International Journal of Computational Geometry & Applications

ON RECONSTRUCTING POLYHEDRA FROM PARALLEL SLICES*

CAROLE GITLIN, JOSEPH O'ROURKE & VINITA SUBRAMANIAN
Department of Computer Science
Smith College, Northampton, MA 01063, USA
orourke@sophia.smith.edu

Received (received date)
Revised (revised date)
Communicated by Editor's name

ABSTRACT

The problem of reconstructing a three-dimensional object from parallel slices has application in computer vision and medicine. Here we explore a specific existence question: given two polygons in parallel planes, is it always possible to find a polyhedron that has those polygons as faces, and whose vertices are precisely the vertices of the two polygons? We answer this question in the negative by providing an example of two polygons that cannot be connected to form a simple polyhedron. One polygon is a triangle, the other a somewhat complicated shape with spiraling pockets.

Keywords: Reconstruction, polyhedron, interpolation, slices, contours.

1. Introduction

Starting with the papers of Keppel[9] and of Fuchs, Kedem, and Uselton,[3] there has been considerable research on reconstructing three-dimensional objects from parallel slices by a series of authors[2,10,13,16,11,1,12]. The primary concerns have been fast heuristics to select a "good" reconstruction among the many that are usually available. Apparently the question of whether reconstruction is always possible has not been previously explored. Thomas Sederberg asked for finding a set of tetrahedra that fill the volume between two polygons,[a] and this led[b] to the question of whether the polygons can always be incorporated into a simple polyhedron.

Note first that the precise question we are exploring requires that no new vertices be added: the vertex set of the polyhedron is exactly the vertex set of the polygons. Second, we are restricting attention to simple polyhedra; Fuchs *et al*[3] explicitly allow "double-cones," which are not simple polyhedra. An example of a legal reconstruction is shown in Fig. 1.

*Research supported by NSF grants CCR-882194 and CCR-9122169.
[a]Communicated to the second author by Kenneth Sloan in June 1990.
[b]In discussions with Kenneth Sloan, and with Raimund Seidel.

a

b

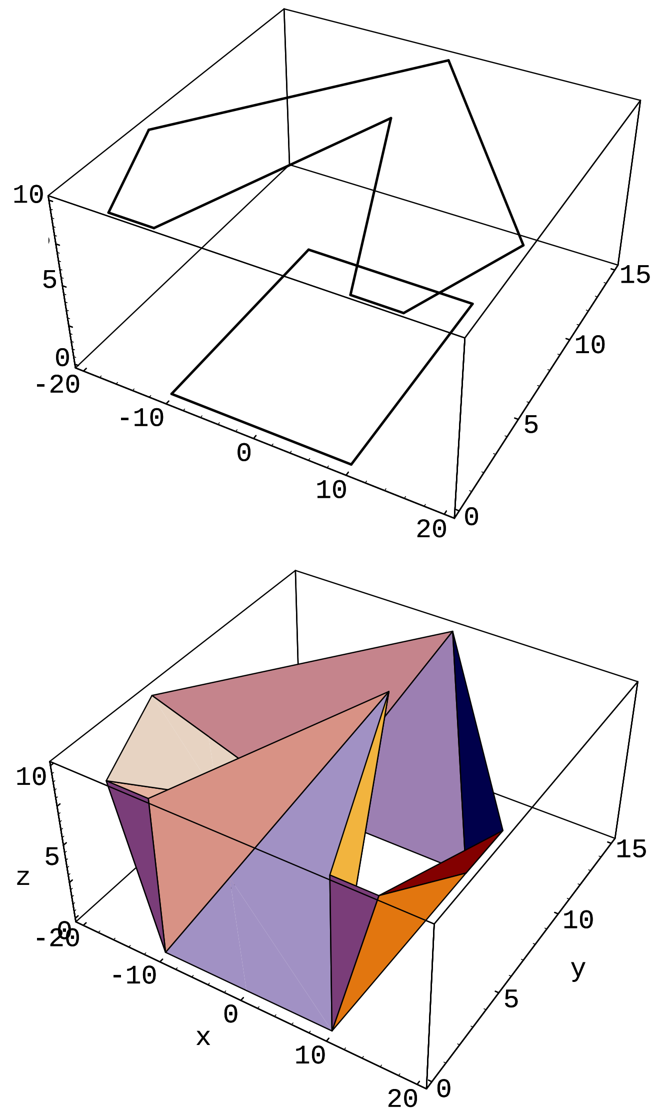

Fig. 1. A mating connection: (a) polygons; (b) connecting triangles.

2. Results

Say that an object A (point, segment, or polygon) in a plane can *mate* with a polygon B in a parallel plane if there is some way to connect A and B to form a polyhedron; precise definitions of "connect" and "polyhedron" will be offered in the next section. The main result of this paper is that there exists a polygon P that cannot mate with a triangle (Theorem 3). We also mention two related, easier results: every polygon can mate with a point, and every polygon can mate with a segment. Preliminary versions of these results appeared in ealier technical reports and theses[14,15,6,7].

After defining terms in the next section, we establish the basic lemmas sufficient to prove the point and segment mating results (Section 4). Next we establish some further conditions necessary for two polygons to mate in Section 5, and finally prove the counterexample to the hypothesis that all pairs of polygons can mate in Section 6. We discuss a computer verification of this counterexample in Section 7.

3. Definitions and Notation

We consider all polyhedra to be triangulated. Then a polyhedron may be defined as a collection of triangles that intersect "properly," and which form a surface homeomorphic to a sphere (we do not consider polyhedra of genus greater than zero). A collection of triangles intersect *properly*[c] if any given pair of triangles are either:

1. disjoint, or
2. have one vertex in common, or
3. have two vertices, and consequently, the entire edge joining them, in common.

A polygon P will be treated as a collection of edges and vertices, so that $x \in P$ means that x is on the boundary of P, not inside. Throughout the paper, the polygon on the upper plane (which we call π) is A, and that on the lower plane (distinct from and parallel to π) is B. The vertices of A are $a_0, \ldots, a_{n-1}$ and those of B are $b_0, \ldots, b_{m-1}$, both in counterclockwise order. A triangle will be represented either by a list of its three corners, or by a list of a base edge and apex.

Define a *connection* $\mathcal{C}$ between two polygons A and B as an assignment that maps each edge of A to a unique vertex of B, and each edge of B to a unique vertex of A. Since each such edge-vertex connection defines a triangle, a connection can be considered a collection of triangles. Every mating between polygons determines a connection, but not every connection yields a mating; in fact most connections will not form polyhedra. We define points of A and B to be "connected" by inheritance from $\mathcal{C}$: given $\mathcal{C}$, say that $a \in A$ is *connected* to $b \in B$ if either $a \in e$ and e is connected to the vertex b by $\mathcal{C}$, or $b \in e$ and e is connected to the vertex a by $\mathcal{C}$. In other words, ab is a subset of some triangle defined by $\mathcal{C}$. Note that it is possible for the segment ab to be a subset of the surface without a being "connected" to

[c]Definition from Giblin.[5]

b by our definition: for example, if the hull of two edges, $e \in A$ and $f \in B$, is a rectangle, then if $\mathcal{C}$ connects e to f with two triangles partitioning the rectangle, one of the two diagonals will not represent a "connection" between its endpoints in our technical sense. Moreover, two interior points of edges of A and B are never "connected."

The main property we need of mating connections is the following.

Lemma 1 *Let A and B mate via a connection $\mathcal{C}$. If $\mathcal{C}$ connects point $a \in A$ to point $b \in B$, and point $a' \in A$ to point $b' \in B$, then the open segments (a, b) and (a', b') are either identical or disjoint.*

Proof: Both segments must belong to some triangles of $\mathcal{C}$, say $ab \in t$ and $a'b' \in t'$. If $t = t'$, then the segments share the apex of the triangle, and otherwise are either identical or disjoint. If $t \neq t'$, then if the open segments intersect without being identical, then t and t' intersect improperly. Thus $\mathcal{C}$ cannot represent a polyhedral mating, contrary to the assumption of the lemma. □

4. Mating with a Point or Segment

This section establishes several easy results, warming up to the more difficult material in later sections.

Mating with a point is obvious:

Theorem 1 (Mating with a point) *A point b mates with any polygon A.*

Proof: There is only one possible connection, mapping every edge of A to b. It is clear that this forms a polyhedron, since slicing it with any plane parallel to π cuts it in a polygon similar to A. □

Mating with a segment requires more argument, but is still not difficult. We employ a lemma of Guibas and Yao on the separability of segments via translation:

Lemma 2 (Guibas-Yao) *Given a direction d and polygon P, there must be at least one edge e of P not parallel to d, that can be translated to infinity along a line parallel to d without intersecting any other edge.*

Proof: Guibas and Yao prove a very similar statement[8] (their Lemma 4.3) for disjoint line segments. Applying their proof to a polygon yields the result. In particular, let d be horizontal without loss of generality. Let e be the edge with the lowest upper endpoint v that is horizontally visible from infinity, in the sense that the horizontal ray from v to the right does not intersect P. Then e satisfies the conditions of the lemma. □

Theorem 2 (Mating with a Segment) *A segment $B = (b_0, b_1)$ mates with any polygon A.*

Proof: Let e be the edge of A guaranteed by Lemma 2 that may be freely translated in π along the direction d parallel to B. Connect e to b_0 with the triangle $t = (e, b_0)$. Connect every other edge of A to b_1. Finally, connect $b_0 b_1$ to both endpoints of $e = (e_0, e_1)$ with triangles $t_0 = (B, e_0)$ and $t_1 = (B, e_1)$. See Fig. 2. We now argue that these connections avoid self-intersections.

Suppose to the contrary that there is an illegal surface intersection. Then, it is clear that this intersection cannot occur among edges of P connected to the same

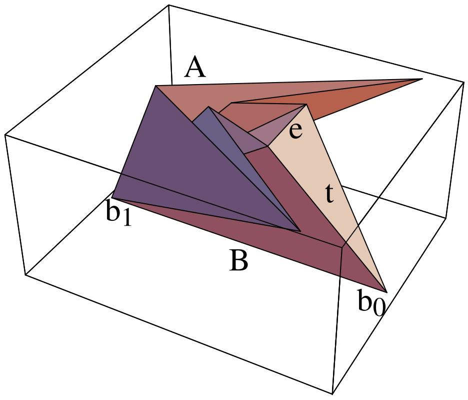

Fig. 2. Polygon A mating with segment B. Only the boundary of A is shown for clarity.

vertex of B, by Theorem 1. Since only e is connected to vertex b_0, any illegal intersection that occurs must be between one of the three triangles incident to e (t, t_0, t_1), and some point $a \in A$ connected to b_1. Consider first $t = (e, b_0)$. The set of all points S on π such that ab_1 intersects t are precisely the points swept out by e as it is moved to infinity along the direction $d = b_0 - b_1$. But we chose e by Lemma 2 so that this "strip" S is empty of points of A. Thus, an illegal surface intersection cannot occur with t.

Consider now $t_0 = (B, e_0)$. The set of all points S on π such that ab_1 properly intersects t_0 form a ray on the boundary of S, which we know is empty of points of A. The case for t_1 is the same, completing all possibilities.

Since the object avoids self-intersections and clearly has the proper topology, it is a polyhedron. □

That connecting e to b_0 forbids points of A in a certain strip to connect to b_1 is an important principle that will be formalized in the next section.

5. Necessary Conditions

We next turn our attention to mating with a triangle. We will need several more lemmas providing conditions that mating connections must satisfy.

5.1. Shadow Lemmas

In a mating of A and B, each point $a \in A$ is connected to some point $b \in B$. Each such point-to-point connection sets up constraints on other possible connections, which are not permitted to intersect the segment ab. We formulate these constraints by imagining a "shadow" on π of forbidden connections. We first prove a point shadow lemma, and then use this to establish strip- and cone shadow lemmas.

Lemma 3 (Point shadow) *Let A and B mate via a connection $\mathcal{C}$. If point $a \in A$ is connected to point $b \in B$, then for any point $b' \in B$, there corresponds* ray shadow *r from a on π parallel to $b - b'$ such that no point a' on r can be connected to b' by*

$\mathcal{C}$.

Proof: Suppose to the contrary that $b' \in B$ is connected to $a' \in r$. Then the segments ab and $a'b'$ intersect at some point x between the planes, a violation of Lemma 1. See Fig. 3. □

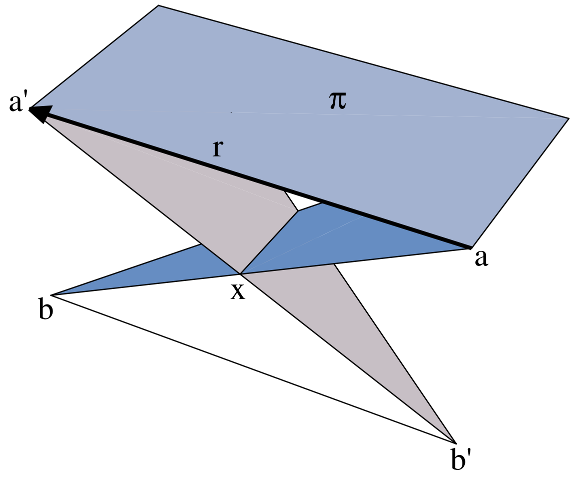

Fig. 3. The point shadow of the ab connection with respect to b'.

Strip shadows arise when an edge of A is connected to a point of B.

Lemma 4 (Strip shadow) *Let A and B mate via a connection $\mathcal{C}$. If edge $e \in A$ is connected to point $b \in B$, then for any point $b' \in B$, there corresponds a closed* strip shadow *S on π, bounded by e and the ray shadows of e's endpoints, such that no point $a' \in S$ can connect to b'.*

Proof: Suppose to the contrary that some point $a' \in S$ is connected to b'. Then apply Lemma 3 to the point $a \in e$ such that $a'a$ is parallel to bb'. Note that a is connected to b because e is, and that a' is on the point shadow of the ab connection with respect to b'. This violates Lemma 3. See Fig. 4. □

We will call a shadow with respect to point b of B a $!b$ shadow, where the "!" is to be read as "not", since b cannot be connected to any point in the shadow without self-intersection. Thus the strip in Fig. 4 is a $!b'$ shadow. Note that there are in general several $!b$ shadows for any particular point $b \in B$, one created by each triangle of the connection $\mathcal{C}$.

Cone shadows arise when a point of A is connected to an edge of B.

Lemma 5 (Cone shadow) *Let A and B mate via a connection $\mathcal{C}$. If point $a \in A$ is connected to edge $e = (b_1, b_2) \in B$, then for any point $b' \in B$, there corresponds a closed* cone shadow *C on π, bounded by the ray shadows emanating from a of the ab_1 and ab_2 connections with respect to b', such that no point in C can be connected to b'.*

Proof: Follows by applying Lemma 3 to every point $b \in e$: the cone C is the union of the point shadows of the ab connections with respect to b', over all $b \in (b_1, b_2)$. □

The shaded region C in Fig. 5 is a $!b'$ cone shadow.

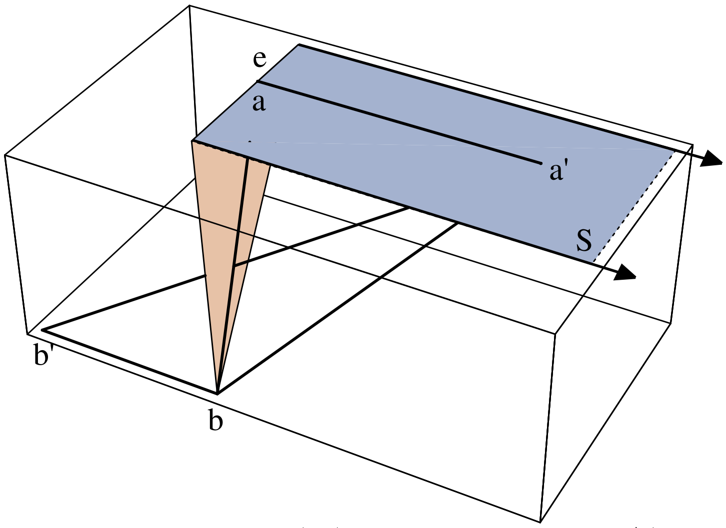

Fig. 4. The strip shadow of the (e, b) connection with respect to b' (e and bb' would not in general be orthogonal, although they are in this figure).

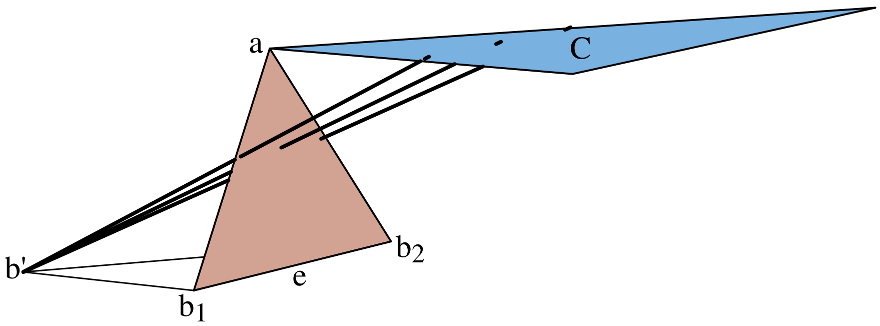

Fig. 5. The cone shadow of the (a, e) connection with respect to b'. The three dark lines show illegal connections from inside C to b'.

5.2. Cycling Constraint

The last tool we need to establish incompatibility is a result proved in Fuchs *et al*[3]: the connections between A and B must cycle exactly once, in a way we now make precise. Recall the vertices of A are $a_0, \ldots, a_{n-1}$ and those of B are $b_0, \ldots, b_{m-1}$.

For a connection $\mathcal{C}$ between A and B, label each edge of A with the unique vertex of B to which it is connected, and label each vertex of A with the labels of all the vertices of B to which it is connected (in counterclockwise order). Recall that a point $a \in A$ is "connected" to a point $b \in B$ iff ab is a subset of a triangle defined by $\mathcal{C}$, which, in the case that a and b are vertices, requires ab to be what is normally called a "diagonal." Now all the labels encountered in a traversal of A form a cyclical sequence of letters from $\{b_0, \ldots, b_{m-1}\}$. Call this sequence a *label sequence*. Since any two sequences that are cyclical rotations of one another are equivalent, we will choose to list b_0 as the first element. Let s^+ be a string of one or more occurrences of the symbol s.

Lemma 6 (Cycle) *Let A and B mate via a connection $\mathcal{C}$. Then the associated label sequence must have the regular expression form*

$$b_0^+ b_1^+ b_2^+ \cdots b_{m-1}^+ \, ,$$

that is, one or more b_0's, followed by one or more b_1's, followed by one or more b_2's, and so on.

Proof: See Fuchs, Kedem, and Uselton[3] (their Theorem 1). If the theorem is violated, an edge of either A or B is shared by two triangles between the planes, which, together with the polygon, makes the edge shared by three faces. This violates the definition of a polyhedron. □

Note that the label sequence cannot be the reverse, $b_{m-1}^+ \cdots b_0^+$, as then the surface would necessarily self-intersect.

6. Counterexample: Unmatable Polygons

In this section we provide a single example of a pair of polygons $\mathcal{A}$ and $\mathcal{B}$ that cannot mate. The polygon $\mathcal{A} = (a_0, \ldots, a_{62})$ is a complicated shape defined by 63 vertices; $\mathcal{B} = (b_0, b_1, b_2)$ is a triangle. Fig. 6 shows an overhead view of the two polygons. The Appendix lists the exact coordinates of $\mathcal{A}$. Although the polygons as shown are unmatable, it will be easier to prove that somewhat idealized forms of these polygons are unmatable. Therefore the proof will concentrate on versions that have the symmetry of the equilateral triangle: rotation by $\pi/3$ leaves each invariant. It was easier to realize the polygons with integer coordinates with only symmetry about the y-axis as drawn. Of course, a scaling in the vertical direction would restore equilateral symmetry. Secondly, in the proof it will be convenient to assume the central spirals wind more than they do in the drawings, as will be detailed at the appropriate place.

We now describe polygon $\mathcal{A}$. We view its edges as being composed of three groups: spikes, bays, and anvils, S_i, B_j, A_k respectively. There are three *spikes*,

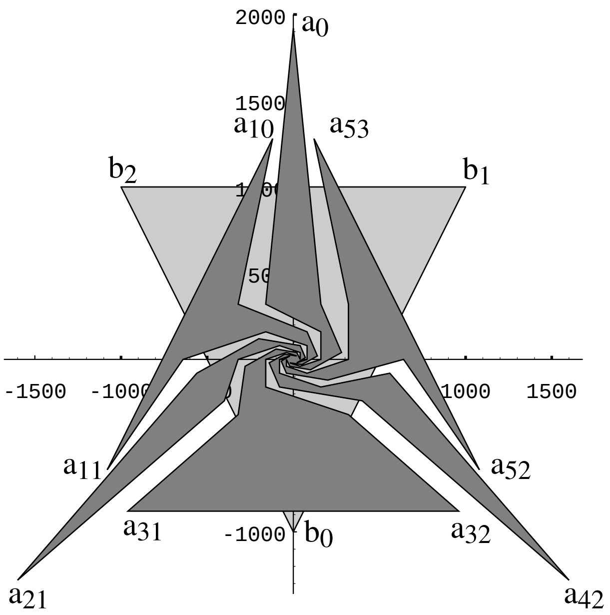

Fig. 6. Overhead view of two polygons that cannot mate.

each consisting of two edges, with spike tips at a_0, a_{21}, and a_{42}. We label $S_0 = (a_{62}, a_0, a_1)$, and S_1 and S_2 similarly. There are three *anvils*, each consisting of three edges. We label $A_0 = (a_9, a_{10}, a_{11}, a_{12})$, and A_1 and A_2 similarly. And there are six *bays*, each consisting of eight edges. We label $B_0 = (a_1, a_2, \ldots, a_9)$, and $B_1, \ldots B_5$ similarly. The structure of the spiraling bays is more visible in the magnifications shown in Figs. 7 and 8. Note that the spikes, bays, and anvils partition the edges of the polygon, and share vertices at their junctures.

The rough logic of the design of $\mathcal{A}$ is that the spike tip connections $(a_0, b_1 b_2)$, $(a_{21}, b_2 b_0)$, $(a_{42}, b_0 b_1)$, are the most "natural," in that the resulting cone shadows do not intersect $\mathcal{A}$. But these connections are incompatible with the spiral bays, which mutually constrain one another.

In any mating of $\mathcal{A}$ and $\mathcal{B}$, each of the three edges of the triangle $\mathcal{B}$ is mapped to a unique vertex of polygon $\mathcal{A}$. Call a vertex of $\mathcal{A}$ to which a $\mathcal{B}$ edge is mapped a *transition vertex*. The transition vertices are the locations in the label sequence of $\mathcal{A}$ at which transitions in labels occur. There are precisely three transition vertices by Lemma 6 (not necessarily distinct).

Lemma 7 *In any mating of $\mathcal{A}$ and $\mathcal{B}$, every edge of at least one bay connects to the same vertex of $\mathcal{B}$.*

Proof: There are six bays, and only three transition vertices. Thus at most three of the bays can have more than one label. $\square$

Let B_i be a bay guaranteed by the lemma, and assume without loss of generality (using here the equilateral triangle symmetry) that all its edges are labeled b_0. We now argue that the tangled structure of the bays implies, via the shadow lemmas, that all bays have a b_0 label.

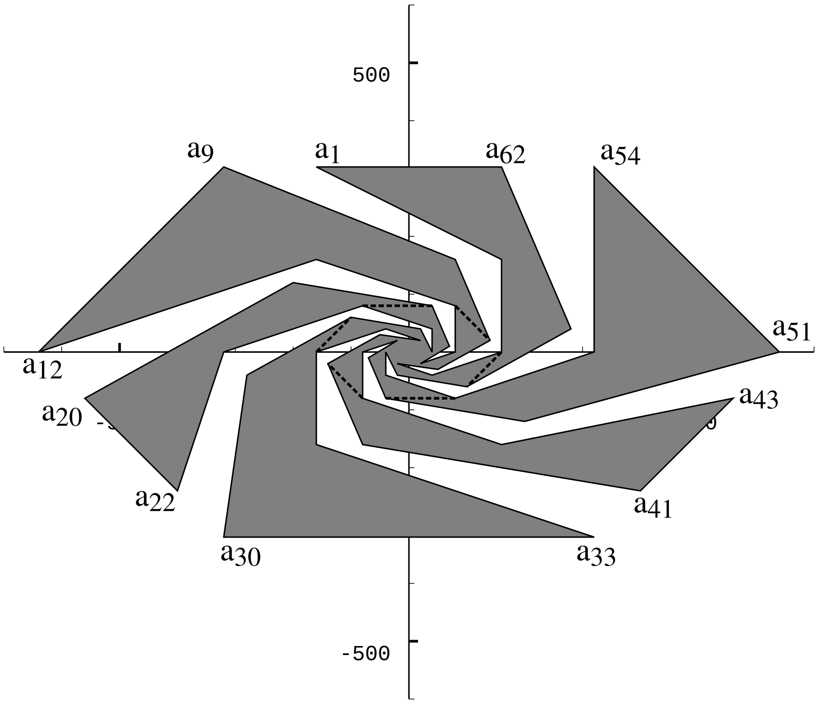

Fig. 7. Magnification of the center of the spirals in Fig. 6. The dashed lines delimit the region further magnified in Fig. 8.

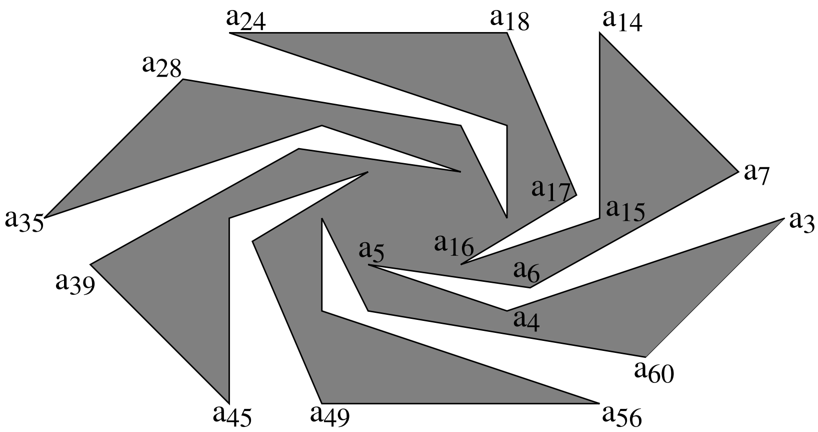

Fig. 8. Magnification of the central portion of Fig. 7.

Lemma 8 *If B_i is labeled solely with b_0 in a mating of $\mathcal{A}$ and $\mathcal{B}$, then all other bays must have an edge labeled b_0.*

Proof: Assume first that $i = 0$; this is a loss of generality. Recall that $B_0 = (a_1, \ldots, a_9)$; see Fig. 8. Each edge of B_0 casts a $!b_1$ strip shadow by Lemma 4 whose bounding rays are parallel to $b_1 b_0$, and a $!b_2$ strip shadow whose rays are parallel to $b_2 b_0$. These strips are shown lightly shaded in Fig. 9. The intersection of these strips (shown dark in the figure) is a region R within which no point of $\mathcal{A}$ may connect to b_1 or b_2: so all points in R must connect to b_0. Call such a region a b_0-region. But now notice that all six bays intersect R: all six bay tips are in R. Therefore all six must have at least one point, and therefore one edge (and many have more) connected to b_0.

The same argument works for $i = 1$, but for some i (notably $i = 5$), the b_0-region R does not intersect all bays. However, it is clear that if the bays spiraled further, for example, one more complete turn around the central hexagon, then every bay B_i would have edges above all the others (playing the role of $a_1 a_2$ in Fig. 9). And then the corresponding R would cast a b_0-shadow on portions of every other bay. Therefore for this idealized version of $\mathcal{A}$ the lemma is proved. □

It is possible to argue that the polygon as drawn satisfies the lemma as well, but the argument needs to exploit secondary effects, and becomes more complicated without adding new insights.

Lemma 9 *If every bay has an edge labeled b_0, then all three vertex transitions must occur on or between two consecutive bays B_j and B_{j+1}.*

Proof: By the cycling constraint (Lemma 6), the b_0's in the labeling sequence must form a contiguous block. Since every bay has a b_0 label, all the non-b_0's must occur somewhere after the beginning of one bay and the end of the next bay. □

In particular, the $b_1 b_2$ transition must occur on some vertex a^* on or between B_j and B_{j+1}. We concentrate on this transition vertex because it casts a $!b_0$-shadow, which we can use to obtain a contradiction. Between the two identified bays lies either an anvil A_k, or a spike S_k. In either case, these are the only anvils or spikes that may have any non-b_0 label. We make this observation a lemma for later reference:

Lemma 10 *In a mating of $\mathcal{A}$ and $\mathcal{B}$ with every bay having an edge labeled b_0, at most one anvil, or at most one spike, may have a non-b_0 label.*

□

Our contradictions will be obtained by showing that $!b_0$ shadows fall on other anvils or spikes. By Lemma 5, the $(a^*, b_1 b_2)$ connection casts a $!b_0$-shadow from a^* bounded by rays parallel to $b_0 b_1$ and $b_0 b_2$; thus it has the same shape as the $\mathcal{B}$ triangle (except of course it is unbounded).

The remainder of the proof falls into three similar cases, depending on whether a^* is on a bay, an anvil, or a spike.

1. a^* on bay B_j (or symmetrically B_{j+1}).

   Every possible location for a^* on any bay results in the $!b_0$-shadow intersecting both an anvil and a spike; but this violates Lemma 10: at most one anvil or

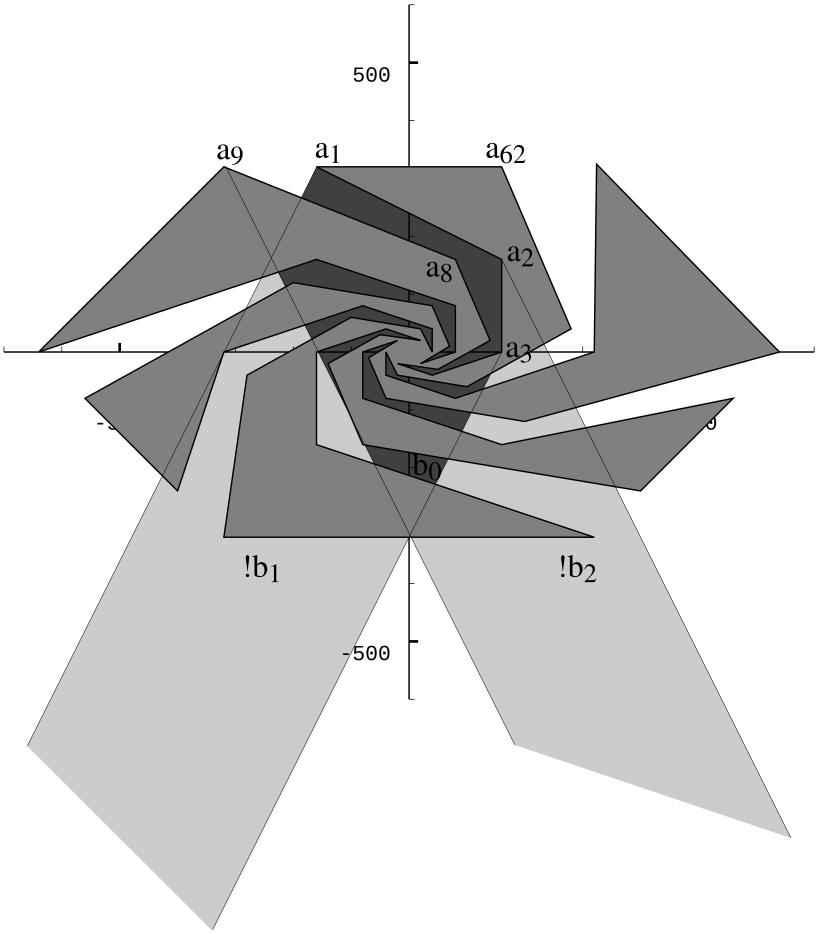

Fig. 9. If B_0 is labeled b_0, then it casts a b_0-shadow over all other bays.

spike can have a non-b_0 label. For example, if $a^* = a_9$, the last vertex of B_0, then the $!b_0$-shadow covers both S_0 and A_0, as shown in Fig. 10.

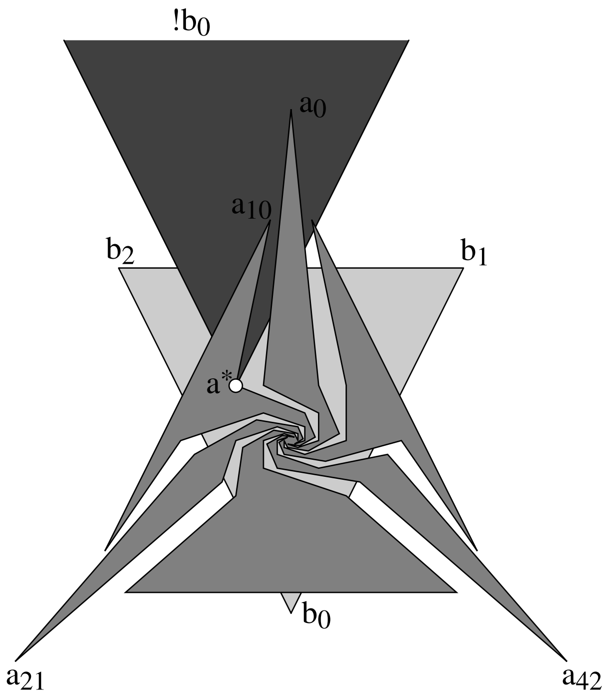

Fig. 10. $a^* = a_9$, last vertex of bay B_0.

2. a^* on anvil A_k.

   If a^* is at a vertex shared between a bay and an anvil, then we fall into Case 1. So it only remains to examine the two vertices on the flat of an anvil. But as is clear from Fig. 11, the spike tips protrude into the $!b_0$-shadow cast by $a^* = a_{11}$ and other locations. (This incidentally reveals why the spikes were arranged to extend beyond the flats of the anvils.) We obtain a contradiction again to Lemma 10.

3. a^* on spike S_k.

   This case is a bit more difficult. If a^* is at a vertex shared between a bay and a spike, then we fall into Case 1. So it only remains to examine the spike tips. The tips a_{21} and a_{42} present somewhat a different case from the tip a_0. We start with $a^* = a_0$, which is the most "natural" connection in any case.

   So assume (a_0, b_1b_2) is part of the mating connection. We now argue that a_0 cannot serve as all three transition vertices. Suppose to the contrary that both edges of A incident to a_0 are labeled b_0, so that a_0 connects to b_0b_1, b_1b_2,

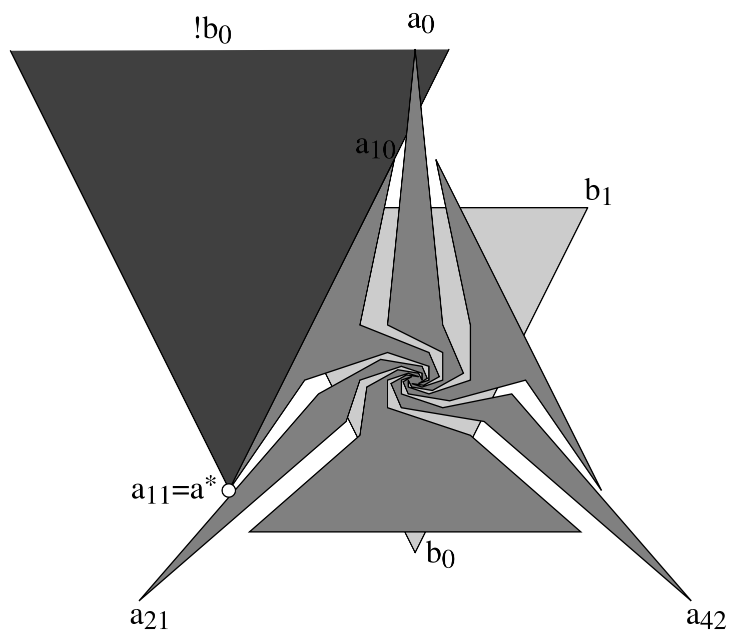

Fig. 11. $a^* = a_{11}$, vertex of anvil A_0.

and b_2b_0, and all other connections from $\mathcal{A}$ are to b_0. Then it is clear that these connections do not constitute a polyhedron, but rather a "double-cone": all of $\mathcal{B}$ connects to a_0, and all of $\mathcal{A}$ connects to b_0, and the surface is not homeomorphic to a sphere.

Therefore we may conclude that in a mating, one of the two edges incident to a_0 is not labeled b_0. Without loss of generality assume a_0a_1 connects to b_2. Then this edge casts a $!b_0$ strip shadow parallel to b_0b_2 by Lemma 4. Fig. 12 shows that this strip intersects anvil A_0, contradicting Lemma 10, for we assumed that S_0 contained non-b_0 labels.

The case when $a^* = a_{21}$ (or symmetrically, a_{42}) is argued in exactly the same fashion. Fig. 13 shows how the contradiction is reached both when the edge $a_{20}a_{21}$ connects to b_2, and when $a_{21}a_{22}$ connects to b_1: the former casts a $!b_0$-shadow parallel to b_0b_2, the latter parallel to b_0b_1. In both cases the shadows intersect an anvil, and we obtain the same contradiction to Lemma 10. (This case shows why the spike's edges were designed to penetrate towards the center of $\mathcal{A}$.)

This completes the proof of the main theorem:

Theorem 3 *The idealized versions of polygons $\mathcal{A}$ and $\mathcal{B}$ cannot mate.*

7. Computer Verification

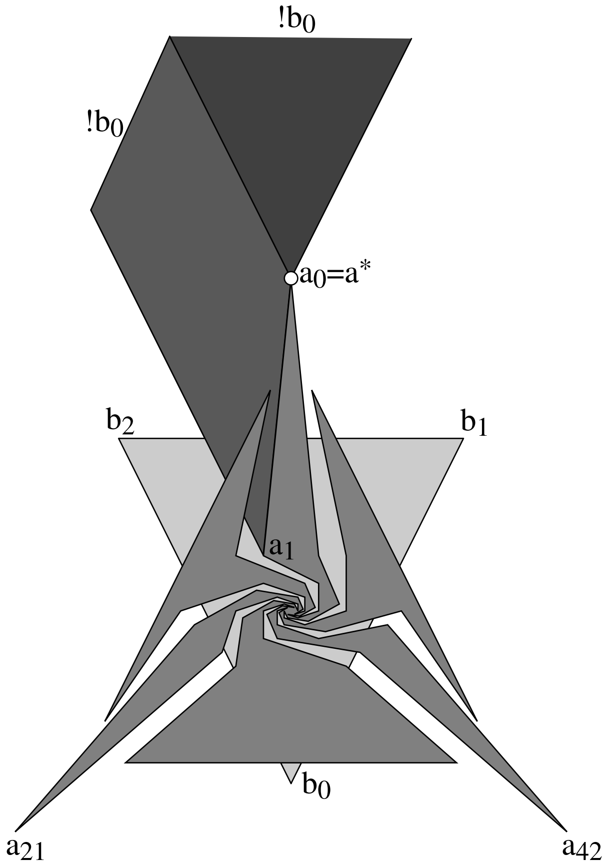

Fig. 12. $a^* = a_0$, tip of spike S_0, and a_0a_1 connects to b_2.

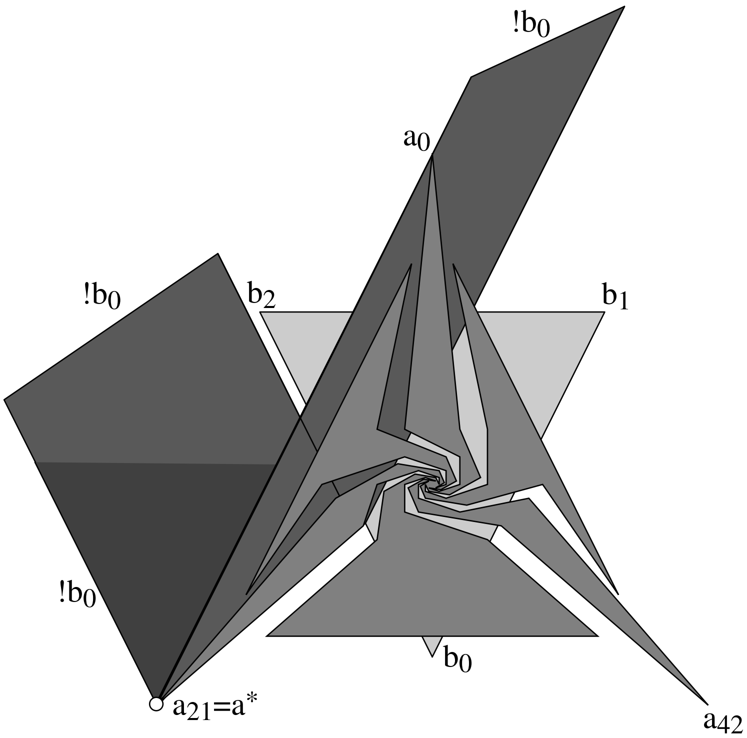

Fig. 13. $a^* = a_{21}$, tip of spike S_1, and $a_{20}a_{21}$ connects to b_2, or $a_{21}a_{22}$ connects to b_1.

As the first presentation of this proof[d] met with some skepticism, we developed a program to check all possible connections between two given polygons, and determine which connections formed polyhedra. The program exclusively uses exact integer calculations, and we were careful to avoid integer overflow.[6] The program verified that the two polygons shown in Fig. 6 are indeed unmatable.

Although it is likely that there is a polygon with fewer vertices than $\mathcal{A}$ that cannot mate with a triangle, there is a sense in which $\mathcal{A}$ is nearly minimal among those of similar design. Using our program, we found that the following modifications lead to matings:

1. If the bay spiral tips of $\mathcal{A}$ are shortened by two vertices each (e.g., a_5 and a_6 in Fig. 8) to produce $\mathcal{A}'$, then $\mathcal{A}'$ mates with $\mathcal{B}$.
2. If one bay is removed from $\mathcal{A}$ to produce $\mathcal{A}''$, then $\mathcal{A}''$ mates with $\mathcal{B}$.
3. If $\mathcal{B}$ is changed to a square, then the polygons can mate.

Fig. 14 shows a mating between $\mathcal{A}'$ and $\mathcal{B}$. One can see in the two lower figures that roughly half the bay edges connect to b_0 (lower left) and half to b_2 (lower right).

8. Open Problems

We have focussed on a very narrow question, whose solution suggests further questions:

1. What is the computational complexity of deciding whether or not two polygons can mate?
2. Are there simple sufficient conditions on the polygons that guarantee mating? In particular, are there natural classes of polygons that can always mate?
3. A *Steiner vertex* is a new vertex that is added on the boundary of a polygon, usually to serve as an anchor for incident diagonals. It is an interesting question whether two shapes can be unmatable regardless of how one sprinkles vertices about their boundary. In an earlier version of this paper[7] we erroneously claimed to prove that the addition of just one Steiner vertex (at a particular location) permits two arbitrary polygons to mate.[e] Geiger proved recently that two Steiner points suffice,[4] but the question of whether one suffices remains open. We have established with our program that one does suffice for the polygon $\mathcal{A}$ in Fig. 6.
4. Dobkin and Suri asked for matings such that the degrees of the vertices in the polyhedron are "balanced" in some sense.

[d]By the second author, at the October 1992 SUNY Stony Brook Computational Geometry Workshop.

[e]We thank a referee for uncovering this error.

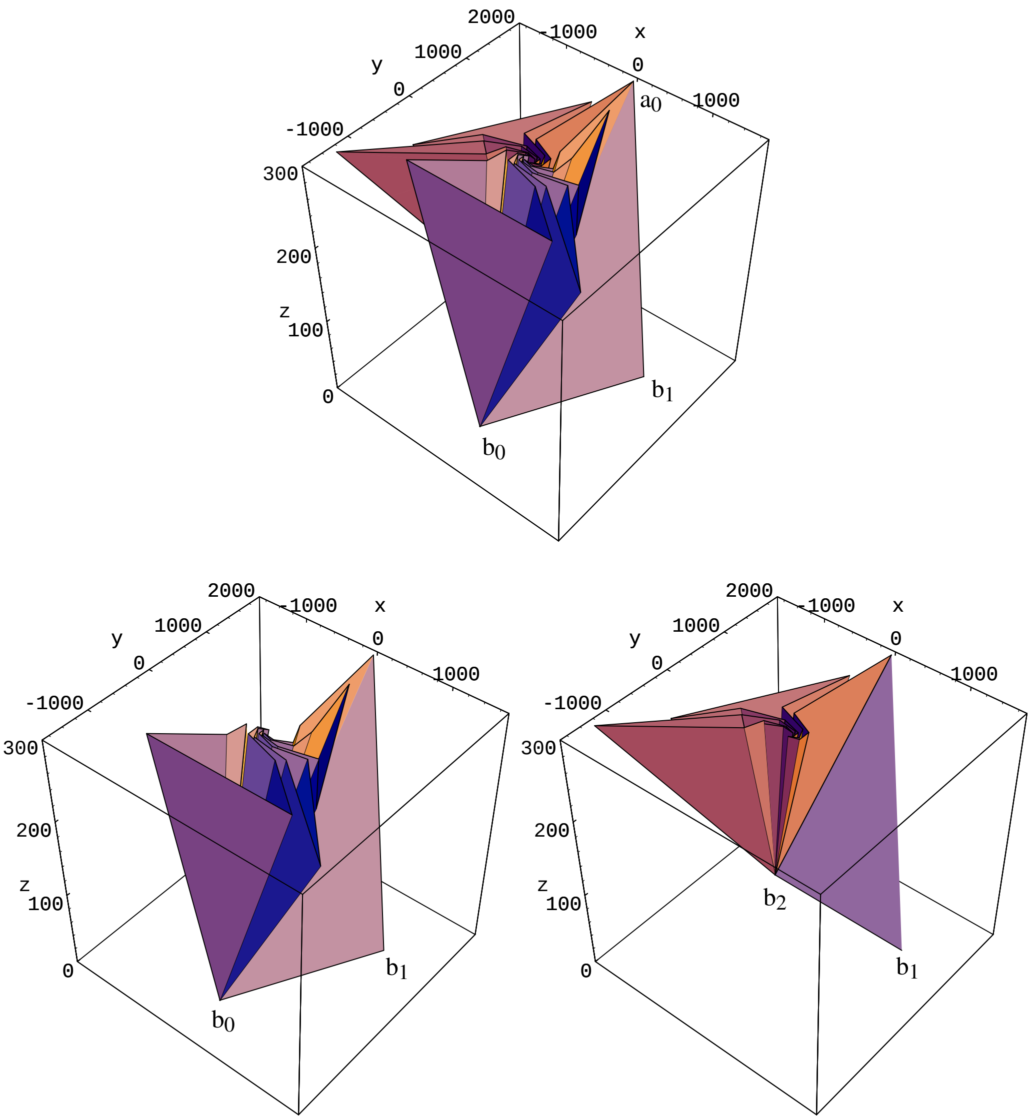

Fig. 14. $\mathcal{A}'$, with each bay's spiral shortened by two vertices, can mate with $\mathcal{B}$. Upper: all connections; lower left: all connections incident to b_0; lower right: all connections incident to b_2.

Acknowledgements. We are grateful for an extraordinarily perceptive referee report that improved the paper significantly, including finding an error in a proof and providing a key reference.

9. Appendix

The coordinates of polygon $\mathcal{A}$ from Fig. 6 are listed in the table below. $\mathcal{B}$ has coordinates $(-1000, 0)$ and $(\pm 1000, 1000)$.

i	(x,y)	i	(x,y)	i	(x,y)
0	(0,1920)	21	(-1600,-1280)	42	(1600,-1280)
1	(-160,320)	22	(-400,-240)	43	(560,-80)
2	(160,160)	23	(-320,0)	44	(160,-160)
3	(160,0)	24	(-80,80)	45	(-80,-80)
4	(40,-40)	25	(40,40)	46	(-80,0)
5	(-20,-20)	26	(40,0)	47	(-20,20)
6	(50,-30)	27	(20,40)	48	(-70,-10)
7	(140,20)	28	(-100,60)	49	(-40,-80)
8	(80,160)	29	(-280,-40)	50	(200,-120)
9	(-320,320)	30	(-320,-320)	51	(640,0)
10	(-120,1280)	31	(-960,-880)	52	(1080,-640)
11	(-1080,-640)	32	(960,-880)	53	(120,1280)
12	(-640,0)	33	(320,-320)	54	(320,320)
13	(-160,160)	34	(-160,-160)	55	(320,0)
14	(80,80)	35	(-160,0)	56	(80,-80)
15	(80,0)	36	(-40,40)	57	(-40,-40)
16	(20,-20)	37	(20,20)	58	(-40,0)
17	(70,10)	38	(-50,30)	59	(-20,-40)
18	(40,80)	39	(-140,-20)	60	(100,-60)
19	(-200,120)	40	(-80,-160)	61	(280,40)
20	(-560,-80)	41	(400,-240)	62	(160,320)

1. J.-D. Boissonnat. Shape reconstruction from planar cross-sections. *Comput. Vision Graph. Image Process.*, 44(1):1–29, Oct 1988.
2. L.T. Cook, P.N. Cook, K.R. Lee, S. Batnitzky, B.Y.S. Wong, S.L. Fritz, J. Ophir, S.J. Dwyer III, L.R. Bigongiari, and A.W. Templeton. An algorithm for volume estimation based on polyhedral approximation. *IEEE Trans. on Biomed. Eng.*, 27(9), September 1980.
3. H. Fuchs, Z.M. Kedem, and S.P. Uselton. Optimal surface reconstruction from planar contours. *Commun. of the ACM*, 20(10):693–702, October 1977.
4. B. Geiger. Three-dimensional modeling of human organs and its application to diagnosis and surgical planning. Thèse de Doctorat en Sciences, Ecole Nationale Supérieure des Mines de Paris, France, 1993.
5. P.J. Giblin. *Graphs, Surfaces and Homology.* Chapman and Hall, 1977.
6. C. Gitlin. Verifying reconstructions of polyhedra from slices. Senior Honors Thesis, Dept. of Comput. Sci., Smith College, 1993.
7. C. Gitlin, J. O'Rourke, and V. Subramanian. On reconstructing polyhedra from parallel slices. Technical Report 025, Dept. Comput. Sci., Smith College, Northampton, MA, March 1993.

8. L.J. Guibas and F.F. Yao. On translating a set of rectangles. *Commun. of the ACM*, pages 154–160, April 1980.

9. E. Keppel. Approximating complex surfaces by triangulation of contour lines. *IBM J. Res. Develop.*, 19:2–11, January 1975.

10. K.R. Sloan, Jr. and L.M. Hrechanyk. Surface reconstruction from sparse data. In *Proceedings of the IEEE Computer Society Conference on Pattern Recognition and Image Processing*, pages 45–48, Dallas, TX, August 1981.

11. K.R. Sloan, Jr. and J. Painter. From contours to surfaces: Testbed and initial results. In *Proceedings: CHI+GI '87*, pages 115–120, Toronto, Canada, April 1987.

12. D. Meyers, S. Skinner, and K.R. Sloan, Jr. Surfaces from contours. *ACM Trans. Graph.*, 11(3):228–258, July 1992.

13. J. O'Rourke. Polyhedra of minimal area as 3d object models. In *Proceedings of Seventh International Joint Conference on Artificial Intelligence*, pages 664–666, Vancouver British Columbia, August 1981.

14. J. O'Rourke and V. Subramanian. On reconstructing polyhedra from parallel slices. Technical Report 008, Dept. Comput. Sci., Smith College, Northampton, MA, June 1991.

15. V. Subramanian. Reconstruction of polyhedra from slices. Senior Honors Thesis, Dept. of Comput. Sci., Smith College, 1991.

16. Y.F. Wang and J.K. Aggarwal. Surface reconstruction and representation of 3-d scenes. *Pattern Recogn.*, 19(3):197–207, 1986.