Illuminating Convex Polygons with Vertex Floodlights

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Abstract

We explore which vertex floodlights suffice to cover any convex polygon of n vertices. Urrutia established that any convex polygon may be illuminated by any three vertex floodlights whose total angle is π. We show that the generalization of this result to k vertex floodlights is false, even when all lights have the same angle, π/k.

1 Introduction

Urrutia established this pleasing result: any convex polygon may be illuminated by any three vertex floodlights whose total angle is π.[1] A *floodlight*[2] is a light within an aperture limited to some fixed angle α; it can illuminate all points with a cone whose apex angle is α. A *vertex floodlight* is one whose apex is located at a polygon vertex. Urrutia's result says that if we are given a convex polygon P, and three angles whose sum is π (or greater), then floodlights of those angles may be assigned to distinct vertices of P and oriented so that the interior of P is completely illuminated. A point is illuminated if it lies in or on the boundary of some floodlight cone, i.e., the light cones are closed.

It is natural to wonder if the following generalization holds:

> **Q0.** Given a convex polygon P of n vertices, a set of $k \leq n$ vertex floodlights whose total angle sum is π, can the lights always be assigned to distinct vertices and oriented to fully illuminate P?

The main result of this paper is that the answer to Q0 is NO. We establish this by showing that the special case of this question when all floodlight angles are equal[3] also has a negative answer:

> **Q1.** Given a convex polygon P of n vertices, a set of $k \leq n$ vertex floodlights each with angle $\alpha = \pi/k$, can the lights always be assigned to distinct vertices and oriented to fully illuminate P?

First we should note that if the floodlight angles are not fixed in advance, the answer to Q0 is YES. More specifically, any convex polygon

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[1] Personal communication, Fall 1994. His proof locates a triangle with the given angles inscribed in the polygon, and argues that the lights can be moved from the triangle corners to nearby vertices while preserving coverage.

[2] The concept was introduced in [BGL+93].

[3] Posed as an open problem by the first author at the Fourth MSI Computational Geometry Workshop, held at Cornell University in Oct. 1994.

P may be covered by some set of vertex floodlights whose total angle is $< \pi$: choose any vertex whose interior angle α is strictly less than π, and place one α-floodlight there. The same holds true for the analog of Q1 if the number of floodlights is fixed to $k < n$ but the angle distribution is not: again place an $\alpha < \pi$ light at some vertex, and distribute $k-1$ lights each of angle $(\pi - \alpha)/(k-1)$ at other vertices.

These results are more difficult to establish if the floodlights only illuminate points "clearly visible" to the floodlight apex. Nevertheless we claim without proof that these two results hold for clear visibility as well.

The more interesting problems involve lights whose apertures are fixed, and equal apertures is an especially inviting case. The answer to Q1 is YES for $k = 2$: any convex polygon may be illuminated by placing two $\pi/2$-lights facing one another at opposite ends of any edge. And Urrutia's result shows that the answer to Q1 is YES for $k = 3$. Although we show that the answer to Q1 is NO for general k, our proof only establishes this for large k. It remains open at this writing whether Q1 is true even for $k = 4$.

2 Fixed Assignment of Lights

We first consider the following easier question, with fixed vertex assignments:

> **Q2.** Given a convex polygon P of n vertices, a fixed set of $k \leq n$ vertex floodlights with angles $\alpha_1, \alpha_2, \ldots \alpha_k$, where $\sum \alpha_i = \pi$, and a fixed assignment of the lights to distinct vertices of P, can the lights always be oriented to fully illuminate P?

Given the tight constraints specified in Q2, it is not surprising that its answer is NO. We demonstrate this with the pentagon $\mathbb{P}$ shown in Fig. 1, and $k = 2$.

Theorem 2.1 *The polygon $\mathbb{P}$ shown in Fig. 1 cannot be covered by two lights, both with angle $\pi/2$, and assigned to vertices v_1 and v_3. Therefore the answer to Q2 is* NO.

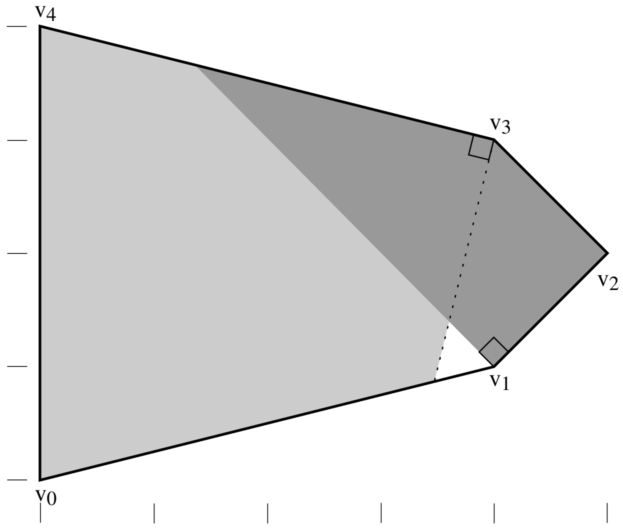

Figure 1: $\mathbb{P}$: vertex coordinates $v_0 = (0, -2)$, $v_1 = (4, -1)$, $v_2 = (5, 0)$, $v_3 = (4, 1)$, $v_4 = (0, 2)$.

Proof: Suppose that the light L_1 at v_1 covers v_2. This is no loss of generality because the polygon is symmetric about the horizontal line through v_2. This requires L_1 to be turned clockwise as far as possible, as illustrated in the figure. Then this light does not illuminate v_4. So the light L_3 at v_3 must shine on v_4, which requires L_3 to be turned clockwise as far as possible, again as illustrated. Because angle$(v_1, v_3, v_4) = \pi/2 + \text{atan}(1/4) \approx 104°$, this leaves points in a neighborhood of v_1 unilluminated. □

One might wonder if the fact that $k < n$ $(2 < 5)$ in our example plays a significant role. That it does not can be seen by changing the example to $k = 5$ by assigning ϵ-aperture lights to v_0, v_2, and v_4 for very small $\epsilon > 0$, and $(\pi - 3\epsilon)/2$ to v_1 and v_3. For sufficiently small ϵ, $\mathbb{P}$ cannot be covered by this assignment of lights.

Rather the key to this example is the fixed assignment between lights and vertices. Without this assignment, this example can be covered, because as we mentioned in the introduction, two $\pi/2$-lights can cover any convex polygon. Our goal is to now diminish the role of the assignment by using lights of equal angles, π/k each for k lights. We proceed in two stages, the first of which permits $k > n$ and multiple assignment of lights to the same vertex.

3 Multiple Equal Lights

Here is the specific question on which we next focus:

> **Q3.** Given a convex polygon P of n vertices, a set of k vertex floodlights each with angle $\alpha = \pi/k$ (perhaps with $k > n$), and a fixed assignment of the lights to vertices of P, can the lights always be oriented to fully illuminate P?

We will show the answer to Q3 is again NO by a modification of the example used to answer Q2.

Theorem 3.1 *The polygon $\mathbb{P}$ in Fig. 1 cannot be covered by k equal $\alpha = \pi/k$ lights, for sufficiently large odd k, under the following assignment: one α-light each at v_0, v_2, and v_4, and $(k-3)/2$ α-lights each at v_1 and at v_3. Therefore the answer to Q3 is* NO.

We choose k to be very large, so that $\alpha = \pi/k$ is correspondingly small. Note that v_1 and v_3 are each assigned a total angle of $\pi/2 - 3\alpha/2$, close to $\pi/2$. The main task of the proof is to show that the freedom to rotate independently each of the many α-lights at these two vertices is insufficient to cover $\mathbb{P}$.

Partition $\mathbb{P}$ into a triangle $T = (v_1, v_2, v_3)$ and quadrilateral $Q = (v_0, v_1, v_3, v_4)$. The proof strategy is as follows. First we concentrate on T, and show in Lemma 3.2 that one of v_1 or v_3 must use nearly 45° of its lights to shine into T. The sense of "nearly" is controlled by α. Lemma 3.3 then shows that what remains from this light, and the entire other light, are not enough to cover Q for sufficiently small α.

We will call a ray from vertex v *black* if the lights at v do not cover it. Note a black ray from v might be partially or wholly covered by a light from some other vertex. The total angular extent of black rays from v play an important role in the proofs to follow.

Lemma 3.2 *For any $\epsilon > 0$, we may choose $\alpha > 0$ so that at least $\pi/4 - \epsilon$ of either v_1 or v_3's lights must shine into T.*

Proof: Suppose that less than $\pi/4 - \epsilon$ of v_3's lights shine into T. Since angle$(v_1, v_3, v_2) = \pi/4$, a total angle of at least ϵ black rays emanate from v_3 into T. We now argue that there must exist at least one black ray B that misses both v_3v_1 and v_3v_2 by $\geq \epsilon/2$.

Suppose to the contrary that all ϵ of the black rays from v_3 aimed into T either form an angle with v_3v_1, or with v_3v_2, of strictly less than $\epsilon/2$. This means that all the black rays from v_3 must fall within the two open wedges shaded dark in Fig. 2. But the sum of the angular extent of these wedges is strictly less than ϵ, contradicting the fact that the black rays constitute $\geq \epsilon$ in angle.

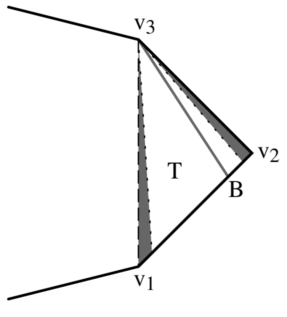

Figure 2: There must exist a black ray B that aims outside of the two shaded $\epsilon/2$-wedges near v_1 or v_2.

Let B be a black ray from v_3 that aims at least $\beta \geq \epsilon/2$ away from either v_1 or v_2. Now we argue that not much of B can be covered by the single α-lights at v_0, v_2, and v_4, so that most of it must be covered by the lights at v_1.

For a fixed β, the fraction of B's length that can be covered by an α-light at v_2 can be given as small an upper bound as desired by choosing α small. This can be seen as follows. First, it is clear that a fixed α-light at v_2 can cover more of B the closer B aims toward v_2. (This is the main reason for bounding B away from v_2 by β.) So consider B oriented as close to v_3v_2 as permitted, at an angle of β from this edge (aimed along the boundary of the shaded $\epsilon/2$-wedge in Fig. 2). Second, it is clear that a fixed α-light can cover more of this particular B by aiming the light toward v_3, the root of B. Third, the fraction x of B that can be covered in this situation approaches α/β as the angles get small, as illustrated in Fig. 3. So for fixed β, this fraction can be made as small as desired by choosing α small.

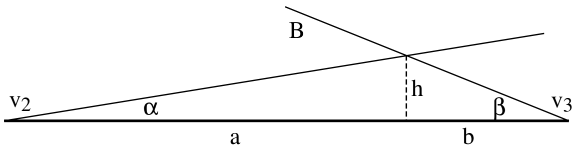

Figure 3: For fixed β, the portion of B covered by the left α-light is roughly proportional to the size of α. $h = a\tan\alpha = b\tan\beta$, so $b/a = \tan\alpha/\tan\beta \approx \alpha/\beta$ for small angles.

The α-lights at v_0 and v_4 cover even smaller portions of B, since they cut B at a relatively large angle. Therefore, given any fraction $x > 0$, we can ensure that the three single α-lights cover together no more than x of B's length, regardless of where B lies in T, by selecting α sufficiently small. Note that α is a function of x and β.

Suppose now that no more than a fraction x of B is covered. Then the remaining $1-x$ portion of B must be covered by the lights at v_1. Note that because B must aim at least β away from v_1, this portion subtends some positive angle $\leq \pi/4$ from v_1. If $x = 0$, B subtends an angle of exactly $\pi/4$ from v_1. For $x > 0$, let $\theta(x)$ be the smallest angle subtended by the $1-x$ uncovered portion of B, over all orientations of B, and over all possible ways that the fraction x covered from the other lights might be arranged along B. Although it would not be easy to compute $\theta(x)$ for a given x, it is clear that $\theta(x) \to \pi/4$ as $x \to 0$ (because when $x = 0$, all of B subtends $\pi/4$).

Let $\epsilon_1(x) = \pi/4 - \theta(x)$; we know that at least $\pi/4 - \epsilon_1(x)$ of v_1's lights must shine into T. But we reached this conclusion from the assumption that less than $\pi/4 - \epsilon$ of v_3's lights shine into T. Notice that for a fixed ϵ, we can make ϵ_1 as small as desired by arranging for x to be as small as necessary, which we can do by choosing α sufficiently small. In particular we can arrange for $\epsilon_1 \leq \epsilon$, in which case we establish that at least $\pi/4 - \epsilon$ of either v_1 or v_3's lights aim into T. □

Without loss of generality, we will assume that nearly 45° of v_1's lights shine into T, which leaves just a bit more than 45° to shine into Q. We now show that this implies that Q cannot be covered.

Lemma 3.3 *If at most $\pi/4 + \epsilon$ of v_1's lights shine into Q, with $\pi/4 + \epsilon < \operatorname{atan}(4/3) \approx 53°$, then we may choose α so that Q cannot be illuminated by the other vertex lights.*

Proof: The angle at v_3 in Q exceeds $\pi/2$ by $\phi = \operatorname{atan}(1/4) \approx 14°$, so even if all of the lights at v_3 (whose total angle is $\pi/2 - 3\alpha/2$) shine into Q, there must be a total of $\phi + 3\alpha/2$ black rays from v_3 in Q. By an argument analogous to that used in the previous lemma, this guarantees the existence of a black ray B from v_3 into Q that makes an angle of at least $\beta = (\phi + 3\alpha/2)/4$ with each of the segments v_3v_0, v_3v_1, and v_3v_4. See Fig. 4. Now we argue that not much of B can be covered by the single α-lights at v_0, v_4, and v_2, so that most of B must be covered by the lights at v_1.

The argument is similar to that used in the previous lemma. Because B is at least β off from v_3v_0 and v_3v_4 (and makes a much larger angle with v_3v_2), the total fraction x of B that can be collectively covered by those three ver-

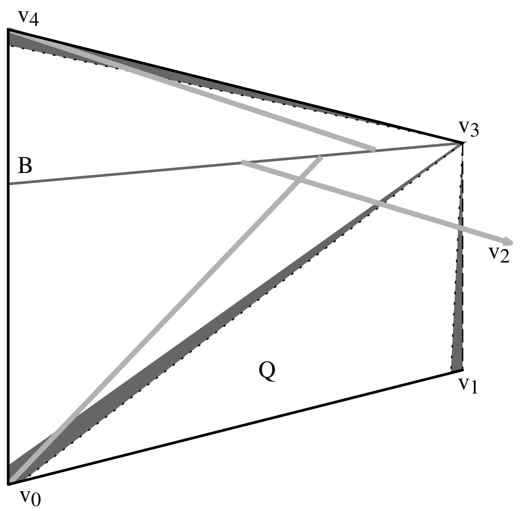

Figure 4: There must exist a black ray B that aims outside of the three darkly-shaded wedges. Only a small fraction x of B can be covered by α lights from v_0, v_4, and v_2 (lightly-shaded beams). The remainder must be covered from v_1.

tex lights can be made as small as desired by choosing α appropriately small.

Now suppose only a fraction x of B is covered by those three lights. The remaining $1-x$ portion of B must be covered by the lights at v_1. By the assumption of the lemma, only a total light angle of $\pi/4+\epsilon$ from v_1 is available to shine into Q. The angle that B subtends from v_1 in Q depends on B's orientation, but its minimum θ_m is larger than $\operatorname{atan}(4/3) \approx 53°$, which is the angle subtended from v_1 by the segment v_3v_4 (larger because B cannot aim directly along this segment). So $\operatorname{atan}(4/3) < \theta_m$. By the assumption of the lemma, v_1 has strictly less than this minimum to offer: $(\pi/4+\epsilon) < \operatorname{atan}(4/3) < \theta_m$.

Let $\theta(x)$ be the smallest angle subtended by the uncovered $1-x$ portion of B from v_1, over all orientations of B and all possible layouts of x along B. Again $\theta(x)$ might be difficult to compute, but as $x \to 0$, $\theta(x) \to \theta_m$. Now choose x (by choosing α small enough) so that $\theta(x) > \pi/4+\epsilon$. Then the uncovered portion of B subtends an angle greater than is available at v_1 for shining into Q, and therefore Q cannot be covered. □

We can finally prove Theorem 3.1:

Proof: Choose $\epsilon = 5°$ in Lemma 3.2. That lemma guarantees that at least $40°$ of v_1's light shines into T. Let the α needed to achieve this be α_T.

This leaves no more than $50°$ of v_1's light for Q, a value which satisfies the premises of Lemma 3.3. Applying that lemma, we know that by chosing α sufficiently small, we can guarantee that the uncovered portion of B subtends more than $50°$ from v_1, and so Q cannot be covered. Let the α determined by this lemma be α_Q.

Finally, choose $\alpha = \min(\alpha_T, \alpha_Q)$. The corresponding value of k is $\lceil \pi/\alpha \rceil$. □

Converting this existence proof to an explicit counterexample would result in a large value of k. However, it is quite likely that the theorem is true for much smaller values of k; perhaps for $k = 45$, $\alpha = 4°$, the theorem holds. The authors have not been successful in finding light orientations that cover $\mathbb{P}$ in this case. Fig. 5 shows one attempt.

4 One Equal Light per Vertex

We now strengthen the previous result by restricting each vertex to receiving just one light, and at the same time fixing $k = n$. So in the following question, the notion of assignment has disappeared entirely:

> **Q4.** Given a convex polygon P of n vertices, a set of n vertex floodlights each with angle $\alpha = \pi/n$, placed one per vertex, can the lights always be oriented to fully illuminate P?

Theorem 4.1 *There is a polygon $\mathbb{P}'$ of n vertices that cannot be covered by n equal $\alpha = \pi/n$ lights, one per vertex. Therefore the answer to Q4 is* NO.

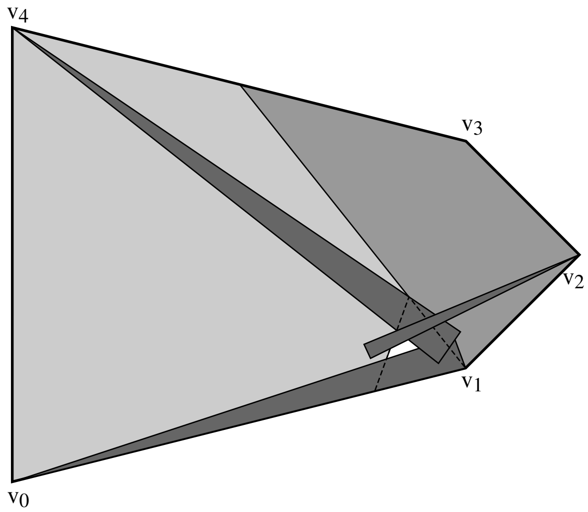

Figure 5: With $k = 45$, $\alpha = 180°/45 = 4°$. Vertices v_1 and v_3 each are assigned 21 lights, with a total angle of 84°, arranged in this case to cover a wedge of that angle. The orientations of the lights at v_0, v_2, and v_4 shown are insufficient to completely cover the dashed triangular region left uncovered by the lights at v_1 and v_3. Compare Fig. 1.

Proof: [Sketch] $\mathbb{P}'$ is a modification of $\mathbb{P}$ as follows. The vertices v_1 and v_3 are each replaced by $j = (k-3)/2$ vertices, where k is a number of lights for which Theorem 3.1 holds. (So note that n here is the same as k in that theorem.) The v_3 replacements are laid out collinear along the v_3v_4 edge of $\mathbb{P}$, bunched closely within a small distance δ of v_3. Call them $v_3 = v_{3_0}, v_{3_1}, \ldots, v_{3_j}$. (See Fig. 6.) The v_1 replacements are placed symmetrically.

Now we follow the proof of Theorem 3.1, arguing that for sufficiently small δ, the "slack" in that proof enables us to show that $\mathbb{P}'$ cannot be covered. One difference is that there may be no entirely black ray from v_3 even when not too much light from v_{3_i} aims into T. This is illustrated in Fig. 6. The important point is that the v_{3_i} lights of $\mathbb{P}'$ cannot cover much more than the v_3 lights of $\mathbb{P}$.

□

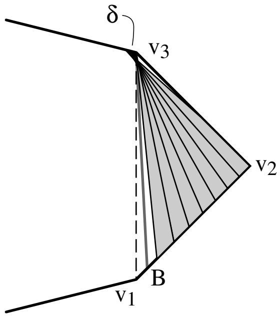

Figure 6: The lights at the v_{3_i} vertices combine to cover a portion of T, leaving some ray B largely uncovered.

This sketch is unsatisfactory in that it only shows the existence of counterexamples. Constructing an explicit counterexample from the proof would be tedious and result in a large number n of vertices, as well as vertices quite close together. But from the evidence of Fig. 5, we suspect that the theorem holds for $n = 45$.

5 Conclusion

Q4 is a specialization of Q1, which is a specialization of Q0. So the NO answer to Q4 implies NO answers to Q1 and to Q0. We mentioned in the Introduction that these questions remain open for small values of k. We have shown that four $\pi/4$ vertex lights suffice to illuminate any quadrilateral (so the answer to Q4 is YES for $n = 4$), but we do not yet know if four $\pi/4$ vertex lights suffice to illuminate all convex polygons of $n > 4$ vertices, or if five $\pi/5$ lights suffice to illuminate all convex pentagons.

References

[BGL+93] P. Bose, L. Guibas, A. Lubiw, M. Overmars, D. Souvaine, and J. Urrutia. The floodlight problem. In *Proc. 5th Canad. Conf. Comput. Geom.*, pages 399–404, Waterloo, Canada, 1993.