Folding and Unfolding in Computational Geometry

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Abstract. Three open problems on folding/unfolding are discussed: (1) Can every convex polyhedron be cut along edges and unfolded flat to a single nonoverlapping piece? (2) Given gluing instructions for a polygon, construct the unique 3D convex polyhedron to which it folds. (3) Can every planar polygonal chain be straightened?

1 Introduction

Rather than survey the use of folding and unfolding throughout computational geometry, this paper will examine three unsolved problems, presenting partial results obtained in their pursuit, discussing relevant related research, and describing applications.

2 Polytope $\rightarrow$ Polygon

The first problem is the oldest and perhaps the most difficult:

> **(P1)** Prove or disprove that every convex polytope's surface $\mathcal{P}$ may be cut along edges and unfolded flat to a single nonoverlapping simple polygon.

The problem is stated explicitly in [She75], but B. Grünbaum believes[1] that it has been implicit since at least the time of Albrecht Dürer (1471-1528). There has been a recent resurgence of interest in this problem, perhaps due to the efforts of K. Fukuda, who developed useful software for exploring the question [NF93], and posed a number of explicit hypotheses.[2]

It is necessary for the cuts to form a tree on $\mathcal{P}$ spanning the vertices: a tree because any cycle would disconnect the unfolding, and a spanning tree because each vertex must be cut in order to lay flat. If the cuts are permitted to be arbitrary segments on $\mathcal{P}$, rather than only edges as in the problem statement, then

[1] Personal commun., 1987. See [D1528].
[2] www.ifor.math.ethz.ch/staff/fukuda/unfold_home/unfold_open.html.

nonoverlapping unfoldings are guaranteed in at least two ways. Fix a "source" point $x \in \mathcal{P}$. The *star unfolding* with respect to x cuts $\mathcal{P}$ along the n shortest paths from x to each of the n vertices of $\mathcal{P}$ [AAOS97]. This is clearly a tree (in fact a star), and clearly spans the vertices. That this unfolding does not overlap is by no means obvious, but has been established [AO92]. Fig. 1 shows an example.

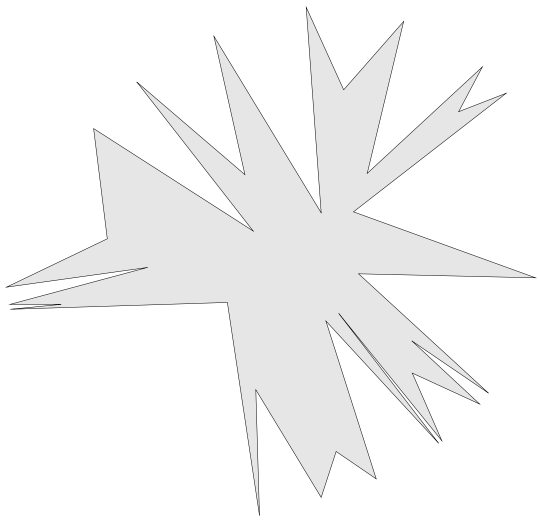

Fig. 1. The star unfolding of a polytope of $n = 18$ vertices.

A second unfolding, more difficult to define but easier to prove nonoverlapping, is the *source unfolding* [SS86] [MMP87]. This cuts the surface along the *ridge tree*, the set of points with two or more equal-length paths from the source x. The unfolding arrays the surface about x so that the shortest paths to the vertices emanate from it like the spokes of a wheel.

However, with the restriction that the cut tree is a subset of the 1-skeleton of $\mathcal{P}$, that is, follows edges only (*edge-cut*), the problem is much more difficult, and remains open. A curious aspect of the problem is that, although it is somewhat difficult to find overlapping unfoldings by "hand" exploration, "most" cut trees (and there are exponentially many) seem to lead to overlapping unfoldings. The data shown in Fig. 2 led to the conjecture that as $n \to \infty$, the probability that a random unfolding of a random polytope of n vertices is overlapping approaches 1 [SO87] [Sch89]. One can see in the figure that 99% of unfoldings of polytopes with $n \geq 70$ vertices lead to overlap. And yet no one has found a specific polytope for which exactly 100% of its unfoldings overlap, nor proven that at least one nonoverlapping unfolding exists for every polytope.

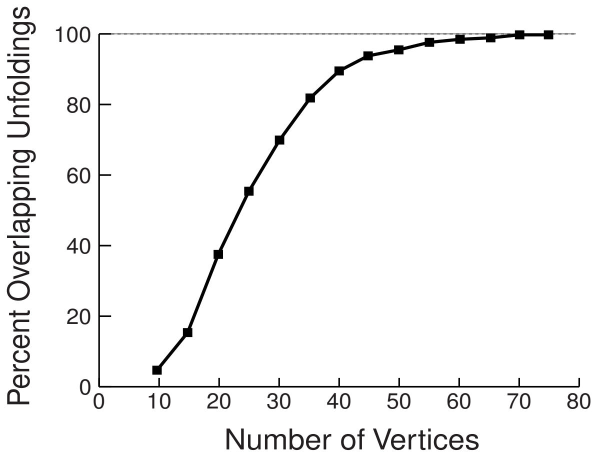

Fig. 2. Each point in this graph represents an average of several ($\sim$ 5) randomly generated polytopes whose vertices lie on a sphere, and the percentage of 1000 randomly selected unfoldings of each which overlap. After Fig. 2.12 in [Sch89, p.30].

This problem is not solely of academic interest: manufacturing parts from sheet metal leads directly to unfolding issues. A 3D part is approximated as a polyhedron, its surface is mapped to a collection of 2D flat patterns, each is cut from a sheet of metal and folded by a bending machine [GBKK98], and the resulting pieces assembled to form the final part. The author of a recent work in this area laments that, "Unfortunately, there is no theorem or efficient algorithm that can tell if a given 3D shape is unfoldable [without overlap] or not" [Wan97, p.81]. Consequently, heuristic methods are used.

Although those in manufacturing would be pleased to have problem **P1** resolved, they are even more keenly interested in unfolding nonconvex polyhedra. This is a relatively unstudied topic. It is easy to construct examples of nonconvex polyhedra that cannot be unfolded without overlap with cuts along edges: for example, the surface formed by placing a small cube on the center of a face of a larger cube. This is easy only because the top face of the larger cube is nonconvex. It is much more challenging to find an example of a nonconvex polyhedron with convex faces that cannot be edge-cut unfolded without overlap. Such an example was only recently constructed [BDEK99]. These authors raise the new question of whether any simplicial polyhedron (all faces triangles) is edge-cut un-unfoldable.

Some special classes of orthogonal polyhedra are known to be unfoldable without overlap [BDD+98b]. In particular, such unfoldings exist for *orthotubes*, orthogonal polyhedra obtained by gluing together a series of rectangular boxes $B_1, \ldots, B_n$ such that $B_i \cap B_{i+1}$ is a face of both B_i and B_{i+1}, and nonconsecutive boxes are either disjoint, or share a vertex or edge. Orthotubes may be closed into a cycle, and might form a topological knot. Another class of orthogonal polyhedra, *orthostacks*, is shown unfoldable in the same paper.

Given the applications to manufacturing and the paucity of results, unfolding nonconvex polyhedra seems an area ripe for exploration.

3 Polygon → Polytope

The inverse of the previous open problem is also open, but likely more tractable: Given an unfolding (a polygon), refold it. There are two issues here, one more open than the other. The first is to find a "legal" gluing of the polygon boundary to itself. What is legal is determined by a powerful theorem of Aleksandrov [Ale58]: Any gluing that (a) uses up the perimeter with matches of equal length portions of the boundary, (b) that glues no more than 2π face angle at any point, and (c) which results in a surface homeomorphic to a sphere—any such gluing corresponds to a unique convex polyhedron. Consequently the decision problem, "Can a given polygon fold to a polyhedron?", may be solved by finding a legal gluing. This problem has been solved in a special case: for *edge-to-edge* gluings, where gluings are restricted to glue whole edges of the input polygon to whole edges. Then a dynamic programming algorithm of time and space complexity $O(n^2)$ answers the decision question [LO96]. This algorithm has been implemented, and has led to some curious discoveries. For example, the familiar Latin cross unfolding of a cube may be refolded edge-to-edge using five different legal gluings, leading to five different polytopes: the cube, a flat, doubly-covered quadrilateral, a tetrahedron, a pentahedron, and an octahedron. The folding that produces a tetrahedron is illustrated in Fig. 3, a few frames from a video [DDL$^+$99].

This work does not entirely settle the decision question, for Aleksandrov's theorem does not demand edge-to-edge gluings. A recent result obtained here [DDLO99] is that every convex polygon may be folded into an infinite number of distinct polytopes.

The second, relatively more open issue concerning refolding is to construct the 3D shape:

> **(P2)** Given a legal gluing of a polygon, construct the unique 3D convex polyhedron to which it folds.

Here I am only aware of my own work on a numerical approximation procedure, which operates as follows. Cauchy's rigidity theorem (see, e.g., [Cro97]) implies that the lengths of the edges of a triangulated polytope uniquely determine the 3D structure of the polytope. We have implemented a numerical relaxation procedure which takes as input a combinatorial triangulated graph, with edge weights representing lengths, and outputs 3D vertex coordinates of a convex polytope that approximately realizes that input information.

Now, given a polygon and a gluing, the polytope vertices may be identified as those points with curvature less than 2π. Using Chen and Han's shortest

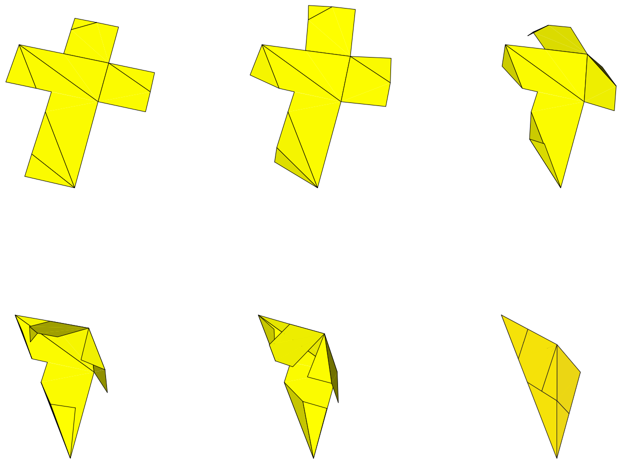

Fig. 3. Folding a Latin cross to a tetrahedron. (Animation courtesy of Erik Demaine.)

path algorithm [CH96], the shortest paths between all pairs of vertices may be computed in the unfolding. The arrangement formed by these paths may be triangulated, forming a triangulation T that is a superset of the 1-skeleton of the unique folded polytope $\mathcal{P}$. This follows because each edge e of $\mathcal{P}$ must be a shortest path between its endpoints; so $e \in T$. Consequently, applying the Cauchy rigidity relaxation to T will reconstruct (an approximation of) the polytope.[3] We have to date carried out this procedure only on polytopes of few vertices, as our relaxation procedure is not (yet) numerically robust.

4 Locked Planar Chain?

Lastly we turn to a tantalizingly simple problem in the plane:

> **(P3)** Given a simple polygonal chain in the plane, may it always be "opened" (i.e., straightened), where all links are rigid, all vertices are joints, and all intermediate configurations are simple?

This problem, known variously as "the paperclip" or "carpenter's ruler" problem, has been posed by several people independently, including J. Mitchell, and

[3] This all-shortest paths approach was suggested by B. Aronov (personal comm., 1998), and is also implicit in [Ale58].

W. Lenhart and S. Whitesides [LW95]. A variation asks for convexification of a closed chain, i.e., a polygon; this is equally unsolved. Note that at no time during a proposed reconfiguration may links cross; it is the noncrossing/simplicity aspect which makes the problem challenging.

Some progress has been made recently on the fringes of problem **P3**, establishing that several specializations lead to positive answers, but a natural generalization leads to a negative answer:

1. Every star-shaped polygon may be convexified [ELR+98].
2. Every monotone polygon may be convexified [BDL+99].
3. Generalizing a chain/path of rigid links to a tree of rigid links embedded in the plane, and generalizing the notion of "straightening" appropriately, it has been established that not every tree can be straightened: Some tree configurations are "locked" [BDD+98a].

The problem can be fruitfully generalized to 3D. Here there are practical applications, for example, to hydraulic tube bending: Given a polygonal chain in 3-space, a "tube design," can it be manufactured by a bending machine?[4]

Several results have been obtained on straightening/convexifying chains in 3D:

1. There exist both open and closed chains in 3D that are locked [CJ98,BDD+99].
2. An open chain in 3D with a simple projection onto some 2D plane may be straightened [BDD+99].
3. A closed chain in a plane, i.e., a planar polygon, may be convexified in 3D by "flipping" out the reflex pockets, i.e., rotating the pocket chain into 3D and back down to the plane. See Fig. 4. This procedure was suggested by Erdős [Erd35] and proved to work by de Sz. Nagy [dSN39]. The number of flips, however, cannot be bound as a function of the number of vertices n of the polygon, as first proved by Joss and Shannon [Grü95]. See [Tou99] for the tangled history of these results.
4. A planar polygon may be convexified in 3D by a different algorithm which uses only $O(n)$ basic "moves" [BDD+99].[5]

In dimensions $d \geq 4$, it has recently been established that neither open nor closed chains can lock: Every open chain can be straightened in $O(n)$ moves, and every closed chain can be convexified in $O(n^6)$ moves [CO99a,CO99b]. Thus less is known about locking in two dimensions than in three and higher dimensions. But with recent increased scrutiny by the community, perhaps **P3** will soon be settled.

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[4] This application was detailed by J. Mitchell at the NSF Workshop on Manufacturing, Apr. 1994. He has solved several variants of the problem (personal comm.).

[5] The cited paper claims $O(n^2)$, but this has been improved in the full version [unpublished].

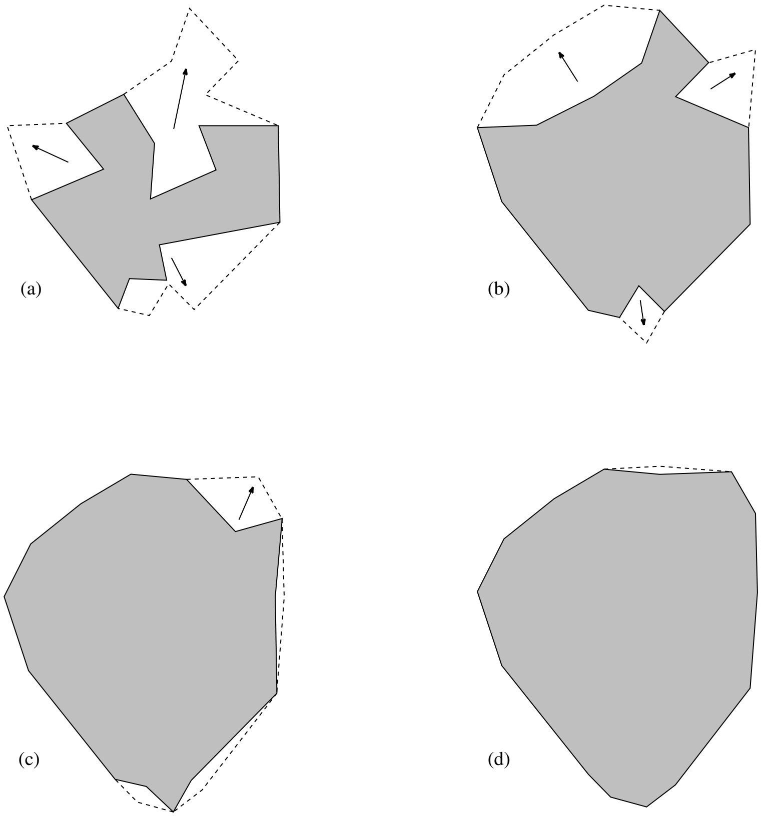

Fig. 4. A nonconvex polygon of $n = 14$ vertices: (a) after three pocket flips; (b) after three more; (c) after four more; (d) the final convexification, after a total of 11 pocket flips.

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