The Foldings of a Square to Convex Polyhedra

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Abstract. The structure of the set of all convex polyhedra foldable from a square is detailed. It is proved that five combinatorially distinct non-degenerate polyhedra, and four different flat polyhedra, are realizable. All the polyhedra are continuously deformable into each other, with the space of polyhedra having the topology of four connected rings.

1 Introduction

If the perimeter of a polygon is glued to itself in a length-preserving manner, and in such a way that the resulting complex is homeomorphic to a sphere, then a theorem of Aleksandrov [1] establishes that as long as no more than 2π face angle is glued together at any point, the gluing corresponds to a unique convex polyhedron (where "polyhedron" here includes doubly-covered flat polygons). Exploration of the possible foldings of a polygon to convex polyhedra via these *Aleksandrov gluings* was initiated in [2], and further explored in [3, 4]. Theorem 1 in [4] established that every convex polygon can fold to a nondenumerably infinite number of incongruent convex polyhedra. Although this set is infinite, it arises from a finite collection of *gluing trees*, which record the combinatorially possible ways to glue up the perimeter. Enumerating the gluing trees leads to an inventory of the possible foldings of a given polygon to polyhedra. These ideas were implemented in two computer programs, developed independently by Anna Lubiw and Koichi Hirata.[1] These programs only list the gluings, not the polyhedra. Even though the polyhedra are uniquely determined by the gluings, there is no known practical algorithm for computing the creases and reconstructing the 3D shape of the polyhedra [5].

The contribution of this paper is to construct the 3D structure of all the polyhedra foldable from one particularly simple polygon: a unit square. Spare remarks on regular n-gons will be ventured in the final Section 4. The polyhedra foldable from a square have from 3 to 6 vertices, and we show they fall into nine distinct combinatorial classes: tetrahedra, two different pentahedra, hexahedra, and octahedra; and a flat triangle, square, rectangle, and pentagon. Each achievable shape can be continuously deformed into any other through intermediate foldings of the square, i.e., no shape is isolated. An illustration of the foldings for

[1] Personal communications, Fall 2000. Hirata's program is available at http://weyl.ed.ehime-u.ac.jp/cgi-bin/WebObjects/Polytope2.

a portion of a continuum are shown in Figure 1. In general the crease patterns vary continuously, as in this figure, with discontinuous jumps at coplanarities to different combinatorial types. The continua fall into four distinct rings (A,

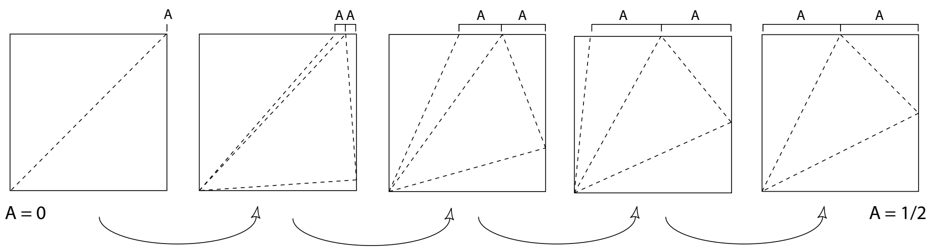

Fig. 1. Creases for a section of a continuum of foldings: as `A` varies in $[0, \frac{1}{2}]$, the polyhedra vary between a flat triangle and a symmetric tetrahedron. See also Figure 7) below.

B, C, D), each corresponding to a single parameter change in the gluing, which join together topologically as depicted in Figure 2. Three of the rings (A, B, D) share and join at the flat $1 \times \frac{1}{2}$ rectangle; rings A and C join at a symmetric tetrahedron.

An animated GIF of the entire set of polyhedra is available at `http://cs.smith.edu/~orourke/Square/animation.html`.

2 Proof

Our goal in this (long) section is to prove that Figure 2 represents a complete inventory of the shapes foldable from a square; but we will not prove formally every detail depicted in the figure. The starting point for the proof is a lemma that limits the combinatorial structure of gluing trees for convex polygons, specialized to $n = 4$:

Lemma 1. *[3, 4]; see also [6]. The possible gluing trees for a convex quadrilateral ($n = 4$) are of four combinatorial types:*

1. *'`|`': a tree of two leaves, i.e., a path.*
2. *'`Y`': a tree of three leaves and one internal degree-3 node.*
3. *'`I`': a tree of four leaves and two internal degree-3 nodes.*
4. *'`+`': a tree of four leaves and one internal degree-4 node.*

We will now make an exhaustive list of the possible gluings of a unit square, applying three facts to increasingly restrict the possibilities:

1. Lemma 1.
2. A square has four corners, each of internal angle $\pi/2$.
3. These corners are separated by edges of length 1, and the perimeter is 4.

Let c_0, c_1, c_2, c_3 be the corners of the square. The proof, an extended case analysis, follows the structure provided by Lemma 1.

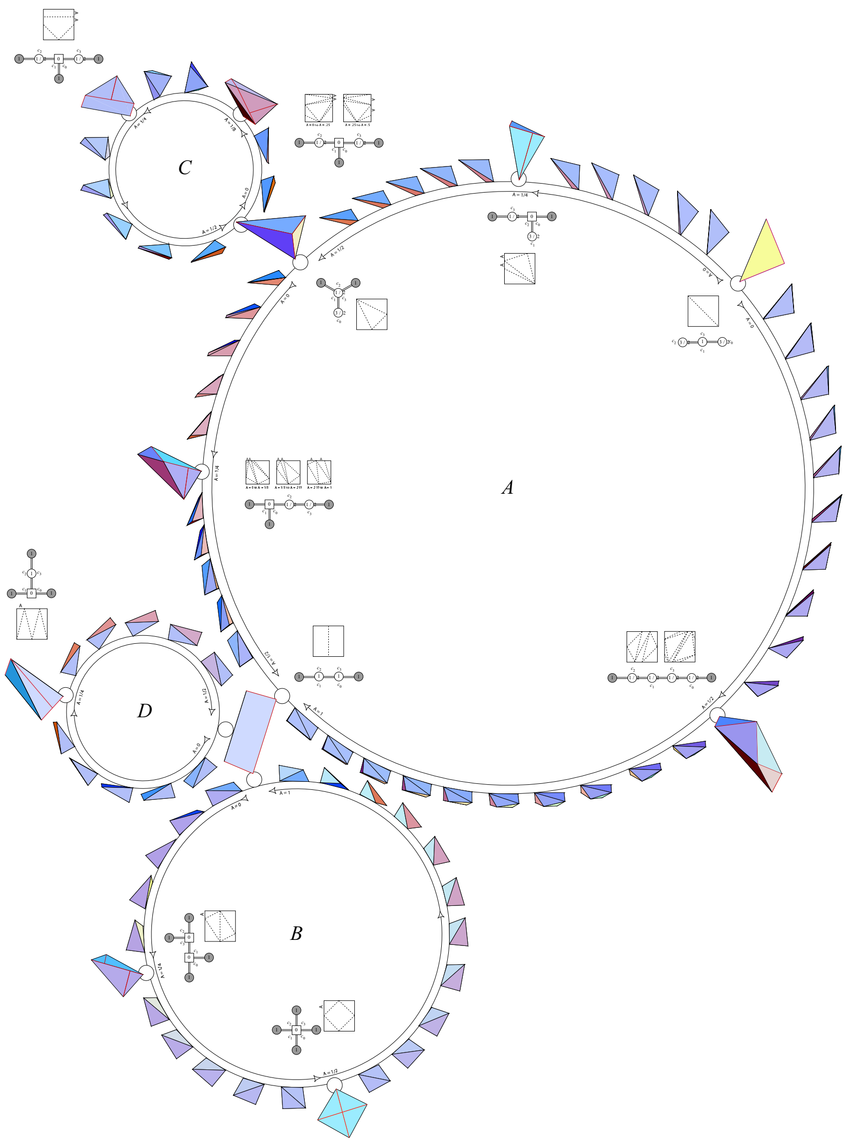

Fig. 2. The foldings of a square to convex polyhedra. Selected polyhedra are shown enlarged (including the four flat shapes), together with their crease patterns and corresponding gluing trees.

Case I. The four corners must be distributed along the path. If no corner is glued to another, and neither leaf is a corner, then we have the structure illustrated in Figure 3(a). Here, and in subsequent figures, nonzero-curvature vertices of the polyhedron are marked by circles, with the curvature (in units of π) indicated inside the circle. (Thus the circled numbers must add to 4 to satisfy the Gauss-Bonnet theorem.) A shaded circle represents a *fold point*—a creased edge—necessarily of curvature π. The corners are labeled c_i, with the position of the labels indicating to which side of the pieces of paper gluing there the corner resides. The staggered distribution of the corners in (a) are necessary, for if, say, c_0 and c_1 were adjacent on the lower side as shown in (b), then two arcs would have to be zero length in order to have a total perimeter of 4, which would force nodes to merge. The gluing tree in (a) is a six-vertex polyhedron, generally an

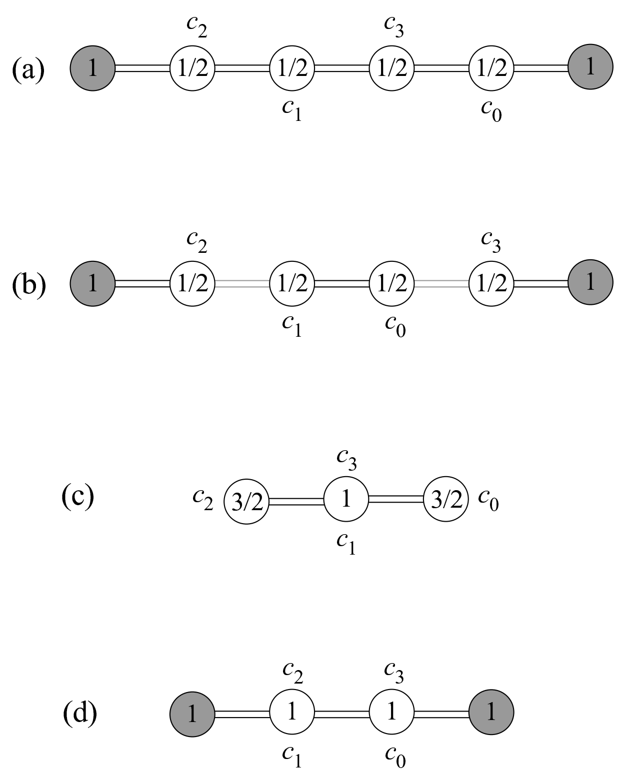

Fig. 3. (a) When all corners are distinct and not leaves; (b) A configuration that forces the lightly shaded arcs to have zero length; (c) A flat right triangle; (d).A flat rectangle.

octahedron, although there one spot in the continuum (with parameter $A \approx 0.35$) where it becomes a pentahedron with three quadrilateral faces: a triangular prism, two nearly parallel triangles joined by four near-rectangles. Figure 4 shows

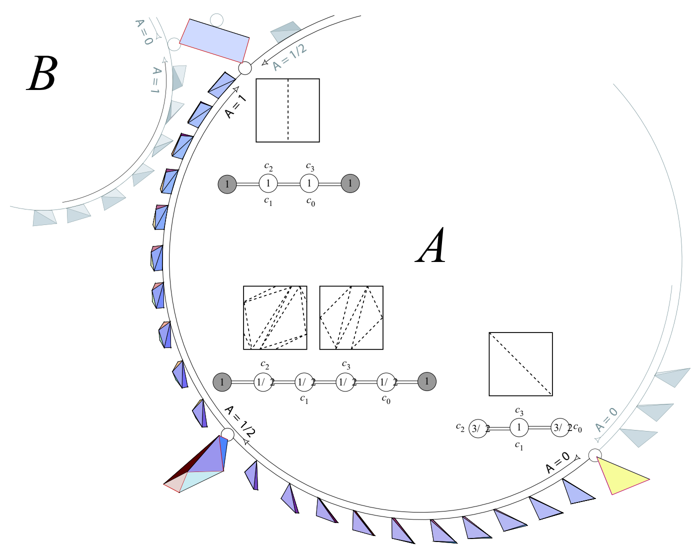

Fig. 4. The octahedra continuum, a subpart of the A-loop.

an enlargement of the A- (triangle) continuum in Figure 2[2] that includes the octahedra.

If one corner of the square is a leaf of the gluing tree, then both leaves are forced to be opposite corners, and the two other opposite corners mate, resulting in the structure shown in (c) of the figure, which corresponds to a flat right triangle, the right end of the octahedron continuum in Figure 4.

If two adjacent corners mate, the structure shown in (d) of the figure is forced, which folds to a flat $1 \times \frac{1}{2}$ rectangle the left end of the octahedron continuum in Figure 4 (where it joins with two other continuua, B and D).

Case `Y`. The single internal node of a '`Y`' must consist of two (cc) or three (ccc) corners, for otherwise the paper there would exceed 2π.

1. *ccc*. The only issue remaining is where the fourth corner lies. If it is a leaf, the structure shown in Figure 5(a) is forced. This represents a "fixed" tetrahedron, fixed in the sense that the two fold-point leaves are forced to be side midpoints and so do not form a "rolling belt" [3, 4]. (This tetrahedron is the shape shared by loops A and C in Figure 2). If the fourth corner is a path node, then the length of the shaded arc is forced to be zero, which reduces this case to that in (a).

[2] All subsequent closeups of the continuua are rotated 90° with respect to the orientation used in Fig. 2.

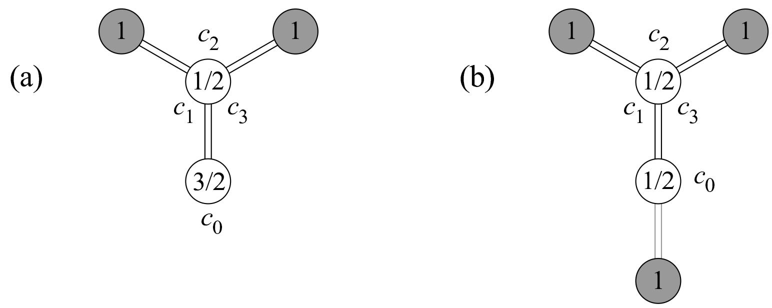

Fig. 5. (a) A fixed tetrahedron; (b) An impossible gluing tree.

2. *cc*. Two corners and an interior point on a square side forms a total angle of 2π, so the internal node of the 'Y' has zero curvature. Thus it is not a polyhedron vertex; we will indicate such nodes with a box enclosing a 0. There are two possibilities: either the two corners at the junction are opposite corners, or adjacent corners.
   (a) Opposite. With c_0 and c_2 (without loss of generality) glued to the junction, c_1 is forced to be at a leaf. This leaves only the location of the fourth corner to be determined. It might be a leaf node or a path node. If c_3 is a path node, we have the tree shown in Figure 6(a). This corresponds to an asymmetric tetrahedron, a continuum with the two shaded leaves forming a rolling belt. If c_3 is a leaf node, then the shaded arc

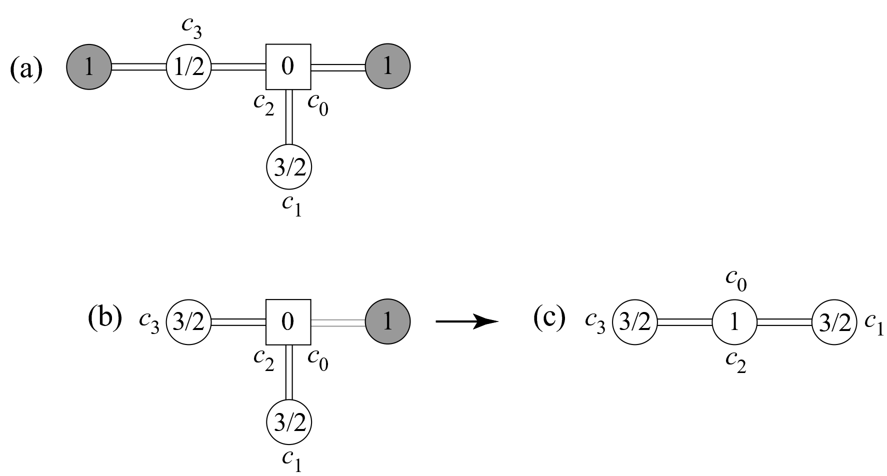

Fig. 6. Opposite corners glued to 'Y'-junction.

in (b) of the figure must be zero length, yielding (c), which is the same as Figure 3(c): a flat right triangle. The corresponding continuum (part of the A-loop) is shown Figure 7; note that its lower end is the flat right triangle and its upper end the fixed tetrahedron.

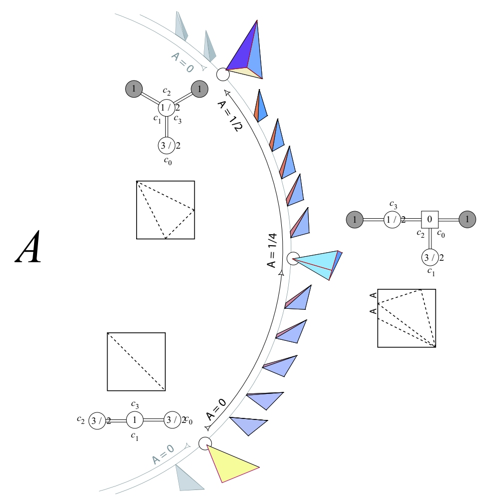

Fig. 7. A portion of the A-loop that includes a continuum of tetrahedra. Compare Figure 1.

(b) Adjacent. With c_0 and c_1 (without loss of generality) glued to the junction, the leaf between is a fixed fold midpoint. The two other corners of the square must be placed on the other two branches of the '`Y`' (which now looks like a '`T`'). We can distinguish four possibilities, determined by whether each corner is a leaf (L), a path node (P), and when both are path nodes, whether to the opposite (o) or same (s) side of the junction.

i. LL. When both corners are leaves, we have the structure shown in Figure 8(a). Knowing that the distance between consecutive corners is 1, this structure must have perimeter 5. So it is unachievable.

ii. LP. When one corner is a leaf and the other a path node, the path node could be to the same or the opposite side of the '`Y`' as the leaf. If it is to the same side, the total perimeter can be calculated to be 5, ruling out this structure. If it to the opposite side, we have the structure shown in (b) of the figure. Here the shaded arc must be zero length to have a perimeter of 4, which reduces this structure to the three-corner '`Y`' of Figure 5(a).

iii. PPo. This structure (Figure 8(c)) corresponds to a five-vertex polyhedron continuum, which at its midpoint is a flat pentagon, and at

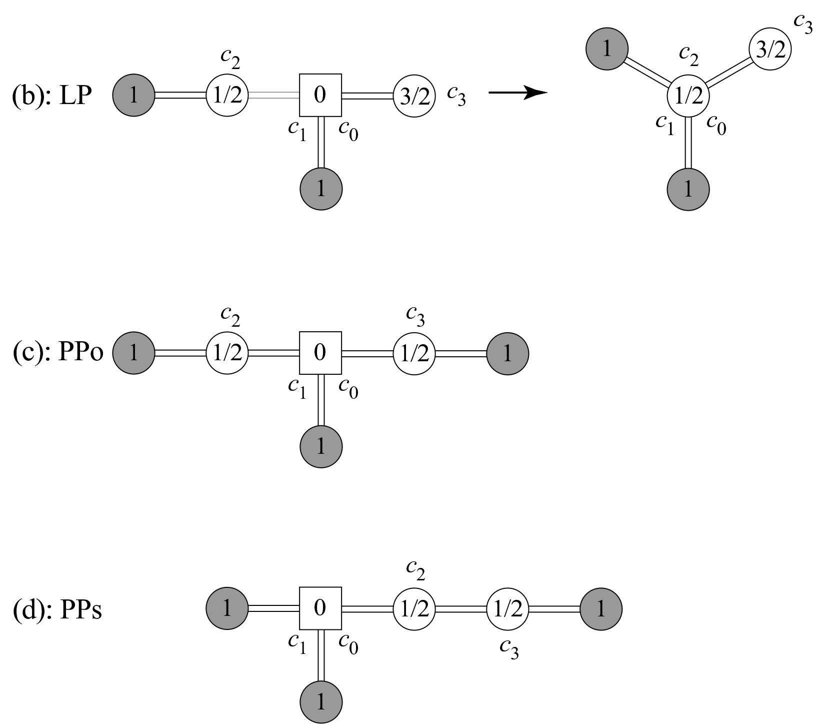

Fig. 8. Adjacent corners glued to '`Y`'-junction. Labels: L=leaf; P=Path; o=opposite; s=same.

either endpoint (when one corner becomes a leaf and the other corner joins the junction) becomes the fixed tetrahedron of Figure 5(a). This forms the C-loop of the continuum, as shown in Figure 9.

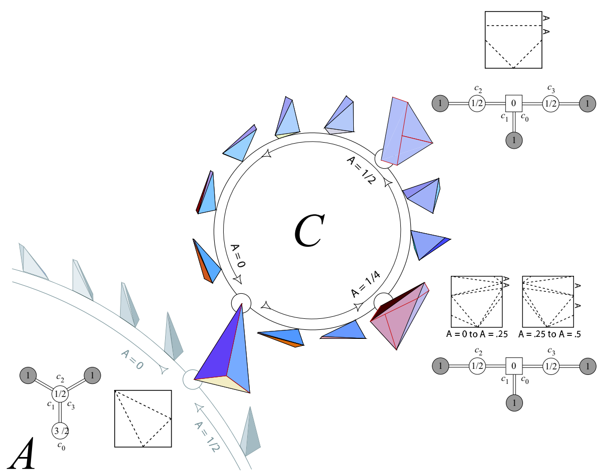

Fig. 9. The C-loop containing the flat pentagon.

iv. PPs. This (Figure 8(d)) corresponds to another five-vertex continuum, which at one endpoint is the same fixed tetrahedron, and at the other the flat rectangle of Figure 3(d). See Figure 10. Note that c_2 and c_3 cannot be on the same side of the branch, as then the perimeter would be too long, so they must be on opposite sides. And c_3 cannot be on the lower side, adjacent to c_0, for the same reason. Thus the structure illustrated is the only one possible.
We will not establish this formally, but the hexahedron continuum in Figure 10 partitions into three sections, separated by two pentahedra which occur when two triangles become coplanar and form a quadrilateral face.

Case `I`. Each junction must have two or three corners, which, because there are two junctions, means that both must have two corners. There are three possible patterns for the distribution of the corners, illustrated in (a,b,d) of Figure 11, which we will call mixed, adjacent, and opposite.

1. Mixed. The shaded edge in Figure 11(a) must be zero length to achieve a perimeter of 4, which reduces this case to (c), which we will discuss below.
2. Adjacent. The pattern of corner distribution shown in Figure 11(b) is possible, producing two independent rolling belts. This is a two-dimensional

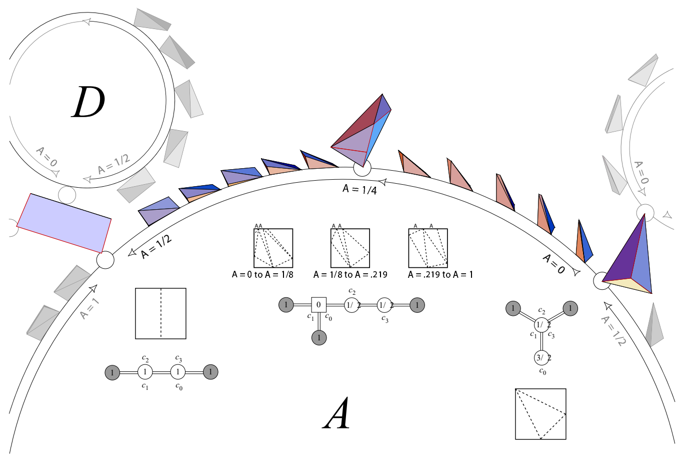

Fig. 10. The portion of loop A consisting of hexahedra.

continuum of tetrahedra, all four of whose vertices have curvature π. It is clear that if we roll the upper belt to one extreme and fix it there, thereby gluing two corners together as in Figure 11(c), the shapes produced by the rolling of the lower belt include all the shapes achievable through the rolling of both belts. This is because it is only the relative rolling of the two belts that is significant. Thus, the 2D continuum of shapes can be captured in the 1D loop D, illustrated in Figure 12. Notice that the polyhedra at symmetric positions with respect to the flat rectangle are reflections of one another.

3. Opposite. The structure forced here, Figure 11(d), is another continuum of tetrahedra with curvature-π vertices. Here the upper and lower fold points form a rolling belt; the two side fold-point leaves are fixed at the midpoint of the folded edge. The continuum forms the B-loop, Figure 13, shapes mirror-symmetric about the flat rectangle and square.

The extremes of both continuua, at both ends, lead to the structures shown in Figure 11(e), which can be recognized as the flat rectangle of Figure 3(d).

Case +. Finally, the degree-4 junction of the '+' can only be realized with all four corners glued together, which forces the structure in Figure 14: the flat square of Figure 13.

This completes the inventory of the polyhedra foldable from a square.

3 Reconstructing the 3D Shapes

We mentioned that it is an unsolved algorithmic problem to compute the three-dimensional coordinates of a polyhedron given the face structure determined by

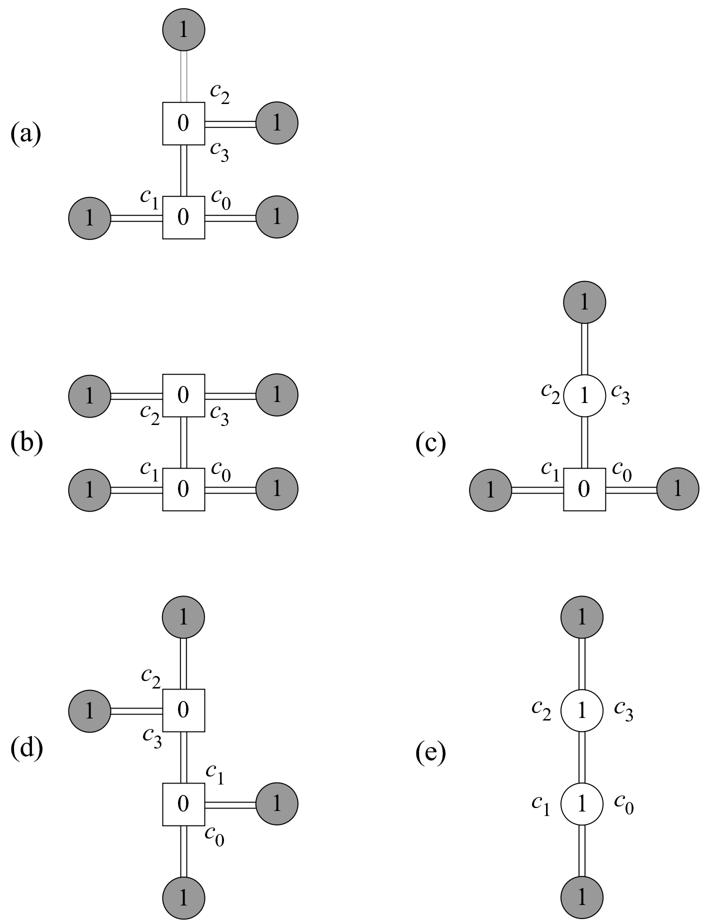

Fig. 11. The gluing trees with two degree-3 nodes: (a) Not possible; (b) Double belt; (c) Equivalent single belt; (d) Another tetrahedron continuum.

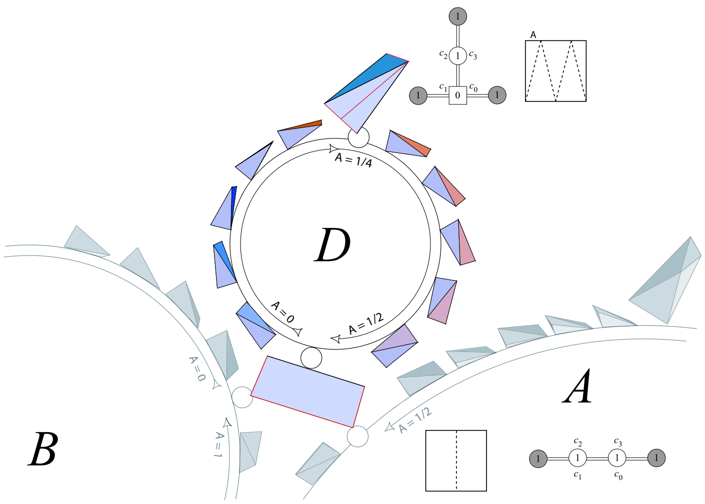

Fig. 12. Twisting tetrahedra: loop D.

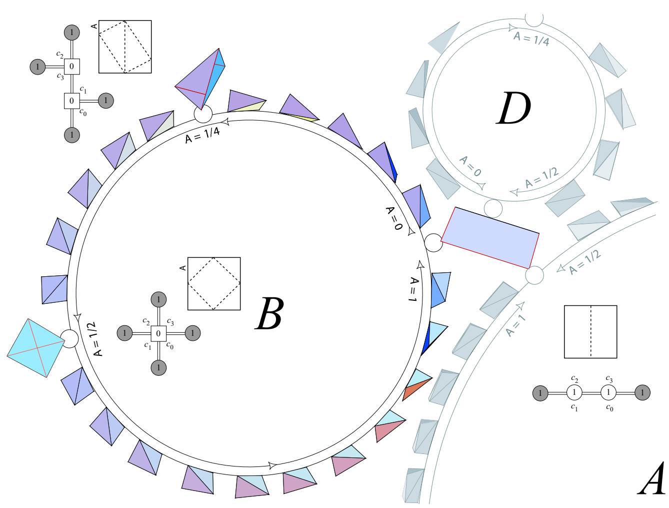

Fig. 13. The B-loop, containing the flat square.

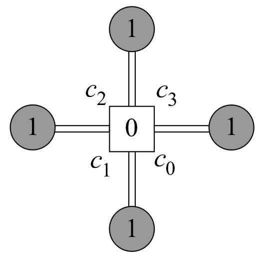

Fig. 14. This gluing tree corresponds to a flat square.

a particular gluing. We now describe how it was possible nevertheless to compute the structure of all the polyhedra foldable from a square, as displayed in Figure 2.

There are two issues: (1) Identifying the creases, and therefore the edge lengths; and (2) reconstructing the 3D shape from the edge lengths.

(1) Although we have no systematic method for identifying the creases, there are only a finite number of possibilities, as Aleksandrov observed [7]. The vertices are determined as the points of nonzero curvature, and we know that every edge is a shortest path between the vertices that are its endpoints. So, lacking any other information, we could try all $\binom{n}{2}$ possibilities. In practice, some of the creases are obvious from physical models, leaving only a few uncertainties. These were resolved by "trying" each, and relying on Aleksandrov's theorem guaranteeing a unique reconstruction: the wrong choices failed to reconstruct, and the correct choice led to a valid polyhedron.

(2) All of the nonflat polyhedra foldable from a square have 4, 5, or 6 vertices. Aside from isolated special cases caused by coplanar triangles merging into quadrilaterals, all faces are triangles. In fact, only three distinct combinatorial types are realized: tetrahedra, hexahedra equivalent to two tetrahedra glued base-to-base (i.e., a "trigonal dipyramid"), and octahedra combinatorially equivalent to the regular octahedron. Reconstructing tetrahedra from their six edge lengths is not difficult. Reconstructing the hexahedra from their nine edge lengths can be accomplished by reconstructing the two joined tetrahedra.

Reconstructing octahedra is more challenging, for the structure is not determined by the union of tetrahedra all of whose edges are on the surface. Any partitioning into tetrahedra leaves edges of tetrahedra as internal diagonals, whose lengths are not determined by creases of the square. The strategy we used is to partition the octahedron into two hexahedra, each a trigonal bipyramid. Consider one half of the octahedron, for which we know eight lengths: the four edges of the quadrilateral "base" (which will not in general be a flat polygon), and the four edges from the base to an apex. These lengths leave the structure with one degree of flexibility. The unknown length x of the "internal" diagonal d splitting the base quadrilateral is one parameter that determines this flex. Two octahedron halves together with the same x may join only if the dihedral angle in each at the edge d sum to 2π. This gives us a method to solve for x: find an x

such that the angles sum to 2π. Although one can express x as a polynomial of some large degree (perhaps 16), we found it possible to solve for x via numerical search.

4 Discussion

Constructing the entire set of polyhedra permits us to answer a special case of a question posed by Joseph Malkevitch[3]: What is the maximum volume polyhedron foldable from a given polygon? We found (by numerical search) that the maximum is achieved by an octahedron along the A-ring (at about the 4-5 o'clock position), with a volume of approximately 0.055849. We expected to observe symmetry here but did not.

The work reported here is part of a larger project to understand the structure of all the convex polyhedra foldable from any given polygon. Although that goal does not appear close with our current understanding, the foldings of regular n-gons seem more approachable. It was established in [3] that, for $n > 6$, there is only one nonflat folding of regular n-gons, the class of "pita polytopes" produced by perimeter halving. (For the square, this corresponds to the path gluing of Figure 3(a).) One might expect a one-ring continuum of polyhedra for these n-gons. The complexity of the multiple rings manifest with the square may be an artifact of small $n \leq 6$. For arbitrary convex polygons, it would be of interest to establish that the space of folded polyhedra is connected, i.e., that no shape is isolated.

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