Open Problems in the Combinatorics of Visibility and Illumination

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Abstract. A compendium of thirty-eight open problems is presented: on point and edge visibility, floodlight illumination, and visibility graphs.

The "art gallery theorem," that $\lfloor n/3 \rfloor$ guards suffice and are sometimes necessary to visually cover a polygon of n vertices [**Chv**] [**O**], typifies the interplay between geometry and combinatorics in a wide variety of subsequent results. The open problems in this collection are intellectual kins to these art gallery-like results. The problems represent a personal selection from the frontier of research on the combinatorics of visibility; no attempt is made to be comprehensive. Updates will be maintained on the Web.[1]

In general, a point x can *see*, is *visible* to, or *illuminates* a point y if the line segment xy is not obstructed. (The metaphors of "visibility" and "illumination" are used interchangeably.) Often this means that xy cannot contain any point of a set of obstacles, but sometimes xy is permitted to contain boundary points of the obstacles (line-of-sight grazing contact). Other nuances to the definition of visibility are appropriate for certain problems, and will be detailed below.

Throughout γ will be used to denote the constant of proportionality for quantities of the form $\gamma n + c$ for some constant c, where n is the primary input variable (e.g., the number of vertices). Precise fractional bounds will be supplemented by their approximate decimal equivalents.

1. Point and edge visibility problems

1.1. Art gallery theorems. Most these problems concern simple polygons. Grazing contact is permitted, so a guard (i.e., a point) g sees an interior point x if gx is nowhere exterior, and sees an exterior point y if gy is nowhere interior. A *vertex guard* is a point located at a vertex of the polygon. An *edge guard* is an edge e of the polygon, which sees a point x if there is a point $y \in e$ such that xy is nowhere exterior. Edge guard problems can be viewed as illumination by fluorescent lightbulbs.

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1. Orthogonal Prison Yard. How many vertex guards are needed to cover both the interior and the exterior of a simple orthogonal polygon? An *orthogonal polygon* is one whose edges meet at right angles. The best lower and upper bounds are $\lceil (5n-10)/16 \rceil$ and $\lfloor 5n/12 \rfloor + 2$ [**HK**]. ($\gamma \in [0.31, 0.42]$.)
2. Edge Guard Problem. How many edge guards are needed to cover the interior of a simple polygon? The best lower and upper bounds (for $n > 11$) are $\lfloor n/4 \rfloor$ and $\lfloor 3n/10 \rfloor$ [**S2**]. ($\gamma \in [0.25, 0.30]$.)
3. How many edge guards are needed to cover the interior of a star-shaped polygon? Toussaint showed that $\lfloor n/5 \rfloor$ are necessary [**O**]. ($\gamma \in [0.2, 0.3]$.)
4. Prove or disprove that $\lfloor n/4 \rfloor$ edge guards suffice to cover a star-shaped polygon of n vertices [**O2**].
5. The problem of finding the minimum number of guards needed to cover a given simple polygon is NP-complete [**LL**] [**O**]. Is there a polynomial approximation algorithm with some guarantee of performance? Since set cover can be approximated within a $O(\log n)$ factor, improving upon $O(\log n) \cdot \textsc{opt}$ could be a goal.
6. In a forest of a finite number of disjoint circular trees whose boundaries are perfect mirrors, it is true that some light rays from a lit match always escape to infinity? [**St**]. The trees are disks which reflect light at its incident angle.
7. How many edge guards are needed to cover an n-vertex polyhedral terrain? A *polyhedral terrain* T is a polyhedral surface that meets every vertical line in at most one point. Point x sees y if the segment xy is nowhere strictly below T. The best lower and upper bounds are $\lfloor (4n-4)/13 \rfloor$ [**BSTZ**] and $\lfloor n/3 \rfloor$ [**ERC**]. ($\gamma \in [0.31, 0.33]$.)

1.2. Exterior boundary illumination. These problems ask for the fewest exterior point light sources to illuminate the boundary of a collection of objects. Rectangles are *isothetic* if their sides are parallel to coordinate axes.

8. How many points are need to illuminate n disjoint isothetic rectangles in the plane? Fejes Tóth proved that $4n-7$ points suffice to illuminate $n \geq 4$ disjoint compact convex sets in the plane [**Fej**], but restricted cases often require fewer lights. For rectangles, examples are known that require $n-1$ lights, and $\lfloor (4n+4)/3 \rfloor$ lights suffice [**CRCU**]. ($\gamma \in [1.00, 1.33]$.) Here visibility excludes grazing contact.
9. How many points are need to illuminate n disjoint triangles in the plane? The bounds known here are exactly the same as for rectangles, for different reasons [**CRCU**].
10. How many points are needed to illuminate n disjoint segments in the plane? Examples are known that require $\lceil 4n/9 \rceil$ lights [**Z**], and for $n \geq 11$, it has been proven that $\lceil 2n/3 \rceil - 3$ suffice [**CRCUZ**]. ($\gamma \in [0.44, 0.67]$; it has been conjectured that $\frac{1}{2}$ is the right value.)
11. Hadwiger-Levi Covering. What is the fewest number of exterior points sufficient to illuminate any closed convex body in three dimensions? The conjecture is that 8 (in general, 2^d) points suffice, but this is only proven for 3-polytopes with some affine symmetry [**Bez**]. Illumination here requires the lines of sight to penetrate to the interior.

2. Floodlights

A *floodlight* of angle α (an α-*floodlight*) is a light that projects from a point in a (contiguous) cone of angle at most α. A floodlight placed in a simple polygon with its apex at a vertex is called a *vertex light*; at most one light may be placed at a vertex.

12. Stage Illumination. Given a line segment (the stage) and a set of n floodlights of fixed apexes and apertures, find a sub-exponential algorithm to decide if the floodlights can be rotated to illuminate the stage [**BGL**], or prove intractability. It is only known that the problem is decidable via an $c^{O(n)}$ algorithm [**Roy**].
13. Do five vertex $\pi/5$-floodlights suffice to illuminate every convex polygon of $n > 5$ vertices? It is known that three $\pi/3$-, or four $\pi/4$-floodlights suffice [**OSS**].
14. For what value of α is it the case that any set of k vertex floodlights, whose total angle sum is at least α, suffices to cover any convex polygon of $n \geq k$ vertices? It is known that $\alpha = 2\pi$ does suffice, as such floodlights can cover the entire plane [**BGL**], and that $\alpha = \pi$ does not suffice [**OSS**].
15. Find a subquadratic algorithm for finding an optimal pair of floodlights to illuminate a convex polygon, i.e., floodlights whose angle sum is as small as possible. An $O(n^2)$ algorithm is available [**ECU**].
16. Are the apexes of an optimal set of k floodlights that illuminate a convex polygon always at vertices of the polygon? Open for $k \geq 3$ [**ECU**].
17. How many π-floodlights (apexes at distinct non-exterior points) are sufficient to cover any polygon of n vertices? It is known that $\lceil \frac{2}{5}(n-3) \rceil$ suffice [**CT**], but it has been conjectured that $\lfloor n/3 \rfloor$ suffice [**U**].
18. A similar question may be asked for vertex π-floodlights: how many suffice to cover any polygon of n vertices? Here it is known that $\lfloor 3n/5 \rfloor$ are sometimes needed [**U**], and that $n-2$ suffice; but no $\gamma < 1$ is known to suffice except for special shapes [**O3**]. ($\gamma \in [0.6, 1.0]$.) No angle $\alpha < \pi$ suffices, even with one light per vertex [**ECOUX**].

3. Visibility graphs

3.1. Polygon vertex visibility graphs. Here visibility permits boundary contact: x sees y if the segment xy is nowhere exterior to the polygon. The *polygon vertex visibility graph* of a simple polygon P has a node for each vertex of P, with an arc between two nodes if the corresponding vertices are visible to one another. Note the definition of visibility implies that every edge of P is included as an arc of the visibility graph (and therefore the graph is Hamiltonian). For problems where the graph is the input, usually it is assumed given labeled with vertex indices in the order of a counterclockwise boundary traversal.

19. How many different labeled vertex visibility graphs are there of polygons of n vertices? The number is in $\Omega(n^{(2-\epsilon)n})$ and $O(n^{3n})$ [**HN**].
20. Develop an algorithm to recognize whether a polygon vertex visibility graph is planar. Necessary and sufficient conditions are known [**LC**], but they seem not to lead to a polynomial algorithm.

21. Do general-position polygons realize all visibility graphs? I.e., is there a polygon vertex visibility graph that can only be realized with collinear vertices? There are polygon vertex-edge visibility graphs that do require collinearities [**OS2**].
22. Do Ghosh's four necessary conditions [**Gho**] suffice to characterize polygon vertex visibility graphs?

3.2. Recognition complexity.

23. Is recognition of polygon vertex visibility graphs in NP? It is only known to be in $PSPACE$ [**Eve**].
24. Can any polygon vertex visibility graph be realized by a polygon drawn on an exponential-size grid? If so, visibility graph recognition is in P [**LS**].
25. Is recognition of polygon vertex pseudo-visibility graphs in P? It is only known to be in NP [**OS**]. A *pseudo-visibility* graph is based on a pseudoline arrangement determined by the vertices, where lines of sight are only along the pseudolines.
26. What is the computational complexity of computing the chromatic number of a polygon vertex visibility graph G? Finding a maximum independent set or a minimum vertex cover are both NP-complete problems when restricted to visibility graphs [**LS**]. Of course $\chi(G) \geq 3$ because G can be triangulated.

3.3. Segment visibility graphs. The most-studied version of segment visibility graph is the *endpoint visibility graph* G: for a set S of closed, disjoint line segments, G has a node for each segment endpoint, and an arc between two nodes x and y if they can see each other in the sense that $[x, y] \cap S = \{x, y\}$, or $[x, y]$: the intersection is either just the two endpoints, or the entire closed segment. Note that here visibility is blocked by grazing contact with a segment, but that G contains an arc corresponding to each segment in S.

27. Does the endpoint visibility graph for a set of disjoint non-collinear segments in the plane always contain a Hamiltonian cycle [**Mir**]?
28. Does the endpoint visibility graph for a set of shellable disjoint non-collinear segments in the plane always contain a Hamiltonian cycle [**OR**]? A set of segments is *shellable* when it is possible to order the segments $s_1, s_2, \ldots$ such that s_i lies in the exterior of the convex hull of $\{s_1, s_2, \ldots, s_{i-1}\}$.
29. What is the complexity of recognizing whether a set of disjoint non-collinear segments in the plane admits a simple circuit? A *simple circuit* is a simple (non-crossing) Hamiltonian circuit in the segment endpoint visibility graph that includes every segment. This is known to be an NP-complete problem when the segments are non necessarily disjoint, and therefore may form polygonal chains [**R**].

3.4. Rectangle visibility graphs. There are several varieties of "rectangle visibility graphs," sometimes known as RVG's. G is a *rectangle visibility graph* if it can be realized by closed isothetic rectangles in the plane, with pairwise disjoint interiors, with nodes corresponding to rectangles, and arcs corresponding to visibility, under the following definition of visibility: two rectangles see one another iff there is either a horizontal or a vertical unobstructed "beam" of visibility of finite width connecting them. The "beam"-visibility requirement is often called "ϵ-visibility" in the literature. Variations are obtained by permitting this beam to be a segment, or demanding that the rectangles be entirely disjoint, or that they have no collinear sides.

30. Is every rectangle visibility graph a doubly-linear graph [**HSV**]? A graph is *doubly-linear* if it can be realized in the plane as the union of two plane graphs drawn with straight lines. (All rectangle visibility graphs are the union of two planar graphs, for the horizontal and for the vertical visibility edges.)
31. Is every graph with aboricity two a rectangle visibility graph [**BDHS**]? The *aboricity* of a graph G is the minimum k such that G can be decomposed into k forests.
32. Is every maximum-degree 4 graph a collinear rectangle visibility graph [**BDHS**]? A rectangle representation is called *noncollinear* if no line contains two distinct rectangle sides. It is *collinear* if collinearities are permitted. $K_{4,4}$ is not a noncollinear rectangle visibility graph, although it is a collinear one [**DH**].

3.5. Visibility graphs in $2\frac{1}{2}$D and 3D. Here it is reasonable to extend the ϵ-visibility definition to require a finite-volume beam of visibility for an arc of the graph.

33. What is the largest complete graph realizable by disjoint isothetic rectangles in $2\frac{1}{2}$D with vertical visibility (all rectangles are parallel to the xy-plane)? It is known that K_{22} is realizable and K_{56} is not [**BEH**] [**FHW**].
34. What is the largest complete graph that is a box visibility graph? Such a graph is realized by disjoint isothetic boxes in three-dimensions with lines of sight parallel to a coordinate axis. It is known that K_{42} is realizable and $K_{10^{75}}$ is not [**BJMO**].
35. Is it true that in every collection of isothetic boxes in three dimensions, there is one whose removal reduces the number of edges in the associated box visibility graph? This property holds for rectangles in the plane [**JMO**].

3.6. Point visibility graphs. A *point visibility graph* [**S**] for a simple polygon P is a "continuous" graph with an uncountable number of nodes, one for each point in P, with an arc between nodes x and y iff the segment xy is nowhere exterior.

36. Find an algorithm to decide whether or not two polygons have isomorphic point visibility graphs. An $O(n^2)$ algorithm to detect such isomorphism between spiral polygons is known [**MS**].
37. Is it true that if two polygons have isomorphic point visibility graphs, then there is an isomorphism between the graphs that is also a homeomorphism? This is a conjecture in [**MS**].
38. Can a node corresponding to a reflex vertex be identified in a given polygon point visibility graph [**MS**]?

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