odd, even > 0} in the number of polygon edges stabbed. Whereas the weak stabbing information's relation to visibility remains an open problem, we completely settle the analogous questions for zero-parity information, with three results: (1) ZP information is insufficient to distinguish internal from external visibility graph edges; (2) but it does suffice for all polygons that avoid a certain complex substructure; and (3) the natural generalization of ZP information to the continuous case of smooth curves does distinguish internal from external visibility.

1 Introduction

It is natural to connect the geometric shape of an object to its combinatorics. The *polygon vertex visibility graph* has been closely studied, but the relationship between this graph and the shape remains open [O'R93]. *Stabbing* information—how lines cross the polygon—has developed into a key concept both in discrete geometry [Gar95, Wen97] and geometric algorithmics [Aga91, Ski97]. The work of Everett et al. [EHN96, EHN97] connects these two worlds, showing how different varieties of stabbing information determine visibility and other combinatorial information.

We now introduce enough notation to state our results. Polygon vertices are assumed in general position and labeled by indices increasing in a counterclockwise boundary traversal. The line L through two vertices x and y of P is partitioned into three components $L \setminus \{x, y\}$, and we count the number of edges of P that cross each component: Tail(x, y), Body(x, y), and Head(x, y). The

Stab Info	Cnv/Rfl	Hull	I/E Vis	OrdTyp
Labeled	YES	YES	YES	YES
Strong	YES	YES	YES	NO
Weak	YES	YES	?	NO
Zero-Parity	YES	YES	NO	NO
Pure Parity	YES	NO	NO	NO

weak stabbing information consists of these three quantities for all pairs of vertices (x, y). Richer information leads to the *strong* and *labeled* stabbing information, which we will not pause to define. We define *pure parity information* to only retain the parity of Tail, Body, and Head. As this is too weak to even identify hull edges (see Fig. 1), we introduce the *zero-parity (ZP) information*, which records three values: zero (which enables visibility edges to be identified), odd, and even > 0. The re-

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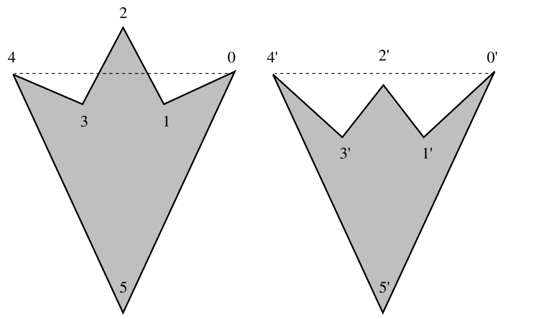

Figure 1: Same pure parity information but different hull and visibility edges.

sults of [EHN96, EHN97] are summarized in the first three lines of Table 1, whereas the last two lines display our contributions, completing the table in a natural way.

We concentrate on the "I/E Vis" column, distinguishing internal (I) from external (E) visibility edges. It is natural to hypothesize that ZP information suffices to make this distinction, due to the connection to the well known ray-crossings point-in-polygon algorithm [Hai94] [O'R98, Sec. 7.4], which depends only on parity.

2 ZP Counterexample

A counterexample to this hypothesis is shown in Fig. 2.[1] Let $[x, y]$ be the chain counterclockwise from x to y. The two $n = 12$ vertex polygons differ in the subchains $[1, 7]$ and $[1', 7']$: the former lies below the I-edge $(0, 8)$, and the latter above the E-edge $(0, 8)$. And yet all $\binom{12}{2} \times 3 = 196$ pieces of ZP information are identical (as checked by a program): $\mathrm{Tail}(1, 6) = 3$ and $\mathrm{Tail}(1', 6') = 1$; $\mathrm{Head}(0, 2) = 3$ and $\mathrm{Head}(0', 2') = 5$; and so on.

[1] The three hull vertices $(9, 10, 11)$ are so far away that their lines of sight to the others are nearly vertical.

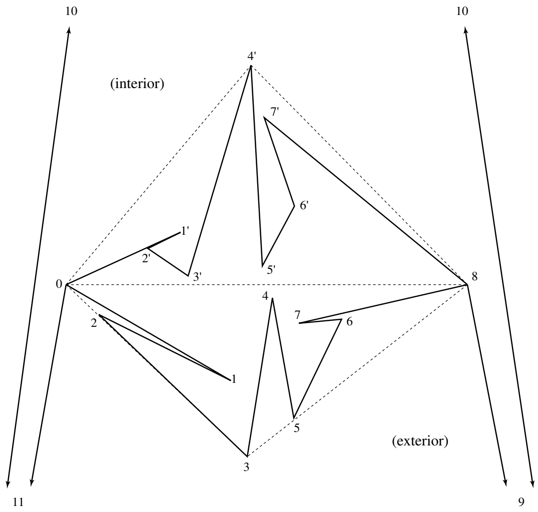

Figure 2: ZP counterexample.

3 Nontriangular Polygons

The structures of the chains in Fig. 2 are not accidental, but rather the precise obstruction to distinguishing I- from E-edges. We prove that for any polygon that is (≥ 8)-*nontriangular*, ZP does suffice to make the distinction. Call a chain $[x, y]$ *k-triangular* if: it contains k vertices; its hull is a triangle; and x and y are seen by a vertex z_I via I-edges, and by a vertex z_E via E-edges. Note the $[0, 8]$ chain in Fig. 2 is 9-triangular, with $z_I = 10$ and $z_E = 9$ or 11. A polygon is $(\geq k)$-nontriangular if it contains no m-triangular chain for any $m \geq k$; thus it avoids all long triangular chains. Our proof depends on a series of six lemmas, whose proofs we only sketch. Let $\mathrm{Even}(x, y, z)$ be the property that $\forall w \in [x, y]$, $\mathrm{Head}(w, z)$ is even, and let $\mathrm{Odd}(x, y, z)$ be the corresponding odd property.

Lemma 1 *The ZP information identifies convex/reflex and hull vertices.*
Proof: The proof of Lemma 4.1 in [EHN97] carries through without change. □

Lemma 2 *Two vertices of a convex (resp. reflex) chain are visible only via an I-(resp. E)-edge.*

Proof: Suppose for a contradiction that two vertices x and y of a convex chain C are visible via an E-edge. Let $C' \subseteq C$ be the convex subchain partitioned off by xy. Orient xy horizontal. Then C' must include a point above or below xy. Assume the latter. Then the rightmost lowest point v of C' is a reflex vertex, a contraction. $\square$

Lemma 3 *If xy is an I-edge, there is a convex z in both chains bounded by x and y such that Even(x, y, z) holds* (10 and $3/3'$ in Fig. 2.) *If xy is an E-edge, there is a reflex z in one of the chains such that Odd(x, y, z) holds* ($4/4'$ in Fig. 2.)

Proof: Orient the I-edge xy horizontal, and let z_1 be the highest vertex of $[x, y]$ and z_2 the lowest vertex of $[y, x]$. Because the vertices z_i are extreme, all rays through z_i must exit the polygon there, so Even(x, y, z_i) holds. The reasoning for an E-edge is similar. $\square$

Lemma 4 *If xy is a nonhull I-(resp. E)-edge, it is shared by two I-(resp. E)-$\triangle$s.*

Proof: Let xy be an I-edge. It must be part of a triangulation of P. By the nonhull assumption, xy must be an internal diagonal and so shared by two triangles of the triangulation; see Fig. 3a. The edges of these triangles must be I-edges. $\square$

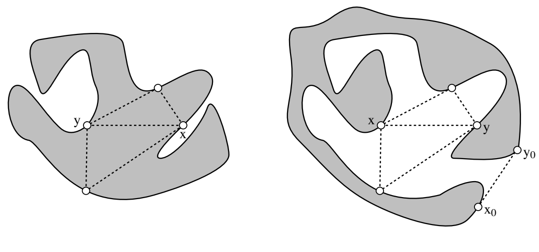

Figure 3: xy is an internal I- or E-edge.

Lemma 5 *The previous lemmas I/E-distinguish all but visibility edges spanned by triangular chains.*

Proof: Suppose a nonhull visibility E-edge xy satisfies both the I- and E-halves of Lemma 4. Then one of the two I-$\triangle$s that share xy must have its apex $z \in [x, y]$ (the other apex is in $[y, x]$). With both xz and zy I-edges and xy an E-edge, it must be that the chain $[x, y]$ remains inside $\triangle xzy$. Thus z is on the hull, and $[x, y]$ is triangular. $\square$

Lemma 6 *For a chain to be I/E-ambiguous, it must contain at least 8 vertices.*

Proof: Let $[x, y]$ be an I/E-ambiguous chain. Lemma 5 shows it must be triangular regardless of whether xy is an I- or an E-edge. The vertices on the hull must be different in the two cases: a convex vertex c if xy is an I-edge, and a reflex vertex r if an E-edge. ($3/3'$ and $4/4'$ respectively in Fig. 2.) Assume without loss of generality that the vertices occur in the order (x, c, r, y) in the I-chain, i.e., the one with xy an I-edge; as in Fig. 2, we label the E-chain vertices with primes. We now argue that there must be at least two vertices between x and c, and between r and y.

From the point of view of x, the chain $[r', y]$ starts out below the ray xr' and ends up above the ray xy, whereas the chain $[r, y]$ starts out below the ray xr and ends up again below the ray xy. This difference demands at least one "flip vertex" w, at which the polygon's relationship to the ray xw differs in the two chains. In Fig. 2, $w = 6/6'$: 5 and 7 are both below ray 06 but $06'$ splits $5'$ and $7'$. To achieve this difference with identical ZP information requires in turn that the ray from x lie on different sides of the edge incident to x: above 01 but below $01'$ in the figure. This forces this edge to aim so that it splits vertices in $[r, y]$. Ad hoc reasoning shows that neither of these can be r or y, so there must be two additional vertices in this chain.

Applying the same argument to $[x, c]$ from the point of view of y leads to 6 interior vertices, and so 8 including x and y. $\square$

We believe this lemma can be strengthened to ≥ 9.

We have embodied these lemmas in a Java applet that accepts a user-specified polygon, computes the ZP information, and then classifies all visibility edges as I or E, except for those spanned by triangular chains. For example, all 2566 visibility edges in Fig. 4 are correctly classified.

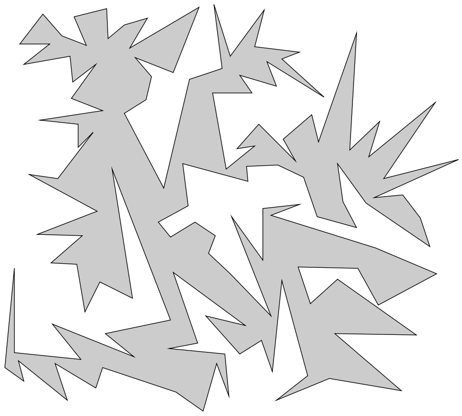

Figure 4: A ($\geq$8)-nontriangular polygon of $n = 136$ vertices.

4 Continuous ZP Info.

Shermer introduced the notion of *point visibility graphs (PVGs)* [She92, MS96], a natural continuous generalization of vertex visibility graphs. We can generalize ZP information to continuous graphs and smooth curves as follows. Let $\mathcal{P} : [0,1] \to \mathbb{R}^2$ be a Jordan curve parameterized by $t \in [0,1]$, a piecewise algebraic curve smooth except at no more than n points, and with nonzero curvature everywhere. This latter condition ensures that every line meets the curve in a finite number of points. The parameter t plays the role of the vertex label. For each point $x \in \mathcal{P}$, define a function $B_x : [0,1] \to \{z, o, e\}$ so that $B_x(y)$ is the zero-parity of $\mathrm{Body}(x,y) = |xy \cap \mathcal{P}|$. Define $T_x()$ and $H_x()$ to similarly depend on $\mathrm{Tail}(x,y)$ and $\mathrm{Head}(x,y)$. The collection of these functions for all $x \in \mathcal{P}$ constitute the continuous ZP information.

For a fixed x, each of the three functions $\{B_x(), H_x(), T_x()\}$ is discontinuous only at points of tangency between the line through xy and $\mathcal{P}$. The assumption that $\mathcal{P}$ is piecewise algebraic assures that each function has at most $O(sn)$ discontinuities, where s the maximum degree of the algebraic pieces. And as x varies over $\mathcal{P}$, the combinatorial structure of $B_x()$ changes only with double tangencies, of which there are at most $O(s^2 n^2)$. In fact, the visibility complex [PV96] records all the relevant critical lines. Thus the continuous ZP information may be finitely represented.

Let xy be a visibility edge, i.e., one for which $B_x(y) = z$. The I/E status of xy may be determined by examination of the ZP functions in the local neighborhood of xy. Care must be taken to deal with tangencies/discontinuities. We label a discontinuity at y by a value of the function at $y-\epsilon$, y, and $y+\epsilon$, for small $\epsilon > 0$: $o/z/e$, etc.

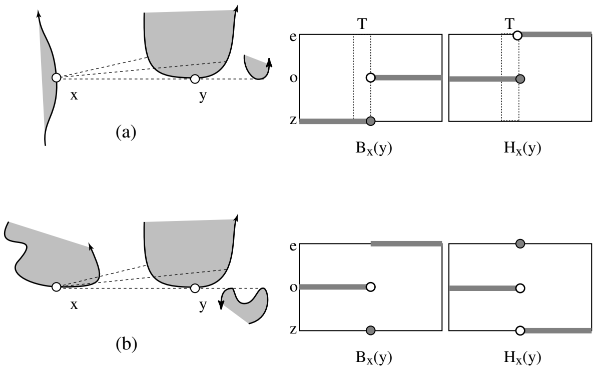

Figure 5: Discontinuities in $B_x()$ and $H_x()$.

Lemma 7 *If $B_x(y)$ is either continuous at y, or has a $z/z/o$ or $o/z/z$ discontinuity at y* (see Fig. 5a)*, then the I/E status of xy may be determined.*

Proof: Let $T = (y, y+\delta)$ or $T = (y-\delta, y)$ be an interval incident to y in which, for $t \in T$, (a) both $B_x(t)$ and $H_x(t)$ are continuous, and (b) $B_x(t) = z$. Our assumptions guarantee such an interval. Then,

E: If $H_x(t) = o$, xy is an E-edge (Fig. 5a).

I: If $H_x(t) = z$ or e, xy is an I-edge.

When $B_x(y)$ is continuous at y, it can be shown that T on either side of y leads to the same conclusion. □

Lemma 8 *If $B_x(y)$ has a $o/z/e$ discontinuity at y, then xy is an E-edge; if it has an $e/z/o$ discontinuity, then xy is an I-edge.*

Proof: See Fig. 5b for an $o/z/e$ discontinuity. Achieving $B_x(t) = e$ requires rays to intersect the curve $\mathcal{P}$ both near x and near y. The direction of the curve at y is determined by the need to achieve e after y. This forces the direction of the curve at x as shown; otherwise we could not have $B_x(y) = z$. The local situation then forces xy to be an E edge. The $e/z/o$ discontinuity is the same with all directions reversed. □

In addition it must be argued that the discontinuities covered by the previous two lemmas are the only ones possible. The final conclusion, that continuous ZP information distinguishes I- from E-edges, justifies the intuition based on the point-in-polygon algorithm.

5 Open Problems

A judicious addition of two vertices to the chains in Fig. 2 produces a "near" counterexample to the hypothesis that the weak stabbing information determines I/E, leaving "only" the Head()'s and Tail()'s of 55 vertex pairs to be equalized. If this could be accomplished, the one '?' in Table 1 could be replaced with NO.

The continuous stabbing information introduced in Section 4 raises a number of new questions. Generalization to Jordan curves with points of zero curvature (including polygons) would be pleasing. Connections to point X-ray theory [Gar95, Ch.5] should be developed. Identifying the equivalence class of curves that share the same ZP information might be possible. And it remains unclear what additional information is gained by having the absolute stabbing numbers rather than only their zero-parity information.

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