An Extension of Cauchy's Arm Lemma with Application to Curve Development

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Abstract. Cauchy's "Arm Lemma" may be generalized to permit nonconvex "openings" of a planar convex chain. Although this (and further extensions) were known, no proofs have appeared in the literature. Here two induction proofs are offered. The extension can then be employed to establish that a curve that is the intersection of a plane with a convex polyhedron "develops" without self-intersection.

1 Introduction

Cauchy used his famous "Arm Lemma" to establish the rigidity of convex polyhedra [Cro97, p. 228]. Cauchy's lemma says that if $n-2$ consecutive angles of a convex polygon of n vertices are opened but not beyond π, keeping all but one edge length fixed and permitting that "missing" edge e to vary in length, then e lengthens (or retains its original length). This lemma has numerous applications, including at least one to curve development [OS89]. Here we describe an apparently little-known generalization of Cauchy's lemma to permit opening of the angles beyond π, as far reflex as they were originally convex. The conclusion remains the same: e cannot shorten. We then apply this extension to another result on curve development, that "slice curves" develop without self-intersection.

The first part of this paper (Section 2) concentrates on the generalization of Cauchy's lemma. The issue of self-intersection is addressed in Section 3, and the curve development result is discussed in Section 4.

2 Cauchy's Arm Lemma Extended

Let $A = (a_0, a_1, \ldots, a_n)$ be an n-link polygonal chain in the plane with n fixed edge lengths $\ell_i = |a_i a_{i+1}|$, $i = 0, \ldots, n-1$. We call the vertices a_i the *joints* of the chain, a_0 (which will always be placed at the origin) the *shoulder*, and a_n the *hand.* Define the *turn angle* α_i at joint a_i, $i = 1, \ldots, n-1$ to be the angle in $[-\pi, \pi]$ that turns the vector $a_i - a_{i-1}$ to $a_{i+1} - a_i$, positive for left (counterclockwise) and negative for right (clockwise) turns.

Define an open polygonal chain A to be *convex* if its joints determine a (nondegenerate) convex polygon, i.e., all joints are distinct points (in particular, $a_n \neq a_0$), all joints lie on the convex hull of A and they do not all lie on a line.

Note there is no chain link between a_n and a_0. The turn angles for a convex chain all lie in $[0, \pi)$; but note this is not a sufficient condition for a chain to be convex, for it is also necessary that the angles at a_0 and a_n be convex.

We can view the configuration of a polygonal chain A to be determined by two vectors: the fixed edge lengths $L = (\ell_0, \ldots, \ell_{n-1})$ and the variable turn angles $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$, with the convention that a_0 is placed at the origin and a_n horizontally left of a_0. Let $\mathcal{C}_L(\alpha) = A$ be the configuration so determined. We use α to represent the angles of the initial configuration, and β and γ to represent angles in a reconfiguration.

Let $D(A)$ be the open disk of radius $|a_n a_0|$ (the length of the "missing" link) centered on the shoulder joint a_0. We will call $D(A)$ the *forbidden (shoulder) disk*. We may state Cauchy's arm lemma in the following form:

Theorem 0. Cauchy's Arm Lemma. *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [0, \alpha_i] \tag{1}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(A)$.

Cauchy's lemma is sometimes known as Steinitz's lemma, because Steinitz noticed and corrected an error in the proof a century after Cauchy [Cro97, p. 235]. Many proofs of Cauchy's lemma are now known, e.g., [SZ67,Sin97] and [AZ98, p. 64].

Our generalization of Cauchy's lemma replaces the 0 in Eq. (1) by $-\alpha_i$, and is otherwise identical:

Theorem 1. Cauchy Extension. *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [-\alpha_i, \alpha_i] \tag{2}$$

we must have $|b_n b_0| \geq |a_n a_0|$, i.e., the hand cannot enter the forbidden disk $D(A)$.

The intuition, illustrated in Fig. 1, is perhaps best captured by Chern [Che67, p. 36]: "if an arc is 'stretched,' the distance between its endpoints becomes longer." See also our Java applet illustrating the theorem and its corollaries.[1] Although the chain may become nonconvex, Eq. (2) ensures that the movement constitutes a form of straightening. Note that Theorem 1 makes no claim about steadily increasing hand-shoulder separation during some continuous movement to B; indeed a continuous opening could first increase and later decrease the separation. Rather the claim is that a final configuration satisfying Eq. (2) cannot place the hand in the forbidden disk.

[1] `http://cs.smith.edu/~orourke/Cauchy/` .

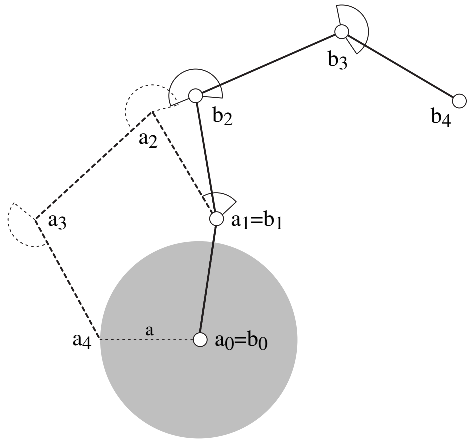

Fig. 1. A reconfiguration of a 4-link convex chain A to chain B, satisfying Eq. (2), leaves b_n outside the forbidden disk (shaded). The valid angle ranges are marked by circular arcs.

As with Cauchy's arm lemma, one may expect many different proofs of such a fundamental result. We sketch three proofs of Theorem 1 here; for full details, see [O'R00]. First we discuss a proof by S. S. Chern, followed by two induction proofs.

2.1 Chern's Proof of Theorem 1

There are three possible generalizations of Cauchy's Theorem 0: to smooth curves, to space curves, and to nonconvex angles. All three generalizations hold, and there is some confusion in the literature on when each was established. Certainly the primary source is acknowledged to be a 1921 paper by Axel Schur [Sch21]. Connelly mentions in [Con82, p. 30] that Schur generalized Cauchy's lemma to the smooth case. Some other mentions of Schur's theorem in the literature, e.g., in [Gug63, p.31], phrase it as the smooth, planar equivalent of Cauchy's lemma, which does not capture the nonconvexity permitted in the statement of Theorem 1. But others, notably the exposition by Chern [Che67], state it as a generalization also to space curves, and employ the absolute value of curvature, implicitly permitting nonconvexity.

Schur states his main result as follows:

> SATZ: "Verbeigt man eine ebene doppelpunktlose Kurve, die mit ihrer Sehne einen konvexen gescholossene doppelpunktlosen Linienzug bildet, so wird dabei die Sehne langer."
> Roughly: "If one deforms a simple curve in the plane that (together with its chord) forms a convex simple closed curve, then the chord becomes longer."

Later he makes clear that his “deformation” is deformation into space without changing the curvature: “By deformation of a curve, we mean that after deforming the tangent planes the lengths and angles of the line segments should remain unchanged. If the curvature of the curve is continuous, then this is equal to demanding that the curvature doesn’t change.” It seems, then, that Schur did not generalize the opening aspect of Cauchy’s lemma (to which he does not refer), but rather extended to deformations into space without angle changes. Moreover, it appears his proof (and claim) are for polygonal curves, not smooth curves.

Chern states Schur’s theorem as follows [Che67, p. 36]:

> THEOREM: Let C be a plane arc with the curvature $k(s)$ which forms a convex curve with its chord AB. Let C^* be an arc of the same length referred to the same parameter s such that its curvature $k^*(s) \leq k(s)$. If d^* and d are the lengths of the chords joining their endpoints, then $d \leq d^*$.

He makes clear that by “curvature” he means the absolute value of the curvature, thus explicitly including nonconvex openings. Although he attributes the theorem to Schur, neither his theorem statement, and certainly not his proof, match Schur’s. Let us then refer to this full generalization of Cauchy’s theorem as the Chern-Schur theorem.

Chern’s proof is carried out largely in the domain of the “tangent indicatrixes” of C and C^*, and relies on smoothness of the curves. He claims the theorem for “sectionally smooth curves” [p. 39], e.g., for polygonal curves, but without proof. Thus, as far as I can determine, no proof of Theorem 1 has appeared in the literature.

However, Chern’s differential geometry proof may be specialized to our planar, nonsmooth instance, and although some of its elegance is lost, it can be carried through to establish Theorem 1 [O’R00]. Therefore Theorem 1 follows in spirit if not in fact from the Chern-Schur theorem.

The next section proves the theorem via an induction proof that makes explicit many relationships only implicit in Chern’s proof.

2.2 First Induction Proof of Theorem 1

Although we impose no restriction on self-intersection of the chain, we will show in Theorem 3 that the chain remains simple. Note that, because we fix a_0 to the origin, and the first turn angle is at joint a_1, in any reconfiguration the first edge of the chain is fixed.

The first induction proof of Theorem 1 requires a few preparatory lemmas, whose proofs may be found in [O’R00]. We start with the simple observation that negating the turn angles reflects the chain.

Lemma 1. *If a chain $A = \mathcal{C}_L(\alpha)$ is reconfigured to $B = \mathcal{C}_L(\beta)$ with $\beta_i = -\alpha_i$, $i = 1, \ldots, n-1$, then B is a reflection of A through the line M containing $a_0 a_1$, and $|b_n b_0| = |a_n a_0|$.* □

Call a reconfiguration $B = \mathcal{C}_L(\beta)$ of a convex chain $A = \mathcal{C}_L(\alpha)$ which satisfies the constraints of Eq. (2) a *valid reconfiguration*, and call the vector of angles β *valid angles*. Define the *reachable region* $R_L(\alpha)$ for a convex chain $A = \mathcal{C}_L(\alpha)$ to be the set of all hand positions b_n for any valid reconfiguration $B = \mathcal{C}_L(\beta)$. One can view Theorem 1 as the claim that $R_L(\alpha) \cap D(A) = \emptyset$. (Again, see our applet[1] for an illustration of this claim.) It is well known [HJW84][O'R98, p. 326] that the reachable region for a chain with no angle constraints is a shoulder-centered closed annulus, but angle-constrained reachable regions seem unstudied.

For the first proof we need two technical lemmas.

Lemma 2. *The configuration of a chain $A = \mathcal{C}_L(\alpha)$ is a continuous function of its turn angles α.* □

Lemma 3. *$R_L(\alpha)$ is a closed set.* □

We use this lemma to help identify, among potential counterexamples, the "worst" violators. Define a configuration $B = \mathcal{C}_L(\beta)$ to be *locally minimal* if there is a neighborhood N of β such that, for all $\beta' \in N$, the determined hand position b'_n is no closer to the shoulder: $|b'_n a_0| \geq |b_n a_0|$. Thus the hand's distance to the shoulder is locally minimal.

Lemma 4. *Let $B = \mathcal{C}_L(\beta)$ be a reconfiguration of convex chain $A = \mathcal{C}_L(\alpha)$ with $b_n \in D(A)$. Then either $b_n = a_0$, or there is some locally minimal configuration $B' = \mathcal{C}_L(\beta')$ with $b'_n \in D(A)$.* □

The above lemma will provide a "hook" to reduce n in the induction step. We separate out the base of the induction in the next lemma.

Lemma 5. *Theorem 1 (Cauchy Extension) holds for $n = 2$.*
Proof: A 2-link chain's configuration is determined by single angle at a_1. The reachable region is a single circular arc exterior to $D(A)$, centered on a_1, of radius ℓ_1. See Fig. 2. □

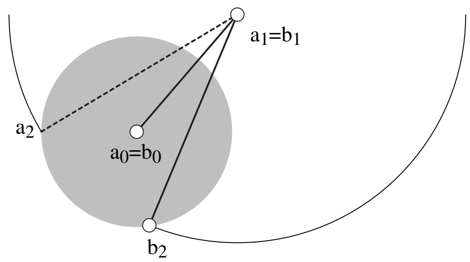

Fig. 2. $R_L(\alpha)$ for a 2-link chain is a circle arc centered on $a_1 = b_1$.

We now prove Theorem 1 by induction.

Proof: Lemma 5 establishes the theorem for $n = 2$. Assume then that the theorem holds for all chains of $n-1$ or fewer links. We seek to establish it for an n-link chain $A = \mathcal{C}_L(\alpha)$, $n > 2$. Assume, for the purposes of contradiction, that A may be reconfigured so that the hand falls inside the forbidden disk $D(A)$. We seek a contradiction on a shorter chain. By Lemma 4, one of two cases holds: the hand reaches a_0, or there is a locally minimal configuration.

1. Suppose $B = \mathcal{C}_L(\beta)$ is such that $b_n = a_0$, as illustrated in Fig. 3(b). We know that $\ell_{n-1} = |a_{n-1}a_n| < |a_{n-1}a_0|$ by the triangle inequality: By definition, A forms a nondegenerate convex polygon, so the triangle $\triangle(a_0, a_{n-1}, a_n)$ is nondegenerate (see Fig. 3(a)). Now consider the chains A' and B' obtained by removing the last links $a_{n-1}a_n$ and $b_{n-1}b_n$. First, A' is a convex chain of $n-1$ links, so the induction hypothesis applies and says that A' cannot be validly reconfigured to place b_{n-1} closer to a_0 than $a' = |a_{n-1}a_0|$. B' places b_{n-1} at distance ℓ_{n-1} from a_0, which we just observed is less than a'. It remains to argue that B' is a valid reconfiguration of A', i.e., that it satisfies Eq. (2). However, this is satisfied for $i = 1, \ldots, n-2$ because these angles are not changed by the shortening, and after shortening there is no constraint on β_{n-1}. Thus B' is a valid reconfiguration of A' but places the hand in the forbidden disk $D(A')$, a contradiction.

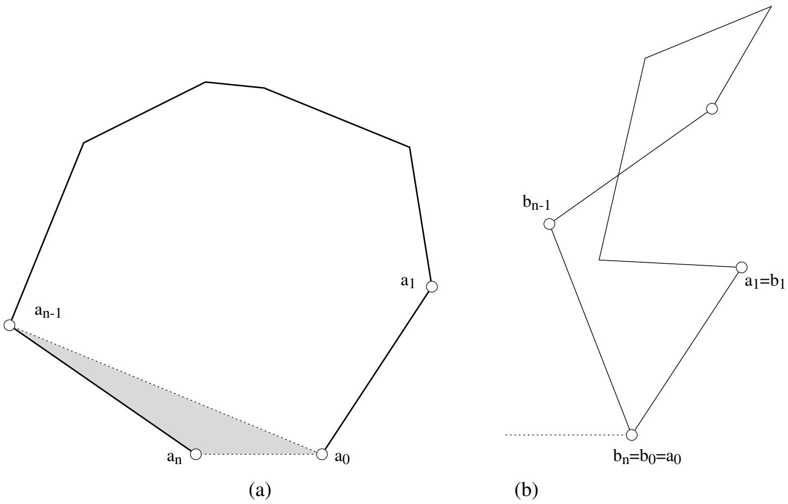

Fig. 3. Case 1: $b_n = a_0$. (Drawing (b) is not accurate.)

2. We may henceforth assume, by Lemma 4, that there is a locally minimal configuration $B = \mathcal{C}_L(\beta)$ that places $b_n \in D(A)$. Again we seek to shorten the chain and obtain a contradiction.

First we establish that at least one[2] β_i is at the limit of its valid turn range: $\beta_i = \pm\alpha_i$. Suppose to the contrary that all β_i, $i = 1, \ldots, n-1$, are strictly interior to their allowable turn ranges: $\beta_i \in (-\alpha_i, \alpha_i)$. Let M be the line containing $b_0 b_n$. Consider two cases:

(a) Some b_i, $i = 1, \ldots, n-1$, does not lie on M. Then because β_i is not extreme, the subchain $(b_{i+1}, \ldots, b_n)$ may be rotated about b_i in both directions. Because b_i is off M, one direction or the other must bring b_n closer to b_0, contradicting the fact that b_n is locally minimal.

(b) All b_i lie on M. Then there must be some b_i which is extreme on M. For this b_i, $\beta_i = \pm\pi$. But $\alpha_i \in [0, \pi)$: the nondegeneracy assumption bounds α_i away from π, and so bounds β_i away from $\pm\pi$.

Henceforth let b_i be a joint whose angle β_i is extreme. If $\beta_i = -\alpha_i$, then reflect B about $b_0 b_n$ so that $\beta_i = \alpha_i$ is convex. By Lemma 1, this does not change the distance from b_n to the shoulder, so we still have $b_n \in D(A)$.

We are now prepared to shorten the chains. Let A' and B' be the chains resulting from removing a_i and b_i from A and B respectively:

$$A' = (a_0, a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n) \tag{3}$$
$$B' = (b_0, b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n) \tag{4}$$

A crucial point to notice is that $|b_{i-1} b_{i+1}| = |a_{i-1} a_{i+1}|$ because $\beta_i = \alpha_i$; this was the reason for focusing on an extreme β_i. Therefore B' is a reconfiguration of A'. Of course both A' and B' contain $n-1$ links, so the induction hypothesis applies. Moreover, because $i \leq n-1$, the b_i removed does not affect the position of b_n. So $b_n \in D(A)$ by hypothesis. To derive a contradiction, it only remains to show that B' is a valid reconfiguration of A', i.e., one that satisfies the turn constraints (2).

Let α'_{i+1} be the turn angle at a_{i+1} in A'. We analyze this turn angle in detail, and argue later that the situation is analogous at a_{i-1}. Let θ be the angle of the triangle $\triangle_i = \triangle(a_i, a_{i+1}, a_{i-1})$ at a_{i+1}; see Fig. 4(a). Because A is a convex chain, cutting off $\triangle_i$ from A increases the turn angle at a_{i+1} in A':

$$\alpha'_{i+1} = \theta + \alpha_{i+1} \tag{5}$$

Now consider the turn angle β'_{i+1} at b_{i+1} in B'. Although here the turn could be negative, as in Fig. 4(b), it is still the case that the turn is advanced by θ by the removal of $\triangle_i$:

$$\beta'_{i+1} = \theta + \beta_{i+1} \tag{6}$$

We seek to prove that $\beta'_{i+1} \in [-\alpha'_{i+1}, \alpha'_{i+1}]$. Substituting the expressions from Eqs. (5) and (6) into the desired inequality yields:

$$-\alpha_{i+1} - 2\theta \leq \beta_{i+1} \leq \alpha_{i+1} \tag{7}$$

And this holds because $\theta > 0$ and $\beta_{i+1} \in [-\alpha_{i+1}, \alpha_{i+1}]$ (because B is a valid reconfiguration of A). The intuition here is that the nesting of the turn angle

[2] In fact I believe that all must be extreme, but the proof only needs one.

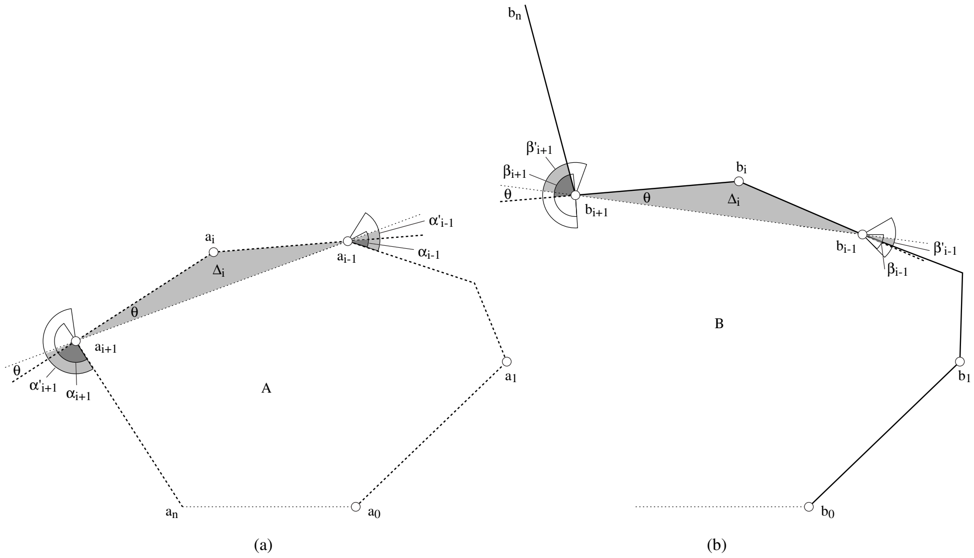

Fig. 4. (a) Shortening the chain A by removal of a_i determines new, larger turn angles α'_{i+1} and α'_{i-1} at a_{i+1} and a_{i-1} respectively. (b) Here the turn angles β_{i+1} and β'_{i+1} are negative.

ranges at a_{i+1} in A and A' (evident in Fig. 4(a)) carries over, rigidly attached to $\triangle_i$, to B, so that satisfying the tighter constraint in B also satisfies the looser constraint in B'.

Although the situation is superficially different at a_{i-1} because our definition of turn angle depends on the orientation of the chain, it is easily seen that the turn constraint is identical if the orientation is reversed.

We have thus established that B' is a valid reconfiguration of A'. By the induction hypothesis, the hand b_n of B' cannot enter the forbidden disk $D(A')$. By assumption $b_n \in D(A)$. But note that $D(A') = D(A)$, because, as mentioned above, deleting a_i and b_i does not affect the positions of a_n and b_n respectively. This contradiction shows that our assumption that $b_n \in D(A)$ cannot hold, and establishes the theorem.

□

The following corollary extends the distance inequality to every point of the chain.

Corollary 2 *Let $A = \mathcal{C}_L(\alpha)$ be a convex chain as in Theorem 1, and let p_1, $p_2 \in A$ be any two distinct points of the chain. Then in any valid reconfiguration B, the points q_1, $q_2 \in B$ corresponding to p_1 and p_2 satisfy $|q_1q_2| \geq |p_1p_2|$, i.e., they have not moved closer to one another.* □

2.3 Second Induction Proof of Theorem 1

We now sketch a second proof, which avoids reliance on locally minimal configurations. Unlike the previous proof, this one employs Cauchy's Arm Lemma (Theorem 0), rather than proving it along the way. The proof is again inductive, by contradiction from a shortened chain, and relies on the same detailed argument concerning the turn angle ranges. None of those details will be repeated.

Proof: Let $A = \mathcal{C}_L(\alpha)$ be the given convex chain, and $C = \mathcal{C}_L(\gamma)$ a valid reconfiguration that places $c_n \in D(A)$, in contradiction to the theorem. We first construct an "intermediate" configuration $B = \mathcal{C}_L(\beta)$ with $\beta_i = |\gamma_i|$ for all $i = 1, \ldots, n-1$, i.e., B is a convex chain formed by flipping all turns in C to be positive. Note that, because γ is a valid angle vector for A, $\gamma_i \in [-\alpha_i, \alpha_i]$, and so $\beta_i \in [0, \alpha_i]$. As this is exactly the Cauchy arm opening condition, Eq. (1), we may apply Theorem 0 to conclude that $b = |b_n b_0| \geq |a_n a_0| = a$.

Now we focus attention on chains B and C. Because $\gamma_i = \pm\beta_i$, $\gamma_i \in [-\beta_i, \beta_i]$. Therefore, C is a valid reconfiguration of B. But here is the point: every angle γ_i of C is extreme with respect to B, and so there is no need to invoke local minimality.

Choose an i and remove b_i from B and c_i from C, obtaining shorter chains B' and C'. Applying the argument from the previous section verbatim, we conclude that C' is a valid reconfiguration of B'. But because B' has $n-1$ links, the induction hypothesis applies and shows that c_n cannot enter the forbidden disk $D(b)$, with $b = |b_n b_0|$. Because $b \geq a$, c_n cannot be in $D(A)$ either. This contradicts the assumption and establishes the theorem. $\square$

3 Noncrossing of Straightened Curve

Define a polygonal chain to be *simple* if nonadjacent segments are disjoint, and adjacent segments intersect only at their single, shared endpoint. By our nondegeneracy requirement, convex chains are simple. In particular, any opening of a convex chain via Cauchy's arm lemma (Theorem 0) remains simple because it remains convex. We now establish a parallel result for the generalized straightening of Theorem 1. We generalize slightly to permit the convex chain to start with the hand at the shoulder.

Theorem 3. *If $A = (a_0, \ldots, a_n) = \mathcal{C}_L(\alpha)$ is a closed convex chain with n fixed edge lengths L and turn angles α, closed in the sense that $a_n = a_0$, then any valid reconfiguration to $B = \mathcal{C}_L(\beta)$ is a simple polygonal chain.*

Proof: Suppose to the contrary that B is nonsimple. Let q_2 be the first point of B, measured by distance along the chain from the shoulder b_0, that coincides with an earlier point $q_1 \in B$. Thus q_1 and q_2 represent the same point of the plane, but different points along B. See Fig. 5. Because B is nonsimple, these "first touching points" exist, and we do not have both $q_1 = b_0$ and $q_2 = b_n$ (because that would make B a simple, closed chain). Let p_1 and p_2 be the points of A corresponding to q_1 and q_2.

Corollary 2 guarantees that $|q_1q_2| \geq |p_1p_2|$. But $|q_1q_2| = 0$, and because the q's do not coincide with the original hand and shoulder, $|p_1p_2| > 0$. This contradiction establishes the claim. □

One could alternatively prove this theorem by induction on the length of the chain, showing that in a continuous motion to B, the first violation of simplicity is either impossible by the induction hypothesis, or directly contradicts Theorem 1.

Corollary 4 *A valid reconfiguration of an open convex chain remains simple.*
Proof: The proof of Theorem 3 guarantees that even the final missing edge is not crossed, so the corollary is obtained by simply ignoring that last edge. □

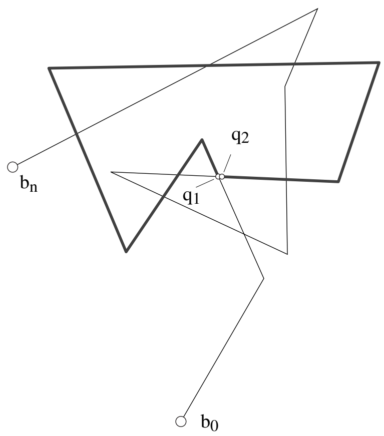

Fig. 5. Violation of Theorem 1. $q_1 = q_2$ is the first point of self-contact; the initial portion of B, up to q_2, is highlighted.

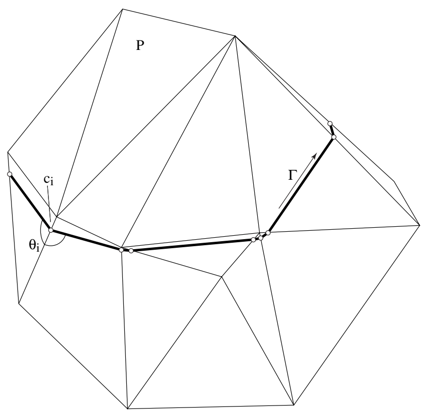

Fig. 6. Γ is the intersection of a plane (not shown) with polyhedron P.

4 Application to Curve Development

Although the intersection of a plane Π with a convex polyhedron P is a convex polygon Q within that plane, on the surface of P, this *slice curve* (Fig. 6) can be nonconvex, turning both left and right. The "development" of a curve on a plane is determined by its turning behavior on the surface. Thus slice curves develop (in general) to nonconvex, open chains on a plane. An earlier result is that a closed convex polygonal curve on a convex polyhedron, i.e., one whose turns are *all* leftward on the surface, develops to a simple polygonal chain [OS89]. That proof relied on Cauchy's Arm Lemma (Theorem 0). We claim that slice curves also develop without self-intersection, despite their nonconvexity. The proof relies on Theorem 1, the extension of Cauchy's lemma.

Let Γ be an oriented slice curve on the surface of P. Let $c_0, c_1, \ldots, c_n$ be the *corners* of Γ, the points at which Γ crosses a polyhedron edge with a dihedral angle different from π, or meets a polyhedron vertex. Define the *right surface angle* $\theta(p)$ at a point $p \in \Gamma$ to be the total incident face angle at p to the right of the directed curve Γ at p. Only at a corner c_i of Γ is the right surface angle θ_i different from π. Note that θ_i could be greater or less than π, i.e., the slice curve could turn right or left on the surface.

Define the *right development* of Γ to be a planar drawing of the polygonal chain Γ as the chain $B = (b_0, b_1, \ldots, b_n)$ with the same link lengths, $|b_i b_{i+1}| = |c_i c_{i+1}|$ for $i = 0, \ldots, n-1$, and with exterior angle θ_i to the right of b_i the same as the surface angle to the right of Γ at c_i on P, for all $i = 1, \ldots, n-1$. Left development is defined similarly, with any curve "between" considered *a* development of Γ. If Γ avoids all polyhedron vertices, then the left and right developments are identical, and so the development of Γ is unique.

In [O'R01] I prove that the developed chain B is a valid reconfiguration (in the sense used in Section 2.2, i.e., satisfies Eq. (2)) of the chain A representing Q in Π. Via Corollary 4, this leads to:

Theorem 5. *A slice curve $\Gamma = P \cap \Pi$, the intersection of a convex polyhedron P with a plane Π, develops on a plane to a simple (noncrossing) polygonal curve.*

Because the Chern-Schur Theorem encompasses smooth curves, Theorem 5 should generalize to slice curves for any convex body B.

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