On the Development of the Intersection of a Plane with a Polytope

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Abstract

Define a "slice" curve as the intersection of a plane with the surface of a polytope, i.e., a convex polyhedron in three dimensions. We prove that a slice curve develops on a plane without self-intersection. The key tool used is a generalization of Cauchy's arm lemma to permit nonconvex "openings" of a planar convex chain.

Key words: polyhedron, polytope, development, unfolding
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1 Introduction

Let P be a 3-polytope, a convex, bounded polyhedron in $\mathbb{R}^3$, and Π a (two-dimensional) plane in $\mathbb{R}^3$. The intersection $\Gamma = P \cap \Pi$ is either a vertex, edge, or face of P, or it is a closed curve we will call a *slice curve*. Viewed in $\mathbb{R}^3$, a slice curve is a convex polygon, but viewed as a curve on P it is more complex. The main result of this paper is that a slice curve "develops" or "unfolds" to a simple (i.e., non-self-intersecting) curve in the plane.

Development of a curve Γ on the surface of a smooth convex body may be viewed as the trace of Γ when the convex body is rolled on a plane without slippage so that Γ is always the point of contact. For a polytope, development may be viewed as the result of unfolding flat a narrow strip surrounding Γ. More formally, let $c_0, c_1, \ldots, c_n$ be the *corners* of Γ, the points at which Γ crosses a polytope edge with a dihedral angle different from π, or at which Γ meets a polytope vertex. Orient Γ so each of its edges has a direction. There

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are two *surface angles* defined at each point $p \in \Gamma$: $\rho(p)$, the total incident face angle at p to the right of the directed curve Γ at p, and $\lambda(p)$, the angle to the left. Only at a corner c_i of Γ is either surface angle different from π. If c_i is not a vertex of P, then $\rho(p) + \lambda(p) = 2\pi$; at a vertex the two angles sum to less than 2π. Define the *right development* of Γ to be a planar drawing of the polygonal chain Γ as the chain $B = (b_0, b_1, \ldots, b_n)$ with the same edge lengths, $|b_i b_{i+1}| = |c_i c_{i+1}|$ for $i = 0, \ldots, n-1$, and with exterior angle ρ_i to the right of b_i the same as the right surface angle $\rho(c_i)$, for all i. Define *left development* similarly. If Γ avoids vertices of P, the right and left developments are congruent, and are called *the* development of Γ. If Γ includes one or more vertices, then any equal-lengths polygonal chain that at b_i turns an angle between $\rho(c_i)$ to the right and $\lambda(c_i)$ to the left is called a *development* of Γ. Our main theorem (Theorem 5) is that every development of a slice curve is non-self-intersecting.

Although a slice curve Γ is a convex polygon in $\mathbb{R}^3$, its development is in general not convex, as illustrated in Fig. 1, which shows two slices through a cube. This figure also makes clear the connection to unfolding a polyhedral surface in the plane, a process of considerable practical importance in manufacturing, and yet mathematically unresolved [1]. In particular, it remains unclear whether every polytope may be unfolded without overlap by cutting along edges (*edge-unfolded*). The result of this paper may provide insight into this fundamental open problem, a point to which we return in the Discussion (Section 5). We will also see in that section a connection to polymer folding.

Our proof relies on a little-known generalization of "Cauchy's arm lemma," which is explained in Section 2. The issue of self-intersection is addressed in Section 3, and the curve development result is proved in Section 4.

2 Cauchy's Arm Lemma Extended

We start by explaining Cauchy's arm lemma, which says, roughly, that if a convex chain is opened, the distance between its endpoints gets longer. Let $A = (a_0, a_1, \ldots, a_n)$ be an n-link polygonal chain in the plane with n fixed edge lengths $\ell_i = |a_i a_{i+1}|$, $i = 0, \ldots, n-1$. We call the vertices a_i the *joints* of the chain, a_0 the *shoulder*, and a_n the *hand*. It will be convenient for the needed generalization to phrase Cauchy's lemma in terms of "turn angles." Define the *turn angle* α_i at joint a_i, $i = 1, \ldots, n-1$ to be the angle in $[-\pi, \pi]$ that turns the vector $a_i - a_{i-1}$ to $a_{i+1} - a_i$, positive for left (counterclockwise) and negative for right (clockwise) turns.

Define an open polygonal chain A to be *convex* if the convex hull of its vertices is a (nondegenerate) convex polygon with all joints on the hull. The turn angles

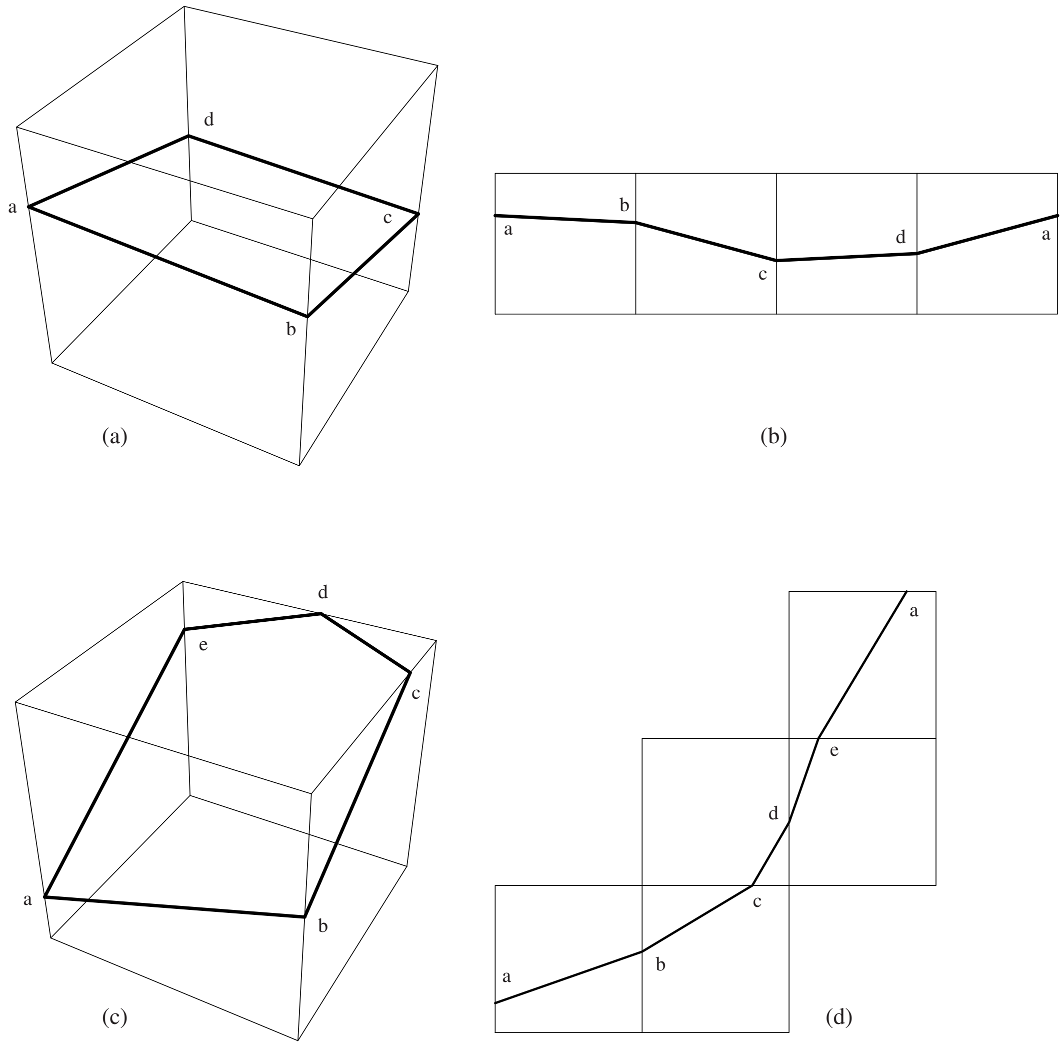

Fig. 1. (a) Slice curve (a, b, c, d); (b) its development. (c) slice curve (a, b, c, d, e); (d) its development.

for a convex chain all lie in $[0, \pi)$.

We can view the configuration of a polygonal chain A to be determined by two vectors: the fixed edge lengths $L = (\ell_0, \ldots, \ell_{n-1})$ and the variable turn angles $\alpha = (\alpha_1, \ldots, \alpha_{n-1})$. Let $\mathcal{C}_L(\alpha) = A$ be the configuration so determined.

We may state Cauchy's arm lemma in the following form:

Theorem 1 *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [0, \alpha_i] \tag{1}$$

we must have $|b_n b_0| \geq |a_n a_0|$.

Cauchy's lemma is sometimes known as Steinitz's lemma, because Steinitz noticed and corrected an error in the original proof [2] a century after Cauchy [3, p. 235]. Many proofs of Cauchy's lemma are now known, e.g., [4, p. 64].

Our main tool is a generalization of Cauchy's lemma that replaces the 0 in Eq. (1) by $-\alpha_i$, and is otherwise identical:

Theorem 2 *If $A = \mathcal{C}_L(\alpha)$ is a convex chain with fixed edge lengths L, and turn angles α, then in any reconfiguration to $B = \mathcal{C}_L(\beta)$ with turn angles $\beta = (\beta_1, \ldots, \beta_{n-1})$ satisfying*

$$\beta_i \in [-\alpha_i, \alpha_i] \tag{2}$$

we must have $|b_n b_0| \geq |a_n a_0|$.

In both of the above theorems, both chains A and B are planar. Call a reconfiguration $B = \mathcal{C}_L(\beta)$ of a convex chain $A = \mathcal{C}_L(\alpha)$ which satisfies the constraints of Eq. (2) a *valid reconfiguration*, and call the vector of angles β *valid angles*. The intuition, illustrated in Fig. 2, is perhaps best captured by Chern [5, p. 36]: "if an arc is 'stretched,' the distance between its endpoints becomes longer." See also our Java applet illustrating the theorem and its corollaries. [2]

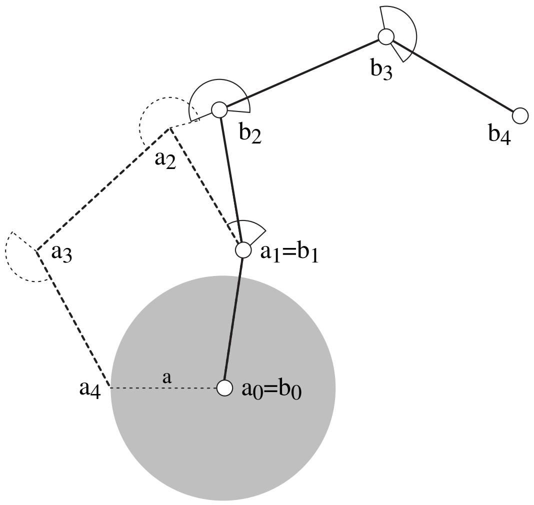

Fig. 2. A valid reconfiguration of a 4-link convex chain A to chain B leaves b_n outside the "forbidden shoulder disk" of radius $|a_0 - a_n|$. The valid angle ranges are marked by circular arcs.

This theorem was first stated by S. S. Chern [5, p. 39], who proved a generalization to smooth space curves based on an earlier theorem of Schur [6]. He only claims without proof the specialization to "sectionally smooth curves,"

[2] `http://cs.smith.edu/~orourke/Cauchy/` .

which include polygonal chains. Chern's differential geometry proof for smooth curves, however, can be modified to establish Theorem 2 in the plane. In addition, independent inductive proofs are possible, as I show in [7].

We will need the following easy extension of the distance inequality to every point of the chain.

Corollary 3 *Let $A = \mathcal{C}_L(\alpha)$ be a convex chain as in Theorem 2, and let $p_1, p_2 \in A$ be any two distinct points of the chain. Then in any valid reconfiguration B, the points $q_1, q_2 \in B$ corresponding to p_1 and p_2 satisfy $|q_1 q_2| \geq |p_1 p_2|$, i.e., they have not moved closer to one another.*

Proof: Without loss of generality, assume that p_1 occurs before p_2 on A, i.e., is a shorter length along A from a_0. Let A' be the chain A with the portion prior to p_1, and the portion after p_2, removed: $A' = (p_1, \ldots, a_i, \ldots, p_2)$. Let B' be the corresponding clipped version of B: $B' = (q_1, \ldots, q_2)$. If p_1 and p_2 lie on the same edge of A, then the claim follows trivially. Otherwise, A' is a convex chain because A is and the convex hull is nondegenerate. B' is a valid reconfiguration of A', for none of the angle ranges satisfied by B have been altered. Applying Theorem 2 with p_1 and q_1 playing the role of the shoulder, and p_2 and q_2 the role of the hand, establishes the claim. $\square$

3 Noncrossing of Straightened Curve

Define a polygonal chain to be *simple* if nonadjacent segments are disjoint, and adjacent segments intersect only at their single, shared endpoint. By our nondegeneracy requirement, convex chains are simple. In particular, any opening of a convex chain via Cauchy's arm lemma (Theorem 1) remains simple because it remains convex. We now establish a parallel result for the generalized straightening of Theorem 2. We generalize slightly to permit the convex chain to start with the hand at the shoulder.

Theorem 4 *If $A = (a_0, \ldots, a_n) = \mathcal{C}_L(\alpha)$ is a closed convex chain with n fixed edge lengths L and turn angles α, closed in the sense that $a_n = a_0$, then any valid reconfiguration to $B = \mathcal{C}_L(\beta)$ is a simple polygonal chain.*

Proof: Suppose to the contrary that B is nonsimple. Let q_2 be the first point of B, measured by distance along the chain from the shoulder b_0, that coincides with an earlier point $q_1 \in B$. Thus q_1 and q_2 represent the same point of the plane, but different points along B. See Fig. 3. Because B is nonsimple, these "first touching points" exist, and we do not have both $q_1 = b_0$ and $q_2 = b_n$ (because that would make B a simple, closed chain). Let p_1 and p_2 be the points of A corresponding to q_1 and q_2.

Corollary 3 guarantees that $|q_1q_2| \geq |p_1p_2|$. But $|q_1q_2| = 0$, and because the q's do not coincide with the original hand and shoulder, $|p_1p_2| > 0$. This contradiction establishes the claim. □

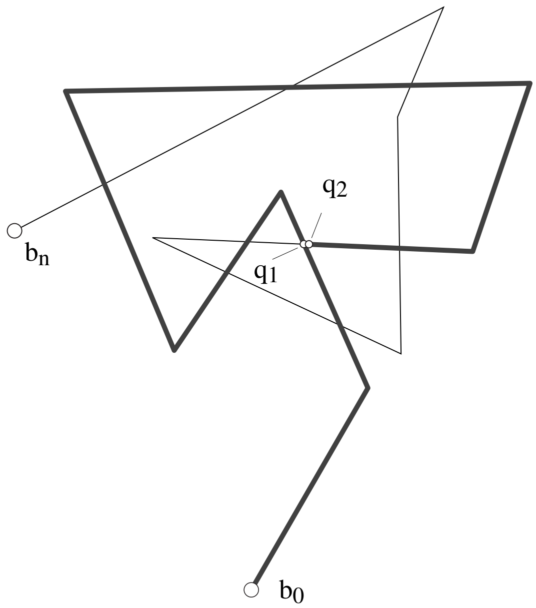

Fig. 3. Violation of Theorem 2. $q_1 = q_2$ is the first point of self-contact; the initial portion of B, up to q_2, is highlighted.

4 Development of Slice Curves

We first relate the current task to earlier work on curve development. Let $\Gamma = (c_0, c_1, \ldots, c_n)$ be an oriented polygonal curve on a polytope P. One can define a curve to be *convex* if the left angle $\lambda(c_i) \leq \pi$ for all i. (Such curves bound "geodesically convex" regions.) It was established in [8] that convex curves develop to simple curves. There the main tool was Theorem 1, in that it was shown that development of a convex curve can be viewed as opening a convex chain according to Cauchy's arm lemma. Now we employ a similar strategy to slice curves Γ. As we have seen (Fig. 1), these are not convex, but we show below that development corresponds to a valid reconfiguration of the chain, permitting Theorem 2 and Theorem 4 to establish that development is again simple.

Theorem 5 *Let $\Gamma = P \cap \Pi$ be a slice curve on a polytope P. Then Γ develops on a plane to a simple (noncrossing) polygonal curve.*

Proof: We first dispense with the degenerate intersections, where there is zero volume of P to one side of Π. Then Π must intersect P in either a face, an edge, or a vertex. In all cases, Γ develops as is, and there is nothing to prove. Henceforth we assume that the slice is nondegenerate.

Again let $\Gamma = (c_0, c_1, \ldots, c_n)$, with c_i the corners. We will first prove that the right development of Γ satisfies the theorem. Let Q be the convex polygon in plane Π bound by Γ. Let $\phi_i \in (0, \pi)$ be the internal convex angle of Q at c_i. See Fig. 4. Our aim is to prove that these internal angles are related to the

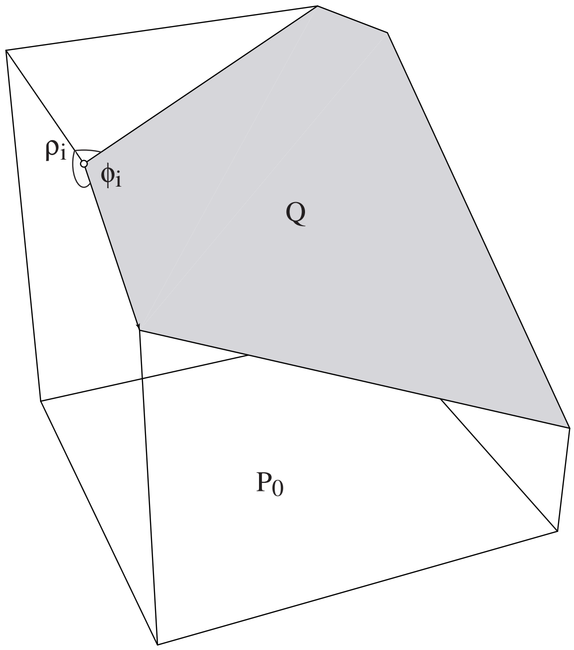

Fig. 4. Q is a slice through a cube P. Angles are shown at corner c_i of $\Gamma = \partial Q$.

right surface angles $\rho_i = \rho(c_i)$ as follows:

$$\phi_i < \rho_i < 2\pi - \phi_i \ . \tag{3}$$

First note that, by our nondegeneracy assumption, the intersection of P with the halfspace below (and including) Π is a polytope; call it P_0. P_0 has Q as a top face, and the corners of Γ as vertices. The total angle incident to vertex c_i of P_0 is $\phi_i + \rho_i$, because P_0 includes the entire right surface angle at c_i; see Fig. 4. Because P_0 is a polytope, and c_i is a (nonflat) vertex of P_0, this sum must be less than 2π, and from $\phi_i + \rho_i < 2\pi$ the right hand inequality of Eq. (3) follows.

Repeating the argument to the other side, let P_1 be the intersection of P with the halfspace above Π. Because the left surface angle $\lambda(c_i)$ on P is no more than $2\pi - \rho_i$ (with equality if c_i is not a vertex of P), then the total angle incident to vertex c_i of P_1 is no more than $\phi_i + 2\pi - \rho_i$. Again because P_1 is a polytope, this sum must be less than 2π, and from $\phi_i + 2\pi - \rho_i < 2\pi$ the left hand inequality of Eq. (3) follows.

Let $A = (a_0, a_1, \ldots, a_n)$ be a polygonal chain representing convex polygon Q, with a_i corresponding to c_i. The turn angle α_i at a_i is $\alpha_i = \pi - \phi_i$, i.e.,

$$\phi_i = \pi - \alpha_i \ . \tag{4}$$

Let $B = (b_0, b_1, \ldots, b_n)$ be the development of Γ in the plane, again with b_i corresponding to c_i. The turn angle β_i at b_i is determined by the right surface angle: $\beta_i = \rho_i - \pi$, i.e.,

$$\rho_i = \pi + \beta_i \ . \tag{5}$$

Substituting Eqs. (4) and (5) into Eq. (3) yields

$$\begin{array}{ccccc} \pi - \alpha_i & < & \pi + \beta_i & < & 2\pi - (\pi - \alpha_i) \\ -\alpha_i & < & \beta_i & < & \alpha_i \end{array}$$

i.e., $\beta_i \in (-\alpha_i, \alpha_i)$. Thus we see that B is a valid reconfiguration (Sec. 2) of A, and Theorem 4 applies and establishes the claim that it is simple.

Reversing the orientation of Γ and reapplying the argument settles the same claim for the left development of Γ. Finally, note that any other development of Γ turns at each joint an angle somewhere between that determined by the left and the right surface angles (when they differ, at vertices of P). Thus, that turn angle must also fall within $(-\alpha_i, \alpha_i)$, and so the theorem holds for all developments of Γ. $\square$

Because the Chern-Schur Theorem encompasses smooth curves, Theorem 5 should generalize to slice curves for any convex body B.

5 Discussion

Nonoverlapping unfoldings of polyhedra have applications to manufacturing, for such unfoldings may be bent to form three-dimensional shapes [1]. As mentioned, it is unknown whether every convex polyhedron may be edge-unfolded without overlap, although it was recently established that some nonconvex, triangulated polyhedra cannot be so edge-unfolded [9]. Any simple polygonal curve drawn inside a nonoverlapping unfolding of a polytope P folds up to a curve Γ on P that develops to a simple curve. Thus slice curves (and convex curves via [8]) identify portions of the surface that can unfold without overlap, and may provide some guidance for edge-unfolding a polytope. For example, it is natural to wonder if a vertex-free strip sandwiched between two parallel slicing planes may be unfolded without overlap. That this question is not settled by Theorem 5 is illustrated by the example[3] in Fig. 5, which demonstrates that not every edge-unfolding of such a strip avoids overlap. However,

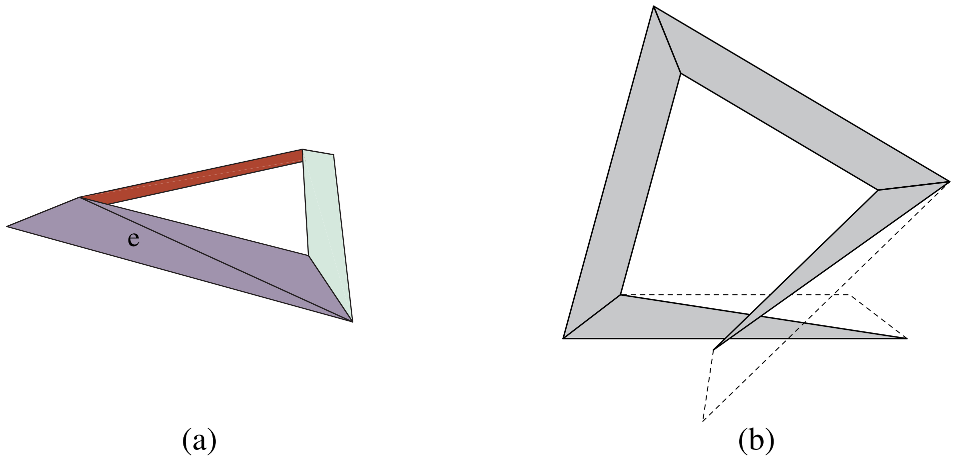

Fig. 5. (a) Parallel slices through a tetrahedron. (b) An edge-unfolding of the strip obtained by cutting e.

I conjecture that cutting a shortest path between the two slice curves develops the strip without overlap.

A second possible application of the line of investigation in this paper is to the folding of molecules in space. For example, Soss and Toussaint have investigated reconfigurations of polymer-like linkages in $\mathbb{R}^3$ [10]. One of the chemically natural constraints they consider is fixing the bond angles (the angles at joints between consecutive edges), allowing motions they call "edge spins." Such edge-spin reconfigurations satisfy the Chern-Schur theorem, one version of which is Theorem 2, but with the reconfiguration into space (rather than in the plane), and with bond angles β. Keeping the angles fixed trivially satisfies Eq. (2), and so establishes that edge-spin reconfigurations only increase the distance between the polymer end-atoms in comparison to their planar separation.

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[3] Discovered by Erik Demaine and Anna Lubiw.

References

[1] J. O'Rourke, Folding and unfolding in computational geometry, in: Discrete Comput. Geom., Vol. 1763 of Lecture Notes Comput. Sci., Springer-Verlag, 2000, pp. 258–266.

[2] A. L. Cauchy, Sur les polygones et les polyèdres, seconde mémoire, J. École Polytechnique XVIe Cahier, Tome IX (1813) 113–148, oEuvres Complètes, IIe Sèrie, Vol. 1, Paris, 1905, 26–38.

[3] P. Cromwell, Polyhedra, Cambridge University Press, Cambridge, 1997.

[4] M. Aigner, G. M. Ziegler, Proofs from THE BOOK, Springer-Verlag, Berlin, 1998.

[5] S. S. Chern, Curves and surfaces in Eucidean space, in: S. S. Chern (Ed.), Studies in Global Geometry and Analaysis, Vol. 4 of MAA Studies in Mathmatics, Math. Assoc. Amer., 1967, pp. 16–56.

[6] A. Schur, Über die Schwarzche Extremaleigenschaft des Kreises unter den Kurven konstantes Krümmung, Math. Ann. 83 (1921) 143–148.

[7] J. O'Rourke, An extension of Cauchy's arm lemma with application to curve development, in: Discrete Comput. Geom., Vol. 2098 of Lecture Notes Comput. Sci., Springer-Verlag, 2001, pp. 280–291.

[8] J. O'Rourke, C. Schevon, On the development of closed convex curves on 3-polytopes, J. Geom. 13 (1989) 152–157.

[9] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, J. Snoeyink, Ununfoldable polyhedra with convex faces, Comput. Geom. Theory Appl. to appear.

[10] M. Soss, G. T. Toussaint, Geometric and computational aspects of polymer reconfiguration, J. Math. Chemistry 27 (4) (2000) 303–318.