Interlocked Open and Closed Linkages with Few Joints

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Abstract

We study collections of linkages in 3-space that are *interlocked* in the sense that the linkages cannot be separated without one bar crossing through another. We explore pairs of linkages, one open chain and one closed chain, each with a small number of joints, and determine which can be interlocked. In particular, we show that a triangle and an open 4-chain can interlock, a quadrilateral and an open 3-chain can interlock, but a triangle and an open 3-chain cannot interlock.

1 Introduction

Consider a simple polygonal chain, either an open *arc* or a closed *polygon*, that is embedded in 3-space. We view the vertices of the chain (except the endpoints of an open chain) as universal *joints*, and the edges of the chain as rigid *bars*. We call a chain with k bars a *k-chain*. A *motion* of the chain is a motion of the vertices that preserves the length of the bars, and never causes bars to cross. In particular, a *straightening of an open chain* is a motion that makes all joint angles become $180°$. We say that a collection of disjoint, simple chains can be *separated* if, for any distance d, there is a motion whose result is that every pair of points on different chains has distance at least d. If a collection cannot be separated, we say that its chains are *interlocked*. If a single chain cannot be straightened, we say that it is *locked*.

It is known that a single, open chain in 3-space, having as few as 5 bars, can be locked [CJ98, BDD+99]. Other classes of chains are known to be unlocked, but the complexity of deciding whether a given chain can be unlocked is not known. One decision procedure applies the roadmap algorithm for general motion planning [Can87, Can88], which runs in polynomial space but exponential time.

Our work is inspired by a question posed by Anna Lubiw [DO00]: Into how many pieces must a chain be cut so that the pieces can be separated and straightened? This problem is motivated by protein molecules, which can be modeled by polygonal chains, and, according to some theories, temporarily split apart in order to reach the minimum-energy folding.

We can observe easy upper and lower bounds for Lubiw's problem: some n-chains require cutting at least $\lfloor (n-1)/4 \rfloor$ vertices for separation, and no chain requires cutting of more than $\lfloor (n-1)/2 \rfloor$

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vertices. The lower bound is obtained by concatenating many copies of the 5-bar "knitting needles" example from [CJ98, BDD+99], each sharing one bar with the next as in Fig. 1. Observe that each copy of the locked 5-bar chain must have one of its four interior vertices cut. The upper bound

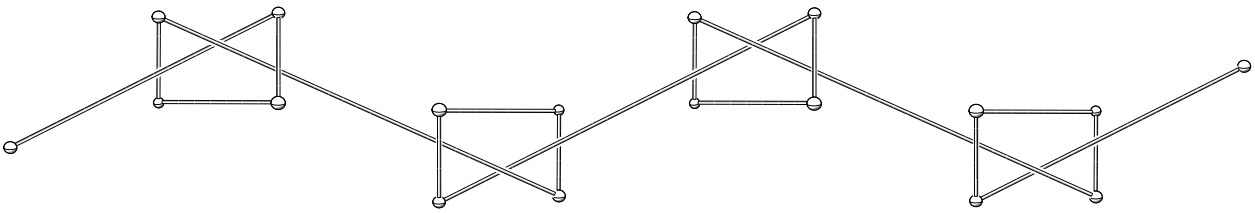

Figure 1: An $n = 17$ bar chain that requires cutting at least $\lfloor (n-1)/4 \rfloor = 4$ vertices to separate.

is obtained by cutting every second joint of a chain, and observing that the resulting 2-bar pieces ("hairpins") can be rigidly separated arbitrarily far by blowing up from a point, because the pieces are starshaped sets. This argument dates back at least to de Bruijn in 1954 [dB54]; we use a variant to prove Lemma 3. See also [Daw84, SS94, Tou85].

While Lubiw's problem motivated our original interest in interlocked open chains, we explore here interlocking for combinations of open and closed chains. In the next section, we resolve how many bars are needed by each chain in order to obtain an interlocked pair, as summarized in Table 1.

Sec	Chain 1	Chain 2	Result
2.1	closed triangle	open 4-chain	Can Interlock
2.2	closed quadrilateral	open 3-chain	Can Interlock
3	closed triangle	open 3-chain	Cannot Interlock

Table 1: Our results on when an open chain and a closed chain can interlock. A claim that a k-chain can interlock, also holds for any l-chain with $l > k$, and a claim that a k-chain cannot interlock also holds for any l-chain with $l < k$.

2 Interlocked Examples and the Topological Method

Our two proofs that chains are interlocked follow a similar structure in what we call the *topological method*. We imagine tying the two ends of the open chain with a long rope near infinity, which defines a topological *link* (multicomponent knot). For the two chains to separate, they must form the trivial link. First we show that before this happens, the ends of the open chain must get close to the closed chain. Second we argue that this proximity is impossible before changing the topology of the link. Finally we prove that this circularity leads to a contradiction, so the chains are interlocked.

To make connections to known mathematics for links, we will refer to some links by their numbers from standard tables. See [Ada94, p. 287] or [Wei99, p. 1086]. Tables of links are often organized by (minimum) crossing number. For example, the link from Fig. 2 is drawn with six crossings, but the $l_0 : l_3$ crossing could be removed to obtain five. In fact, this link is numbered 5^2_1. The superscript in the link notation is the number of components, for us always 2. The subscript is an arbitrary table index.

2.1 Triangle and 4-chain

We begin with the configuration illustrated in Fig. 2.

Theorem 1 *A triangle and a ($k \geq 4$)-chain can interlock.*
Proof: We choose the following notation for the configuration of Fig. 2: A triangle abc lies in a plane H, with H^+ the halfspace above and H^- the halfspace below H. Let the circumcircle of $\triangle abc$ have center o, and radius r.

The 4-chain alternates points and bars p_0, l_0, p_1, l_1, ..., l_3, p_4 with the following placements: p_0 is in H^-, bar l_0 crosses the interior of $\triangle abc$, and ends at a point p_1 above o. Bar l_1 crosses the interior of $\triangle abc$ again, so $p_2 \in H^-$. Bar l_2 crosses H outside of $\triangle abc$, and l_3 crosses the wedge formed by l_0 and l_1 above H. So $\{p_0, p_2\} \subset H^-$ and $\{p_1, p_3, p_4\} \subset H^+$.

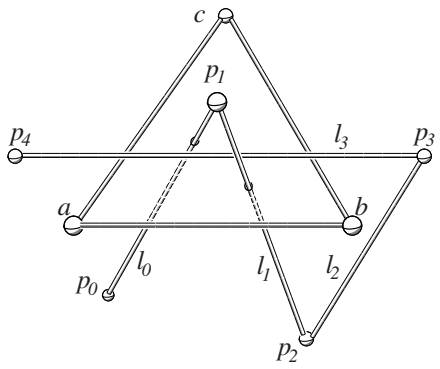

Figure 2: A triangle and a 4-chain can lock.

Let R be the real number $r + |l_1| + |l_2|$, and set the length of l_0 and l_3 to $20R$. Consider the open ball B of radius $15R$, and the ball B' of radius $4R$, both centered at o. Initially, p_0 and p_4 lie outside of B, while a, b, c, p_1, p_2 and p_3 all lie inside $B' \subset B$. As long as p_0 and p_4 stay outside B, we can attach a sufficiently long unknotted string between p_0 and p_4 that remains outside B, and our configuration is equivalent to the link 5_1^2. The non-interlocked configuration corresponds to two separable unknots, so any motion separating this configuration would require p_0 or p_4 to enter the ball B'. Before that happens, p_0 or p_4 will have to enter the ball B.

Consider the first event when p_0 or p_4 touches the boundary of B. Then points p_1, p_2 and p_3 must be out of B' but still inside B; see Fig. 3. Thus, the only elements possibly inside B' are $\triangle abc$ and the two bars l_0 and l_3. Then either one of l_0 and l_3 cross the interior of $\triangle abc$, or both do, or neither do. The first case corresponds to a bar 2_1^2 and the third case to two separable unknots 0_1^2; neither of these are equivalent to our starting configuration.

The second case requires a careful analysis. Because end vertices p_0 and p_4 are still outside of the open ball B, we can replace the string joining them by a great arc γ on the boundary of B. Let T be the plane parallel to l_0 and l_3, and passing through o. Consider the orthogonal projection of the 4-bar linkage onto T. Note that in the projection, the lengths of bars l_0 and l_3 are preserved, and all other segment lengths are at most their original lengths. Then joints p_1, p_2 and p_3 lie in a ball of radius R centered at p_1, and since they are outside B', we can separate the projections of p_1, p_2 and p_3 from the projections of p_0, p_4 and $\triangle abc$ by a line. (This separation is necessary to exclude cases such as the one shown in Fig. 4.) From this we can infer that configuration is equivalent to 0_1^2, which is two separated, unknotted links, or to 4_1^2, which is shown in Fig. 5. But neither of these are equivalent to our starting configuration, so this first event could never happen.

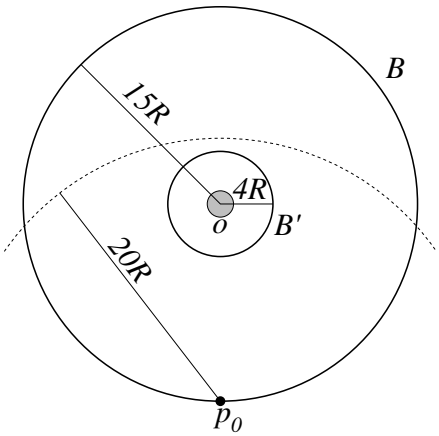

Figure 3: When p_0 touches B, point p_1 must be exterior to B' by at least R, and therefore p_2 and p_3 are also exterior to B'.

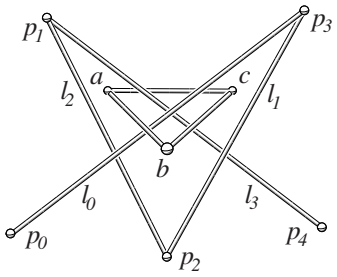

Figure 4: This configuration is incompatible with the fact that p_0 or p_4 touch the boundary of B.

□

Note that an identical argument can be used to show that the chains in Fig. 4 are interlocked as well.

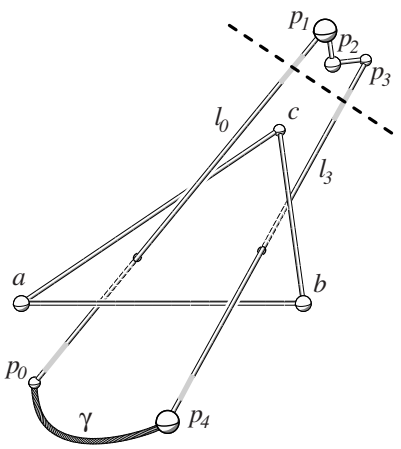

Figure 5: The link 4_1^2, formed when bars l_0 and l_3 both pass through the interior of $\triangle abc$. (Not to scale; gray segments indicate omissions.) Joints $\{p_1, p_2, p_3\}$ can be separated from $\{a, b, c, p_0, p_4\}$.

2.2 Quadrilateral and 3-chain

Theorem 2 *A 4-gon can interlock with a 3-chain.*
Proof: Starting with the configuration of Fig. 6, let $R = |ab| + |bc| + |cd| + |l_1|$ and set the length of l_0 and l_2 to $20R$. Consider the open ball B of radius $15R$, the ball B' of radius $4R$, both centered at a. As in the previous proof, we connect p_0 to p_3 by a string staying out of B. The resulting link is now 7_1^2 (in a variant of this configuration, l_0 is above l_2, and the link is 6_1^2). We again argue that in order to separate the 4-gon from the 3-chain, p_0 or p_3 has to enter the ball B. The first time one of those endpoints touch the boundary of B, p_1 and p_2 are still inside B but out of B' and the only elements possibly inside B' are $abcd$ and the two edges l_0 and l_2.

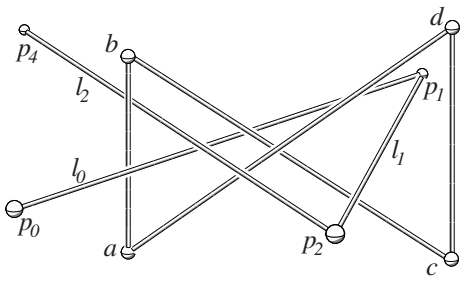

Figure 6: A quadrilateral and a 3-chain can lock.

We now argue that the possible cases are the same as in the previous theorem. Consider the tetrahedron formed by the vertices $abcd$, assign a weight of $+1$ to the faces abd and bcd, and a weight of -1 to abc and acd. The case of an edge not intersecting the tetrahedron and the case of an edge crossing the tetrahedron at two faces of same sign are equivalent to the case of the edge not intersecting the interior of the triangle in the previous theorem, and the case of an edge intersecting faces of opposite sign is equivalent to the case of an edge intersecting the interior of the triangle. The remainder of the proof is identical. □

3 Triangle and 3-chain Cannot Interlock

To show that Theorems 1 and 2 are sharp, we now prove that a 3-chain may always be separated from a triangle.

We begin with a simple case. Define a *T-chain* to be an edge e_1 attached at one end to a point p_1 along another edge e_2. A valid motion for a T-chain preserves edge lengths and the distance between the connection point p_1 and the endpoints of e_2. Note that a 2-chain is a T-chain.

Lemma 3 *A T-chain can always be separated from a triangle.*
Proof: Consider the triangle $\triangle abc$ and the T-chain τ composed of the edge $e_1 = (p_0, p_1)$ attached at p_1 to the edge e_2. Let H be the plane containing $\triangle abc$. Assume that p_1 is not contained in H; otherwise apply a small motion to τ to move p_1 out of H. Choose the coordinate system so that H is the xy plane.

Now, consider the affine transformation $x \to \alpha x$ for any real $\alpha \geq 1$. This transformation is a nonuniform scaling that increases all distances between pairs of points with different x coordinates. So $\triangle abc$ remains unchanged, the distance between p_1 and $\triangle abc$ increases and the length of the edges of τ do not decrease.

Create a motion parameterized by time $t \geq 1$ by placing τ according to the transform for $\alpha = t$, and truncating the edges (edge e_1 at end p_0 and edge e_2 at both ends) to preserve the lengths. No affine transformation can introduce crossings among lines or line segments, so the motion does not introduce any crossings. As t becomes large, $\triangle abc$ and τ can be separated by an arbitrary distance, so they are not interlocked. □

Theorem 4 *An open 3-chain cannot interlock with a triangle.*
Proof: We follow the notation established in Theorem 1: $\triangle abc$ lies in plane H, and the 3-chain C has vertices (p_0, p_1, p_2, p_3) and bars (l_0, l_1, l_2). First assume C is not planar; otherwise, make C nonplanar by a small motion. Define points $q_i = l_i \cap H$.

1. If either bar l_0 or bar l_2 does not intersect $\triangle abc$, then it is possible to move that bar within the plane that it forms with l_1 so that the angle at the joint shared with l_1 is arbitrarily close to either 0 or π because one of the two wedges spanned by these two motions does not intersect any other edge. This configures the chain arbitrarily close to a T-chain, which can be separated from $\triangle abc$ by Lemma 3.

2. If both l_0 and l_2 intersect $\triangle abc$, we consider two further cases.

   (a) If $\{p_0, p_3\} \in H^+$ and $\{p_1, p_2\} \subset H^-$:
   Assume that $\{q_0, p_1, p_2, q_2\}$ do not lie in a common plane (otherwise apply a small motion to C). Translate l_1 in a direction orthogonal to H, away from H, while maintaining l_0 and l_2 through the original points q_0 and q_2. During the entire motion, $\{q_0, p_1, p_2, q_2\}$ never lie in a common plane, which implies that l_0 and l_2 do not intersect. During this motion, l_0 or l_2 will eventually pull out of $\triangle abc$ and we return to the previous case.

   (b) If $\{p_0, p_2\} \in H^+$ and $\{p_1, p_3\} \subset H^-$:
      i. If l_1 intersects $\triangle abc$, then bar l_0 (or bar l_2) can be moved so that the angle with l_1 is arbitrarily close to 0, and we again have a T-chain. This motion is without crossings because the wedge formed by l_0 and l_1 does not intersect any other edge.
      ii. If l_1 does not intersect $\triangle abc$, we have the only "hard" case, to which we now turn.

Because configuration C is non-self-intersecting, we can assume that the points $\{q_0, p_1, p_2, q_2\}$ do not lie on a common plane, or equivalently $\{q_0, q_1, q_2\}$ are not collinear. Denote the line containing q_0 and q_2 by $L_{0,2}$, as in Fig. 7. In fact, for any position of l_1 such that $(l_1 \cap H) \notin L_{0,2}$, the

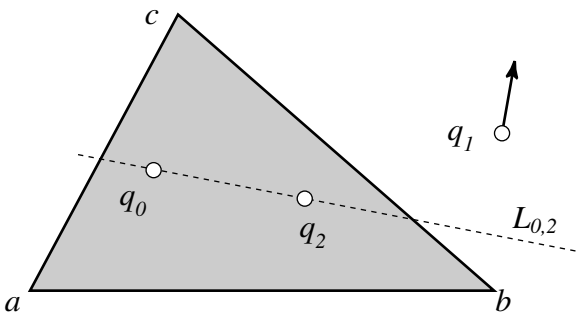

Figure 7: Translate l_1 so that $q_1 = l_1 \cap H$ moves away from $L_{0,2}$. This separates the 3-chain from $\triangle abc$.

lines containing q_0p_1 and p_2q_2 do not intersect, and do not intersect the edges of $\triangle abc$. Thus the

motion that translates l_1 in a direction orthogonal to $L_{0,2}$ and parallel to H, away from $\triangle abc$, while maintaining l_0 and l_2 through the original points q_0 and q_2, will avoid self-intersection, and eventually pull a bar out of $\triangle abc$.[1] Then we fall into a previous case. □

4 Open Problems

Many open problems remain in the context of interlocking pairs of open chains, which have close connections to the motivating problem of Lubiw. For each value of i, what is the smallest j for which an i-chain can interlock with a j-chain?

The topological method of Theorems 1 and 2, where we used a "rope" to close one open chain to form a topological linkage, does not easily extend to pairs of open chains. Two ropes would be needed, and their potential interactions would have to be controlled. To extend this work, therefore, we will be investigating a geometric method that establishes a collection of geometric facts and shows that there can be no first violation. We believe that we can use such a method to establish three conjectures: that a 3-chain can interlock with a 4-chain, that three 3-chains can interlock, but that two 3-chains cannot interlock even in the presence of any finite number of 2-chains.

The proof of Theorem 2 depends upon a tetrahedron formed by the 4-gon, and does not show that a 3-chain and a k-gon can interlock for any $k > 4$. In fact, adding any small edge to the 4-gon would allow the 3-chain to escape. On the other hand, our conjecture that a 3-chain can interlock with a 4-chain, once established, would imply that a 3-chain can interlock with a k-gon for any $k \geq 5$ by connecting the endpoints of the 4-chain with one or more edges.

Chains that model physical phenomena, such as robot arms or protein backbones, often have restrictions placed on the motion of a joint. There are a number of interesting problems for open and closed chains under various restrictions on motions. For example, we conjecture that a rigid, open 3-chain can interlock with a flexible, open 3-chain.

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