Common Edge-Unzippings for Tetrahedra

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Abstract

It is shown that there are examples of distinct polyhedra, each with a Hamiltonian path of edges, which when cut, unfolds the surfaces to a common net. In particular, it is established for infinite classes of triples of tetrahedra.

1 Introduction

The limited focus of this note is to establish that there are an infinite collection of "edge-unfolding zipper pairs" of convex polyhedra. A *net* for a convex polyhedron P is an unfolding of its surface to a planar simple (nonoverlapping) polygon, obtained by cutting a spanning tree of its edges (i.e., of its 1-skeleton); see [DO07, Sec. 22.1]. Shephard explored in the 1970's the special case where the spanning tree is a Hamiltonian path of edges on P [She75]. Such *Hamiltonian unfoldings* were futher studied in [DDLO02] (see [DO07, Fig. 25.59]), and most recently in [LDD+10], where the natural term *zipper unfolding* was introduced. Define two polyhedra to be an *edge-unfolding zipper pair* if they have a zipper unfolding to a common net. Here we emphasize edge-unfolding, as opposed to an arbitrary zipper path that may cut through the interior of faces, which are easier to identify. Thus we are considering a special case of more general *net pairs*: pairs of polyhedra that may be cut open to a common net.

In general there is little understanding of which polyhedra form net pairs under any definition. See, for example, Open Problem 25.6 in [DO07], and [O'R10] for an exploration of Platonic solids. Here we establish that there are infinite classes of convex polyhedra that form edge-unfolding zipper pairs. Thus one of these polyhedra can be cut open along an edge zipper path, and rezipped to form a different polyhedron with the same property. In particular, we prove this theorem:

Theorem 1 *Every equilateral convex hexagon, with each angle in the range $(\pi/3, \pi)$, and each pair of angles linearly independent over $\mathbb{Q}$, is the common edge-unzipping of three incongruent tetrahedra.*

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The angle restrictions in the statement of the theorem are, in some sense, incidental, included to match the proof techniques. The result is quite narrow, and although certainly generalizations hold, there are impediments to proving them formally. This issue will be discussed in Section 6.

2   Example

An example is shown in Figure 1. The equilateral hexagon is folded via a *perimeter halving* folding [DO07, Sec. 25.1.2]: half the perimeter determined by opposite vertices is glued ("zipped") to the other half, matching the other four vertices in two pairs. Because the hexagon is equilateral, the corresponding edge lengths match.

The resulting shape is a convex polyhedron by Alexandrov's theorem ([DO07, Sec. 23.3]), and a tetrahedron because it has four points at which the curvature is nonzero. Choosing to halve the perimeter at each of the three pairs of opposite vertices leads to the three tetrahedra illustrated. Note that each has three unit-length edges that form a Hamiltonian zipper path, as claimed in the theorem.

The main challenge is to show that the edges of the hexagon glued together in pairs become edges of the polyhedron, and so the unzipping is an edge-unzipping.

3   Distinct Tetrahedra

First we show that the linear independence condition in Theorem 1 ensures the tetrahedra will be distinct. We define two angles x and y to be *linearly independent over* $\mathbb{Q}$ if there are no rationals a and b such that $y = a\pi + bx$. We will abbreviate this condition to "linear independence" below.

We will show that the curvatures (2π minus the incident face angle) at the vertices of the tetrahedra differ at one vertex or more, which ensures that the tetrahedra are not congruent to one another. Let the vertices of the hexagon be $v_0, \ldots, v_5$, with vertex angles $\alpha_0, \ldots, \alpha_5$.

We identify the three tetrahedra by their halving diagonals, and name them T_{03}, T_{14}, T_{25}. We name the curvatures at the four vertices of the tetrahedra $\omega_1, \omega_2, \omega_3, \omega_4$. These curvatures for the three tetrahedra (see Figure 1) are as follows:

Tetrahedron halving diagonal	Vertex Curvatures			
	ω_1	ω_2	ω_3	ω_4
T_{03} : $v_0 v_3$	$\frac{1}{2}(2\pi - \alpha_0)$	$\frac{1}{2}(2\pi - \alpha_3)$	$2\pi - (\alpha_1 + \alpha_5)$	$2\pi - (\alpha_2 + \alpha_4)$
T_{14} : $v_1 v_4$	$\frac{1}{2}(2\pi - \alpha_1)$	$\frac{1}{2}(2\pi - \alpha_4)$	$2\pi - (\alpha_0 + \alpha_2)$	$2\pi - (\alpha_3 + \alpha_5)$
T_{25} : $v_2 v_5$	$\frac{1}{2}(2\pi - \alpha_2)$	$\frac{1}{2}(2\pi - \alpha_5)$	$2\pi - (\alpha_0 + \alpha_1)$	$2\pi - (\alpha_3 + \alpha_4)$

Now we explore under what conditions could the four curvatures of T_{03} be identical to the four curvatures of T_{14}. (There is no need to explore other pos-
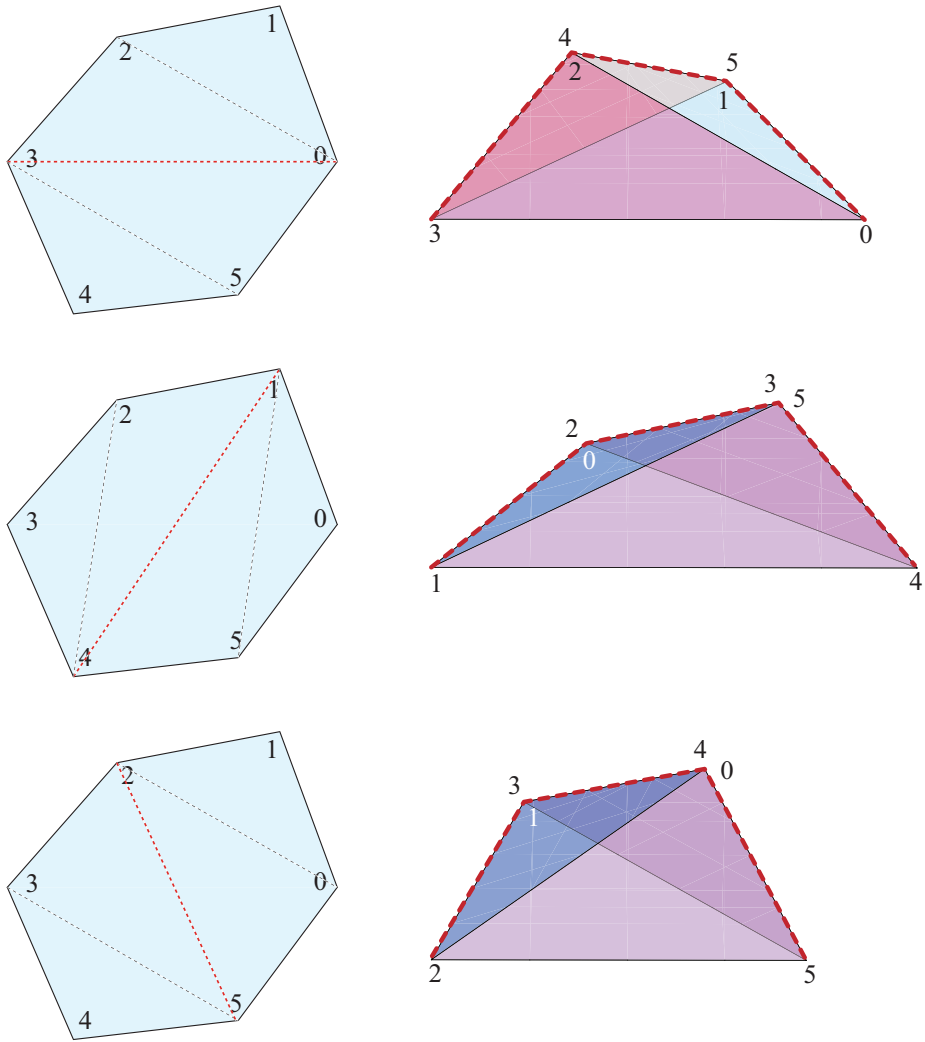
Figure 1: Three tetrahedra that edge-unzip (via the dashed Hamiltonian path) to a common equilateral hexagon.

sibilies, as they are all equivalent to this situation by relabeling the hexagons.) Let us label the curvatures of T_{03} without primes, and those of T_{14} with primes. First note that we cannot have $\omega_1 = \omega_1'$ or $\omega_1 = \omega_2'$, for then two angles must be equal: $\alpha_0 = a_1$ or $\alpha_0 = \alpha_4$ respectively. And equal angles are not linearly independent.

So we are left with these possibilities: $\omega_1 = \omega_3'$, or $\omega_1 = \omega_4'$. The first leads to the relationship $\alpha_2 = \pi - \frac{1}{2}\alpha_0$, a violation of linear independence. The second possibility, $\omega_1 = \omega_4'$, requires further analysis.

It is easy to eliminate all but these two possibilities, which map indices $(1, 2, 3, 4)$ to either $(4', 3', 1', 2')$ or $(4', 3', 2', 1')$:

$$\omega_1 = \omega_4' \ , \ \omega_2 = \omega_3' \ , \ \omega_3 = \omega_1' \ , \ \omega_4 = \omega_2'$$

and

$$\omega_1 = \omega_4' \ , \ \omega_2 = \omega_3' \ , \ \omega_3 = \omega_2' \ , \ \omega_4 = \omega_1'$$

Explicit calculation shows that the first set implies that $\alpha_4 = 2\pi - 2\alpha_2$, and the second implies that $\alpha_1 = \frac{3}{4}\pi - \alpha_0$.

Thus we have reached the desired conclusion:

Lemma 1 *Linear independence over $\mathbb{Q}$ of pairs of angles of the hexagon (as stated in Theorem 1) implies that the three tetrahedra have distinct vertex curvatures, and therefore are incongruent polyhedra.*

We have phrased the condition as linear independence of pairs of angles for simplicity, but in fact a collection of linear relationships must hold for the four curvatures to be equal. So the restriction could be phrased more narrowly. Note also we have not used in the proof of Lemma 1 the restriction that the angles all be "fat", $\alpha_i > \pi/3$. This will be used only in Lemma 3 below.

It would be equally possible to rely on edge lengths rather than angles to force distinctness of the tetrahedra. For example, we could demand that no two diagonals of the hexagon have the same length (but that would leave further work). Another alternative is to avoid angle restrictions, permitting all equilateral hexagons, but only conclude that "generally" the three tetrahedra are distinct. Obviously a regular hexagon leads to three identical tetrahedra.

The form of Theorem 1 as stated has the advantage of easily implying that an uncountable number of hexagons satisfy its conditions (see Corollary 1).

4 Shortest Paths are Edges

Now we know that the hexagons fold to three distinct tetrahedra. So there is a zipper path on each tetrahedron that unfolds it to that common hexagon. All the remaining work is to show that the zipper path is an edge path—composed of polyhedron edges. This seems to be less straightforward than one might expect, largely because there are not many tools available beyond Alexandrov's existence theorem.

It is easy to see that every edge of a polyhedron P is a shortest path between its endpoints. The reverse is far from true in general, but it holds for tetrahedra:

Lemma 2 *Each shortest path between vertices of a tetrahedron is realized by an edge of the tetrahedron.*

Proof: Note that there are $\binom{4}{2} = 6$ shortest paths between the four vertices, and six edges in a tetrahedron, so the combinatorics are correct.

Suppose a path $\rho = xy$ is a shortest path between vertices x and y of a tetrahedron T, but not an edge of T. Because each pair of vertices of a tetrahedron is connected by an edge, there is an edge $e = xy$ of T. Because ρ is not an edge of T, it cannot be realized as a straight segment in $\mathbb{R}^3$, because all of those vertex-vertex segments are edges of T. However, e is a straight segment in $\mathbb{R}^3$, and so $|e| < |\rho|$, contradicting the assumption that ρ is a shortest path. $\square$

The reason this proof works for tetrahedra but can fail for a polyhedron of $n > 4$ vertices is that the condition that every pair of vertices are connected by an edge fails in general. The polyhedra for which that holds are the "neighborly polyhedra" (more precisely, the 2-neighborly 3-polytopes).

5 Hexagon Edges are Tetrahedron Edges

With Lemma 2 in hand, it only remains to show that the pairs of matched unit-length hexagon edges are shortest paths on the manifold M obtained by zipping the hexagon. We use the labeling and folding shown in Figure 2(a), where v_1 and v_1' are identified, as are v_2 and v_2'. We need to show that both $v_0 v_1$ and $v_1 v_2$ are shortest paths ($v_2 v_3$ is symmetric to $v_0 v_1$ and follows by relabeling).

Let D_i be the geodesic disk of unit radius centered on v_i, on the zipped manifold M. If the disks D_0 and D_1, illustrated in Figure 2(a), are empty of other vertices, then the desired shortest paths are established. It is here that we will employ the assumption that the hexagon angles are greater than $\pi/3$.

First, for any unit-equilateral convex hexagon, without constraints on the angles, diagonals connecting opposite vertices are at least length 1: $|v_0 v_3| \geq 1$, $|v_1 v_2'| \geq 1$, and $|v_2 v_1'| \geq 1$. We now argue for this elementary fact (which is likely known in some guise in the literature).

Concentrating on D_0 and $v_0 v_3$ (all others are equivalent by relabeling), consider the quadrilateral (v_0, v_1, v_2, v_3). Because the hexagon is unit-equilateral, $|v_0 v_1| = 1$, $|v_1 v_2| = 1$, and $|v_2 v_3| = 1$. Assume for a contradiction that the diagonal is short: $|v_0 v_3| < 1$. Then it is not difficult to prove that the sum of the quadrilateral angles at the endpoints of this shorter side exceed π: $\angle v_3 v_0 v_1 + \angle v_2 v_3 v_0 > \pi$. Applying the same logic to the other half of the hexagon sharing diagonal $v_0 v_3$, the quadrilateral (v_3, v_2', v_1', v_0), we reach the conclusion that the sum of the hexagon angles at v_0 and v_3 exceeds 2π. Thus one of those angles exceeds π, contradicting the fact that the hexagon is convex.[1]

We may conclude from this analysis is that v_3 cannot lie inside D_0.

Suppose now that v_2' is inside D_0, as in Figure 2(b). Then, the angle α_1' at v_1' must be smaller than $\pi/3$, contradicting the angle minimum assumed in

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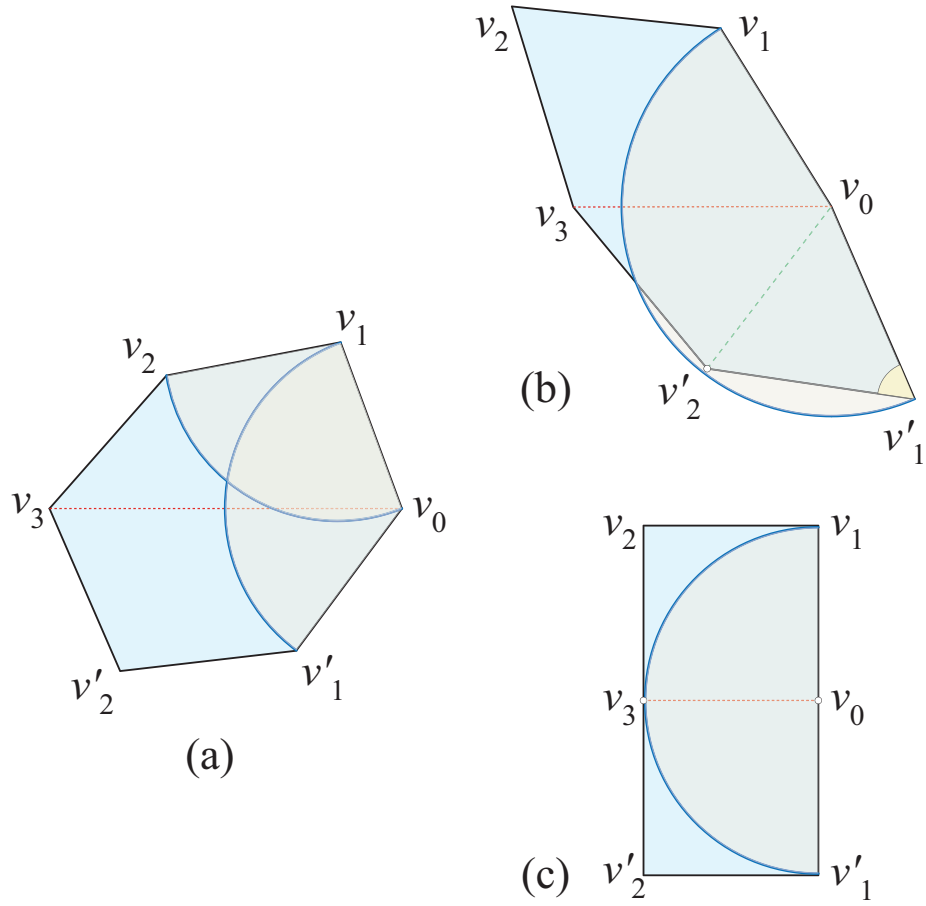[1] I thank Mirela Damian for this argument, which is simpler than my original proof.

Figure 2: (a) Labels for the zipping of a hexagon. The dashed edge shows the perimeter halving line, but there is no assumption that diagonal is a crease that becomes an edge of the tetrahedron. (b) When D_0 is not empty, some angle (here, α_1') is smaller than $\pi/3$. (c) A "degenerate" hexagon, with two angles π, which folds to a doubly covered square.

the theorem. The same clearly holds for v_2 inside D_0, as well as the other combinations. So the assumption that the hexagon is "fat" in the sense that none of its angles are small, guarantees that the disks D_0 and D_1 (and by symmetry, D_2 and D_3) are empty of vertices of the hexagon.

As is evident from Figure 2(b), however, even when no vertex is inside D_0, a portion of D_0 can fall outside the hexagon, which, because it is all zipped to a closed manifold M, means it re-enters the hexagon at the other copy of that edge. However, three facts are easily established. First, the "overhang" has width at most $h = 1 - \sqrt{3}/2$; see Figure 3. Second, no vertex of the "next" hexagon copy can lie inside that overhang (without violating convexity of the hexagon). Third, to even have a hexagon edge be partially interior (and so overhanging the disk into a third hexagon copy) requires some angle (β in the figure) to be very small, violating the $\pi/3$ minimum angle. Thus the overhang of a D_i disk beyond the original hexagon cannot encompass a vertex.
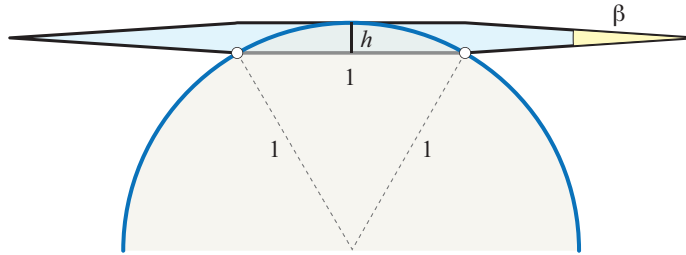

Figure 3: A disk D_i "overhangs" the original hexagon and enters another copy. Here $\beta = 2\sin^{-1}(h/2) < 8°$.

We may conclude:

Lemma 3 *The three unit-length edges of the hexagon, $\{v_0v_1, v_1v_2, v_2v_3\}$ are each shortest paths on the folded manifold M between their endpoint vertices.*

I have firm empirical evidence (via explorations in Cinderella) that Lemma 3 holds just as stated without any assumption that the hexagon is fat. But proving this formally seems difficult. Figure 4 illustrates a path ρ that spirals around the manifold from v_0 to v_1. Each such geodesic path candidate must be established to be at least length 1.

6 Discussion

We have now proved Theorem 1: Lemma 3 shows that the three unit-length edges forming a Hamiltonian path are shortest paths, Lemma 2 then implies that they are edges of the tetrahedron, and Lemma 1 establishes that the three tetrahedra are distinct.

Corollary 1 *There are an uncountable number of hexagons that satisfy the conditions of Theorem 1.*
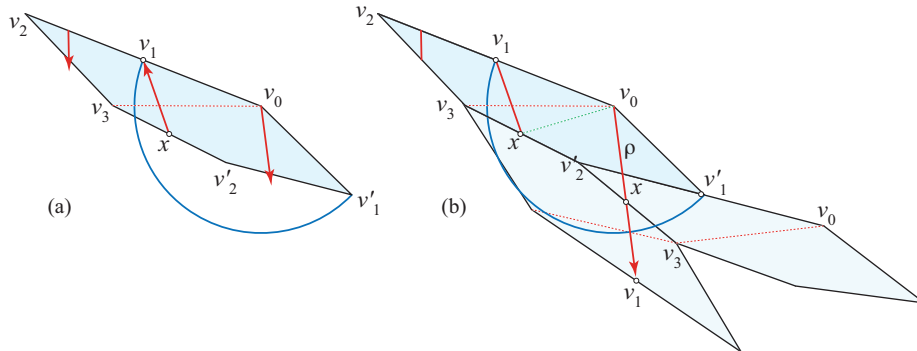
Figure 4: (a) A "spiral" path from v_0 to v_1. (b) The path flattened, crossing several copies of the hexagon.

Proof: The constraints that $\sum \alpha_i = 4\pi$ and $\pi/3 < \alpha_i < \pi$ clearly leave an uncountable number of solutions, in fact a 5-dimensional open set in $\mathbb{R}^5$ (the sixth angle is determined by the other five). (For example, a small 5-ball around

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4) = (\pi - \tfrac{1}{2}, \pi - \tfrac{1}{2}, \pi - \tfrac{1}{2}, \tfrac{\pi}{3} + \tfrac{1}{4}, \tfrac{\pi}{3} + \tfrac{1}{4})$$

with $\alpha_5 = \frac{\pi}{3} + 1$, is inside this set.) The constraint requiring independence over $\mathbb{Q}$ only excludes a countable number of 4-dimensional hyperplanes (e.g., $\alpha_2 = \pi - \frac{1}{2}\alpha_0$). These hyperplanes have zero measure in $\mathbb{R}^5$, and a countable union of sets of zero measure has zero measure,[2] leaving an uncountable number of solutions after excluding the hyperplanes. $\square$

The construction considered here generalizes to arbitary even n, although I do not see how to prove that the zipper path follows edges of the polyhedron:

Proposition 1 *For any even $n \geq 4$, an equilateral convex n-gon is the common zipper-unfolding of $n/2$ generally distinct polyhedra of $(n-2)$ vertices each.*

The regular-polygon version of this construction folds to what were called "pita polyhedra" in [DO07, Sec. 25.7.2].

I conjecture that all the zipper-paths in Proposition 1, for strictly convex equilateral convex n-gons, in fact follow polyhedron edges. Resolving this conjecture would require tools to determine when particular geodesic paths on an Alexandrov-glued manifold are edges of the resulting convex polyhedron.

References

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[2] I thank Qiaochu Yuan and Theo Buehler for guidance here, `http://math.stackexchange.com/questions/41494/`.

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