A Pumping Lemma for Homometric Rhythms

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Abstract

Homometric rhythms (chords) are those with the same histogram or multiset of intervals (distances). The purpose of this note is threefold. First, to point out the potential importance of *isospectral vertices* in a pair of homometric rhythms. Second, to establish a method ("pumping") for generating an infinite sequence of homometric rhythms that include isospectral vertices. And finally, to introduce the notion of *polyphonic homometric rhythms*, which apparently have not been previously explored.

1 Introduction

Both chords of k notes on a scale of n pitches, and rhythms of k onsets repeated every n metronomic pulses, are conveniently represented by n evenly spaced points on a circle, with arithmetic mod n, i.e., in the group $\mathbb{Z}_n$. This representation dates back to the 13th-century Persian musicologist Safi Al-Din [Wri78], and continues to be the basis of analyzing music through geometry [Tou05] [Tym06]. Such sets of points on a circle are called *cyclotomic* sets in the crystallography literature [Pat44] [Bue78]. It is well-established that in the context of musical scales and chords, the intervals between the notes largely determine the aural tone of the chord. An *interval* is the shortest distance between two points, measured in either direction on the circle. This has led to an intense study of the *interval content* [Lew59]: the histogram that records, for each possible interpoint distance in a chord, the number of times it occurs. This same histogram is studied for rhythms [Tou05] and in crystallography.

Of special interest are pairs of noncongruent chords/rhythms/cyclotomic sets that have the same histogram: the sets are *homometric* in the terminology of Lindo Patterson [Pat44], who first discovered them. In crystallography, such sets yield the same X-ray pattern. In the pitch model, they are chords with the same interval content. One of the fundamental theorems in this area is the so-called *hexachordal theorem*, which states that two non-congruent complementary sets with $k{=}n/2$ (and n even) are homometric, whose earliest proof in the music literature is due to Milton Babbitt and David Lewin [Lew59].

Henceforth we specialize to the rhythm model, with each (n, k)-rhythm specified by k beats and $n{-}k$ rests on the $\mathbb{Z}_n$ circle; and we specifically focus on the structure of homometric rhythms. Figure 1 shows a pair of homometric rhythms with $(n, k){=}(12, 5)$.

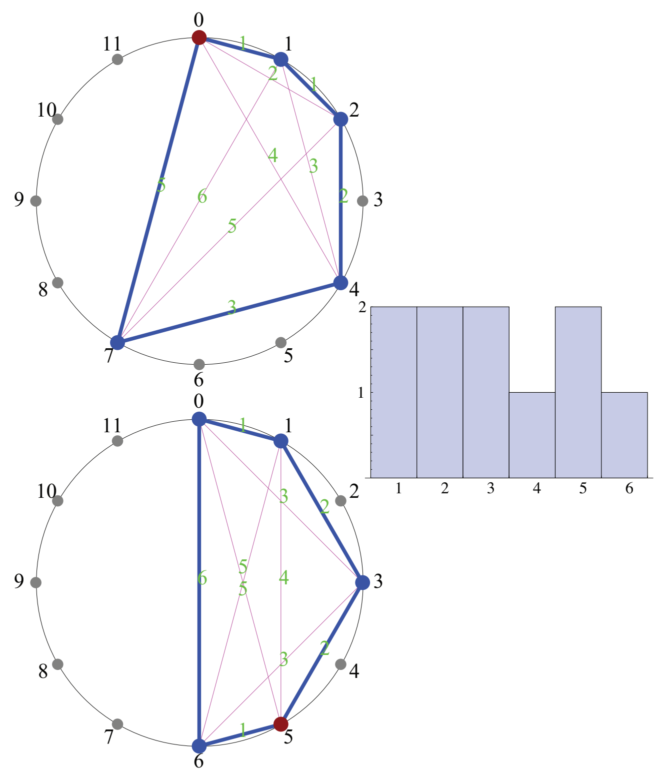

Figure 1: Homometric $(n, k){=}(12, 5)$ rhythms: $(0, 1, 2, 4, 7)$ and $(0, 1, 3, 5, 6)$. Vertices 0 and 5 in the first and second rhythms (respectively) are isospectral.

2 Isospectral Vertices

Let P and Q be two different rhythms, with $p \in P$ and $q \in Q$ vertices (onsets) in each. The vertices p and q are called *isospectral*[1] if they have the same histogram of distances to all other vertices in their respective rhythms. In Figure 1, vertex 0 in the first rhythm,

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[1]The term is used in the literature on Golomb rulers.

and 5 in the second, are isospectral, with spectrum $\{1, 2, 4, 5\}$.

There are two reasons we consider isospectral vertices of potential significance. The first is that removal of a pair of isospectral vertices from a pair of (n, k) homometric rhythms leaves a homometric pair of $(n, k-1)$ rhythms. This raises the possibility of *shelling*: removing a particular onset from a rhythm while retaining a certain property. Shellings of Erdős-deep rhythms are studied in [DGMM+08]. Here we want to perform shelling by removing an onset from each rhythm while keeping the pair homometric.

Shellings of rhythms play an important role in musical improvisation. For example, most African drumming music consists of rhythms operating on three different strata: the unvarying timeline usually provided by one or more bells, one or more rhythmic motifs played on drums, and an improvised solo (played by the lead drummer) riding on the other rhythmic structures. Shellings of rhythms are relevant to the improvisation of solo drumming in the context of such a rhythmic background. The solo improvisation must respect the style and feeling of the piece, which is usually determined by the timeline. A common technique to achieve this effect is to "borrow" notes from the timeline, and to alternate between playing subsets of notes from the timeline and from other rhythms that interlock with it [Ank97][Aga86]. The borrowing of notes from the timeline may be regarded as a fulfillment of the requirements of style coherence, and shellings can be viewed as capturing a particular type of borrowing that achieves coherence through homometricity.

Second, as we show in Lemma 1 below, the presence of an isospectral pair permits "pumping" the rhythms to homometric pairs based on a larger $n' > n$. So isospectral pairs serve as a natural "pivot" from which to generate new homometric pairs from old ones both by removing or adding onsets.

This naturally raises the question of whether every homometric pair of rhythms must contain an isospectral pair of vertices. The answer is NO, as illustrated in Figure 2. We leave further investigation of isospectral vertices and shellings to future work.

3 The Pumping Lemma

Let P and Q be a homometric pair of (n, k)-rhythms on $\mathbb{Z}_n$, with isospectral vertices $p \in P$ and $q \in Q$. We define an (m, r)-*pumping* of P and Q, $m \geq 1$, $r \geq 0$, to be a new pair of (n', k') rhythms P' and Q' on $\mathbb{Z}_{n'}$, with $n' = mn$ and $k' = k + 2r$, obtained by replacing p in P' with $p + \{0, \pm 1, \pm 2, \ldots, \pm r\}$, and similarly replacing q in Q' with $q + \{0, \pm 1, \pm 2, \ldots, \pm r\}$.

Figure 3 shows a (m, r)=$(3, 2)$-pumping of the homometric pair from Figure 1 based on the isospectral pair p=0 and q=5. The original (n, k)=$(12, 5)$ rhythms have been pumped to (n', k')=$(36, 9)$ rhythms. The "pumping" occurs both in $n \rightarrow mn$ and in $k \rightarrow k+2r$, although it may be that m=1 in which case n'=n, or r=0 in which case k'=k.

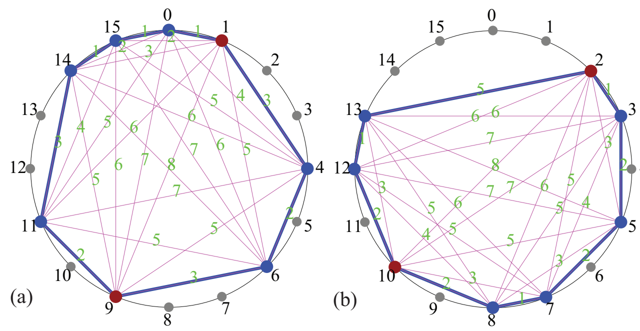

Figure 2: A pair of (n, k)=$(16, 9)$ homometric rhythms that has no isospectral pair of vertices, but does have a pair of two-vertex sets that is isospectral, $\{1, 9\}$ in (a) and $\{2, 10\}$ in (b).

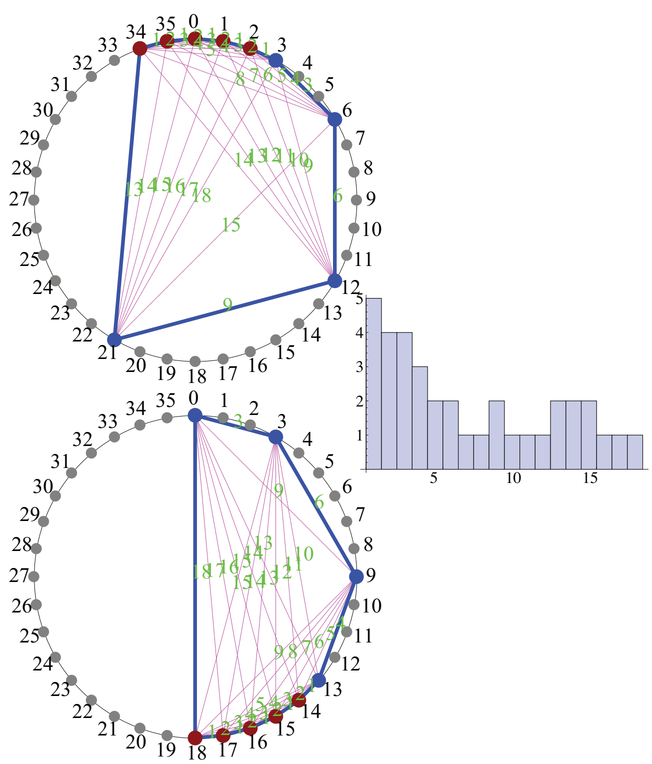

Figure 3: Pumping p=0 and q=5 in Fig. 1 with m=3, r=2, n'=mn=36. The rhythm is monophonic.

The literature focuses on *monophonic* rhythms, those whose vertices form a set with no repeated elements. The pumping lemma can produce *polyphonic* rhythms, ones in which at least one vertex has multiplicity greater than 1, i.e., the onsets form a multiset. These will be discussed further in Section 4.

Lemma 1 (Pumping) *Let P and Q be a homometric pair of (n,k) monophonic rhythms, with isospectral vertices $p \in P$ and $q \in Q$. Then any (m,r)-pumping of P and Q creates a new homometric pair P' and Q', also containing an isospectral pair. If $m \geq r+1$, then the new rhythms are monophonic; if $m \leq r$, the new rhythms could be polyphonic.*

Proof. Call the vertices $p + \{0, \pm 1, \pm 2, \ldots, \pm r\}$ in P'

$$p'_{-r}, \ldots, p'_{-2}, p'_{-1}, p'_0, p'_1, p'_2, \ldots, p'_r$$

and similarly for the q replacements in Q'.

To prove that P' and Q' are homometric, let (x', y') be a segment between two vertices of P'. Consider three cases.

1. Neither x' nor y' is among the p'_i. Then $d(x', y') = md(x, y)$, where x and y are the corresponding vertices in P.

2. $y' = p'_i$. Here there are two subcases. Let $d(x, y) = d(x, p) = d$. Note that the diameter of the circle $\mathbb{Z}_n$ is $n/2$.

   (a) $d = n/2$; or $r \leq n/2 - d$. (Figure 4(a)). Consider the latter inequality. It means that $d \pm r$ does not extend beyond the diameter $n/2$, so that the $p'_{\pm i}$ points and x' all fit inside a semicircle, as in (a) of the figure. Then $d(x', p'_{\pm i}) = md \pm i$ or $md \mp i$, depending on whether the path $x \to p$ or $p \to x$ is shorter, respectively. So, what was the distance d in P between x and y=p becomes the distance set $\{md - r, \ldots, md - 1, md, md + 1, \ldots, md + r\}$ in P'. If d is the diameter $n/2$, then the distance set is $\{md, md-1, md-1, md-2, md-2, ..., md-r, md-r\}$.

   (b) $r > n/2 - d$ (Figure 4(b)). Here $d \pm r$ does extend beyond the diameter $n/2$, at which point its increase or decrease reverses direction. In (b) of the figure, the new distance set is $\{md-2, md-1, md, md+1, md\}$.

3. Both x' and y' are among the p'_i. Here we get a clique of new distances among the p'_i.

In any of these three cases, call the new distance set $D = \{d(x', p'_{\pm i}) \; : \; i = 0, \ldots, r\}$.

So now we see, in the $P \to P'$ transition, either the change $d \to md$ or $d \to D$. But we see exactly the same distance changes in the $Q \to Q'$ transition. For the distances not involving q are stretched by m, and the distances involving q'_i get stretched by $m \pm i$, $i = 0, \ldots, r$. Because P and Q are homometric, all the former changes are identical between them, and because p and q are isospectral, all the latter changes are identical between them. Even in the case where the inflation of (x, p) crosses a diameter in the $P \to P'$ inflation, there is a point $z \in Q$ that achieves the same distance $d(x, p) = d(z, q)$ (because p and q are isospectral), so the crossing-diameter behavior, and the distance set D, is exactly mirrored in the $Q \to Q'$ transition.[2] Therefore, P' and Q' are homometric.

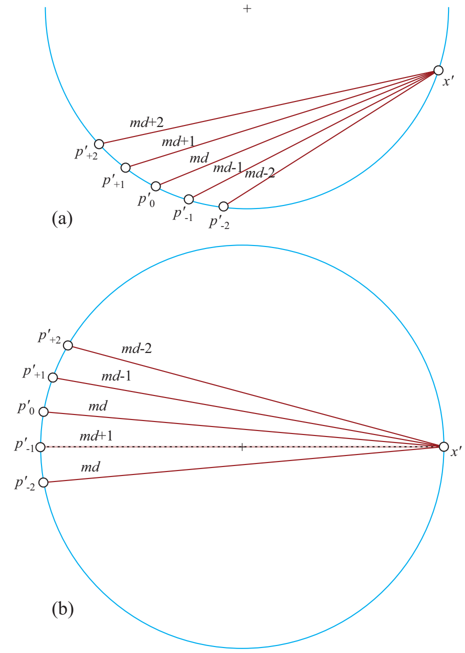

Figure 4: (a) The inflation fits inside a semicircle: r=2, new distances $\{md - r, \ldots, md - 1, md, md + 1, \ldots, md + r\}$. (b) The inflation crosses a diameter, here at (x', p'_{-1}).

The counterparts of p and q, p'_0 and q'_0, are isospectral in P' and Q', because their distance spectra are simply scaled by m (most clearly seen in Figure 3).

We turn now to the mono- and polyphonic claims of the lemma. It should be clear that if we inflate by $m \geq r+1$, then the closest vertices, separated by 1 in P, become separated by $\geq r+1$ in P', which is enough to accommodate the addition of r new vertices to each side of p. (If the closest vertices are separated by more than 1, then even smaller inflation will avoid overlap.) Continuing our example, inflation by m=r+1=3 suffices to avoid overlap and so maintain a monophonic rhythm, as illustrated in Figure 3.

When $m \leq r$, there could be overlap of the newly added vertices on top of the old vertices. So the resulting rhythm may be polyphonic. However, the rhythms are still homometric, where we treat vertices with multiplicity more than 1 as if they were distinct vertices (and distance 0 is ignored). This is illustrated in Figure 5,

[2] An instance of this behavior is illustrated in Figure 5 below, where the set D for segments $(7,0) \in P$ and $(0,5) \in Q$ have inflated distance set $D = \{3, 4, 5, 6, 5\}$.

where we have used m=1, i.e., n'=n. $\square$

The inspiration for the transformation described in this lemma is Property 7 in [AG00], which similarly inflates a particular pair of homometric quadrilaterals by replacing a vertex in each by a sequence of vertices, and increasing n to accommodate. However, their inflation does not rely on isospectral vertices, and appears to only work on that specific quadrilateral pair.

Corollary 2 *From any pair of rhythms satisfying the preconditions of the pumping lemma, we can generate an infinite sequence of increasingly larger homometric pairs.*

Proof. Because (P', Q') again contain an isospectral pair, a pumped pair can be pumped again. $\square$

Given the preconditions of the pumping lemma, it would be useful to characterize the homometric pairs of rhythms that contain an isospectral pair of vertices.

4 Polyphonic Rhythms

As mentioned in the proof above, if we do not pump n enough to accommodate the pumping of k without overlap, i.e., when $m \leq r$, an (m, r)-pumping may convert a monophonic rhythm to a polyphonic rhythm. In general, vertices of a rhythm have integer weights representing their multiplicity. In Figure 5, two vertices have weight 2 whereas all others have weight 1. The interval histogram still makes sense by treating a vertex of weight w as w-distinct colocated vertices. For example, in the histogram of the first rhythm, the distance 6 is achieved three times: by $(10, 4)$ and twice by $(7, 1)$ because vertex 1 has weight 2. And the pumping lemma still guarantees homometricity.

One could interpret onsets of weight greater than 1 as representing greater emphasis, or several drums with different timbre, or several voices sounded in unison in the pitch model, each an octave apart from the others (an elementary form of harmony). Homometricity in polyphonic rhythms is an apparently unexplored topic, which we believe opens new directions for research in music theory. For example, we have established that the hexachordal theorem extends to polyphonic rhythms (and beyond) [BBGM+08]. Shellings of polyphonic rhythms are also a natural topic of investigation.

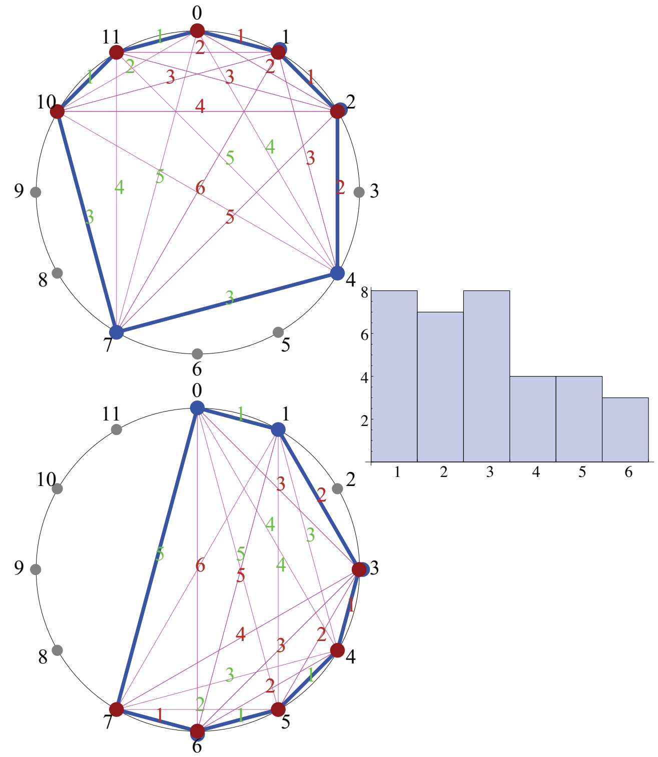

Figure 5: Fig. 1 pumped with m=1, r=2, n'=n=12. The rhythm is polyphonic: $\{1, 2\}$ in the first rhythm, and $\{3, 6\}$, in the second, have multiplicity 2.

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