Quasigeodesics on the Regular Icosahedron

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Abstract. We show that there are exactly eleven simple closed quasigeodesics (excluding geodesics) on the regular icosahedron, up to congruence.

1 Introduction

Let P be a convex polyhedron. A ***geodesic*** is a curve on P that has exactly π surface to each side at every point (and so it cannot pass through a vertex), and if the curve is *non-self-crossing* and closed, we say that it is a ***simple closed geodesic***. For example, there are three simple closed geodesics on the regular icosahedron: the planar equatorial geodesic and two non-planar geodesics: [FF07]. Simple closed geodesics play an important role in the topology of manifolds.

Denote the vertices of the regular icosahedron P as v_i $(i = 1, 2, ..., 12)$, where v_{11} and v_{12} are a pair of antipodal vertices, v_i $(i = 1, 2, ..., 5)$ are adjacent to v_{11} in anticlockwise order, v_i $(i = 6, 7, ..., 10)$ are adjacent to v_{12} in clockwise order, and v_1v_6 is an edge, as shown in Fig. 1, where we use i to stand for v_i.

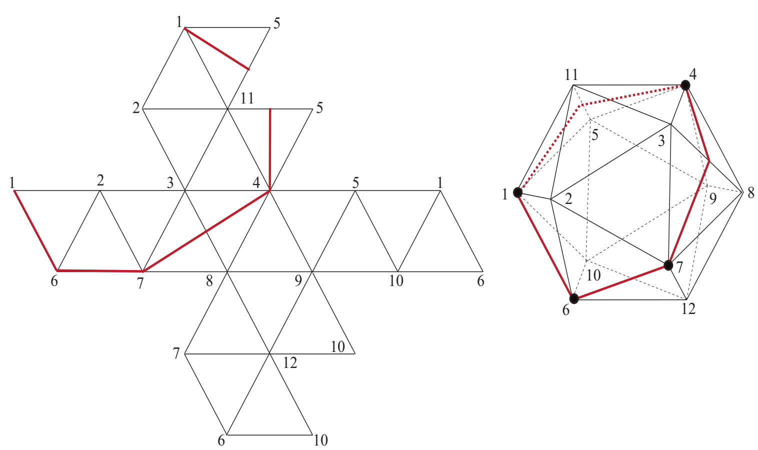

Fig. 1. An icosahedron with vertices labeled $1, 2, ..., 12$, and an unfolding of the surface, showing an example of a quasigeodesic on P depicted by the red line segments. This is Q_5 in Fig. 16.

The three simple closed geodesics on P are shown in Fig. 2 : (a) An equator, (b) a non-planar example, (c) the other non-planar example, (d) the unfolding of the geodesic depicted in (b) of the figure where the geodesic is parallel to the line segment v_2v_8, and (e) the unfolding of the geodesic depicted in (c) of the figure where the geodesic is parallel to the line segment v_2v_9.

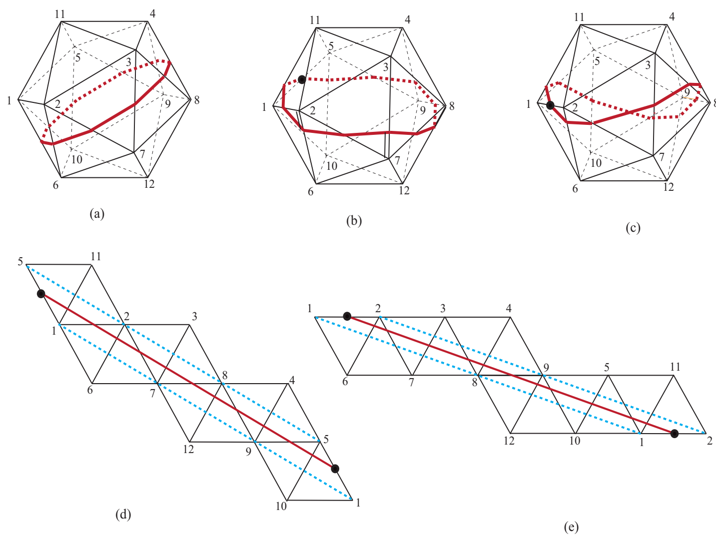

Fig. 2. Simple closed geodesics on the regular icosahedron: (a) An equator; (b) A non-planar example; (c) A second non-planar example; (d) Unfolding of the geodesic depicted in (b), parallel to the line segment v_2v_8; (e) Unfolding of the geodesic depicted in (c), parallel to the line segment v_2v_9.

The notion of a geodesic has been extended to a quasigeodesic. A ***quasi-geodesic*** is a curve on P that has at most $\leq \pi$ surface to each side at every point (and so can include vertices), and if the curve is non-self-crossing and closed, we say that it is a ***simple closed quasigeodesic***. Recently it was proved [MIT$^+$25] that there are exactly 15 simple closed quasigeodesics (excluding geodesics) on the cube. Here we focus on the regular icosahedron. We will show that there are exactly eleven simple closed quaisigeodesics (excluding geodesics) on P, up to congruence, a condition we henceforth take as understood. See Fig. 1 for an example, and Fig. 16 for all eleven $Q_1, Q_2, \ldots Q_{11}$.

2 Quasigeodesics on the regular icosahedron

The following is the our main result.

Theorem 21 *There are exactly eleven distinct simple closed quasigeodesics (excluding geodesics) on the surface of a regular icosahedron.*

Let ***P*** and ***Q*** denote the regular icosahedron and an arbitrary given simple closed quasigeodesic on P, respectively.

A ***geodesic segment*** is a non-self-crossing vertex-to-vertex geodesic. We denote by ***Q(u, v)*** the u-to-v geodesic segment in Q if u and v are adjacent vertices in Q. For example, the quasigeodesic Q in Fig. 1 consists of four segments: $Q(v_1, v_6) \cup Q(v_6, v_7) \cup Q(v_7, v_4) \cup Q(v_4, v_1)$. The following lemma focuses on one geodesic segment, and is a key to proving Theorem 21.[3]

Lemma 22 *Each simple closed quasigeodesic Q on P, excluding simple closed geodesics and one quasigeodesic, consists of geodesic segments each of which has the shortest length among the geodesic segments joining its two adjacent vertices in Q.*

By the definition of a simple closed quasigeodesic Q, we have the following fact.

Fact 1: Angle constraint. Let Q pass through a vertex v. Since the total surface angle at v is $(5/3)\pi$, the surface angle θ_v at v to each side of Q satisfies

$$(2/3)\pi \leq \theta_v \leq \pi.$$

3 Geodesic segments

For two vertices u and v of P that are antipodal, such as v_{11} and v_{12}, the subset of P consisting of ten faces not including both u and v, is called a ***belt*** of P, denoted by ***B(u, v)***, e.g., $B(v_{11}, v_{12})$ is the union of ten triangular faces with vertices in $\{v_i : i = 1, 2, ..., 10\}$. $B(u, v)$ has a boundary consisting of two regular pentagons, and the unfolding of the belt by cutting an edge is a parallelogram. There are six belts in P because $12/2 = 6$. To prove Lemma 22, we divide geodesic segments into two groups, depending on whether each is or is not included in a belt, and provide a precise description for each quasigeodesic segment.

3.1 Geodesic segments included in a belt

Let suppose that a geodesic segment $Q(u, v)$ is included in a belt, e.g., $B(v_{11}, v_{12})$. Consider the infinite parallelogram obtained as a tiling by gluing the shorter sides of infinite copies of the unfolding of $B(v_{11}, v_{12})$, as shown in Fig. 3(a). Then Q can be depicted as a straight-line segment joining two vertices in the boundary. Some of the segments might be edges of P. Then it is easy to see that no geodesic segment includes exactly one vertex (Fact 2 ahead). We will study geodesic segments included in a belt in detail in Section 4.

[3] Note that "geodesic segment" in [FF16] permits self-crossing, whereas our geodesic segments are simple.

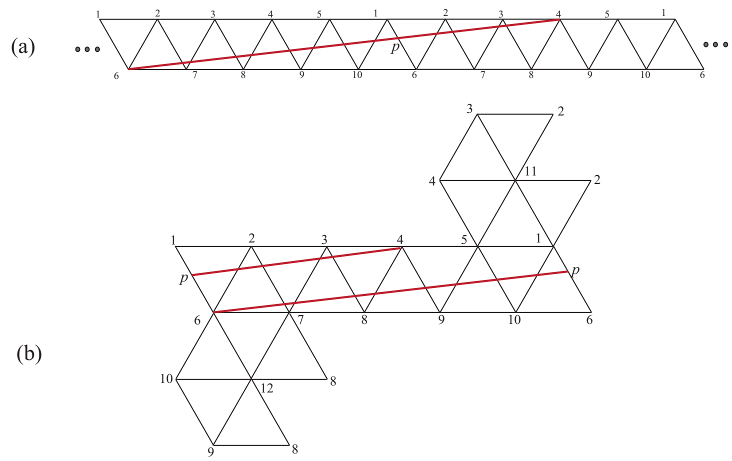

Fig. 3. An example of a v_6-v_4-geodesic segment in $B(v_{11}, v_{12})$: (a) Shown in the unbounded parallelogram; (b) The corresponding geodesic segment on P.

3.2 Geodesic segments not included in any belt

In this subsection we consider geodesic segments not included in any belt. To simplify calculations, we take the edge length of P to be 2. Choose any such geodesic segment. We can assume by symmetry that one end-vertex is v_6 and crosses the belt $B(v_{11}, v_{12})$. Moreover, we can assume that the angle θ between the edge v_6v_7 and the geodesic segment satisfies

$$0 < \theta < \arctan(1/\sqrt{3}).$$

We call such geodesic segment a v_6-**geodesic segment**. We partition θ into five parts as follows:

Part	$\arctan(\cdot) <$	θ	$< \arctan(\cdot)$
(1)	0		$\sqrt{3}/11$
(2)	$\sqrt{3}/11$		$\sqrt{3}/9$
(3)	$\sqrt{3}/9$		$\sqrt{3}/7$
(4)	$\sqrt{3}/7$		$\sqrt{3}/5$
(5)	$\sqrt{3}/5$		$\sqrt{3}/3$

(1) Geodesic segments with $0 < \theta < \arctan(\sqrt{3}/11)$. When the v_6-geodesic segment satisfies $0 < \theta < \arctan(\sqrt{3}/11)$, the segment crosses all ten triangular faces in $B(v_{11}, v_{12})$. Then the segment cannot be a part of any simple closed quasigeodesic of P because it self-crosses.

(2) Geodesic segments with $\arctan(\sqrt{3}/11) < \theta < \arctan(\sqrt{3}/9)$. If there is a v_6-geodesic segment satisfies $\arctan(\sqrt{3}/11) < \theta < \arctan(\sqrt{3}/9)$, the segment crosses the edge v_1v_2 with angle θ, as shown in Fig. 4. Then it has a self-crossing. Thus, there is no v_6-geodesic segment satisfies $\arctan(\sqrt{3}/11) < \theta < \arctan(\sqrt{3}/9)$.

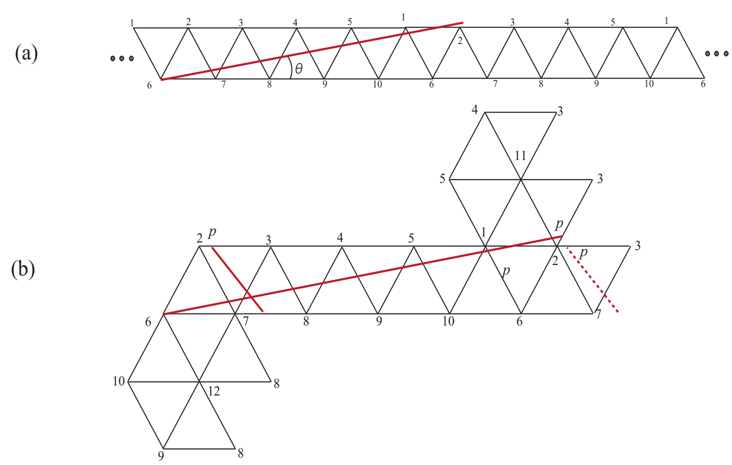

Fig. 4. Suppose there is a v_6-geodesic segments with $\arctan(\sqrt{3}/11) < \theta < \arctan(\sqrt{3}/9)$: (a) Depicted in the unbounded parallelogram; (b) The corresponding geodesic segment on P which has self-crossing.

(3) Geodesic segments with $\arctan(\sqrt{3}/9) < \theta < \arctan(\sqrt{3}/7)$. If there is a v_6-geodesic segment satisfying $\arctan(\sqrt{3}/9) < \theta < \arctan(\sqrt{3}/7)$, the segment crosses the edges v_5v_1, v_1v_{11}, and $v_{12}v_2$, as shown in Fig. 5. Then it has a self-crossing (Fig. 5(b)), so there is no v_6-geodesic segment satisfies $\arctan(\sqrt{3}/9) < \theta < \arctan(\sqrt{3}/7)$.

(4) Geodesic segments with $\arctan(\sqrt{3}/7) < \theta < \arctan(\sqrt{3}/5)$. If there is a v_6-geodesic segment satisfies $\arctan(\sqrt{3}/7) < \theta < \arctan(\sqrt{3}/5)$, the segment crosses the edges v_4v_5, v_5v_{11}, and v_1v_5, as shown in Fig. 6(a). Next, if it crosses the edge v_7v_8, it causes a self-crossing (see Fig. 6(b)). Hence, the only possible extension is to reach v_7 (see Fig. 6(c)). If so, by the similarity of the triangles pv_6v_9 and pv_7v_1, we have

$$(2+x)/6 = (2-x)/4,$$

where x is the length of the line segment pv_5. Then $x = 2/5$, and so p divides the edge v_5v_1 with the ratio 1 to 5. This segment passes the midpoint of v_4v_5 since $(2+2/5)/6 = 2/5$.

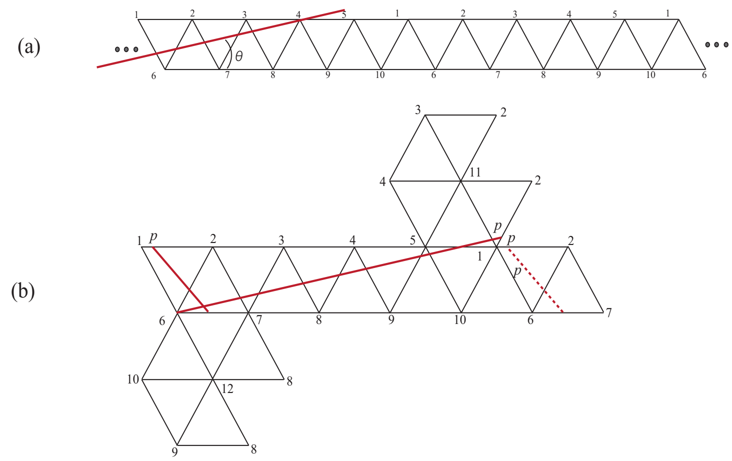

Fig. 5. Suppose there is a v_6-geodesic segment with $\arctan(\sqrt{3}/9) < \theta < \arctan(\sqrt{3}/7)$: (a) Depicted in the unbounded parallelogram; (b) The corresponding geodesic segment on P self-crosses.

Note that this v_6-v_7-geodesic segment cannot be a part of a simple closed geodesic of Q, because of non-self-crossing and Fact 1. More precisely, to extend the v_6-v_7-geodesic segment at v_7, it should reach v_5 and then it cannot extend further. We will employ similar arguments throughout the remainder of this paper.

(5) Geodesic segments with $\arctan(\sqrt{3}/5) < \theta < \arctan(\sqrt{3}/3)$ $(= \pi/6)$. If there is a v_6-geodesic segment that satisfies $\arctan(\sqrt{3}/5) < \theta < \arctan(\sqrt{3}/3)$, the segment crosses the edges v_3v_4, v_4v_{11}, and v_4v_5 at p, as shown in Fig. 7.

We consider three subcases, depending on whether the v_6-segment endvertex is v_{10}, v_7, or v_4.

Case 1: v_{10} is the other endvertex of the v_6-geodesic segment. Let x denote the length the line segment v_4p. By the similarity of two triangles v_6v_8p and $v_{10}v_5p$, we have

$$(2 + x)/4 = (2 - x)/2,$$

and then $x = 2/3$, as shown in Fig. 8(a). This segment passes through the midpoint of v_3v_4 since $(2 + 2/3)/4 = 2/3$.

The v_6-v_{10}-geodesic segment forms a simple closed geodesic Q on P together with the edge v_6v_{10}, as shown on Fig. 8(a) below. Q divides the surface angles at v_6 and v_{10} into $(2/3)\pi + \arctan(\sqrt{3}/4)$ and $\pi - \arctan(\sqrt{3}/4)$. We call this quasigeodesic the ***one-edge quasigeodesic***. It is Q_1 in Fig. 16.

Case 2: v_7 is the other endvertex of the v_6-geodesic segment. Let x denote the length the line segment v_4p. By the similarity of two triangles v_6v_8p and

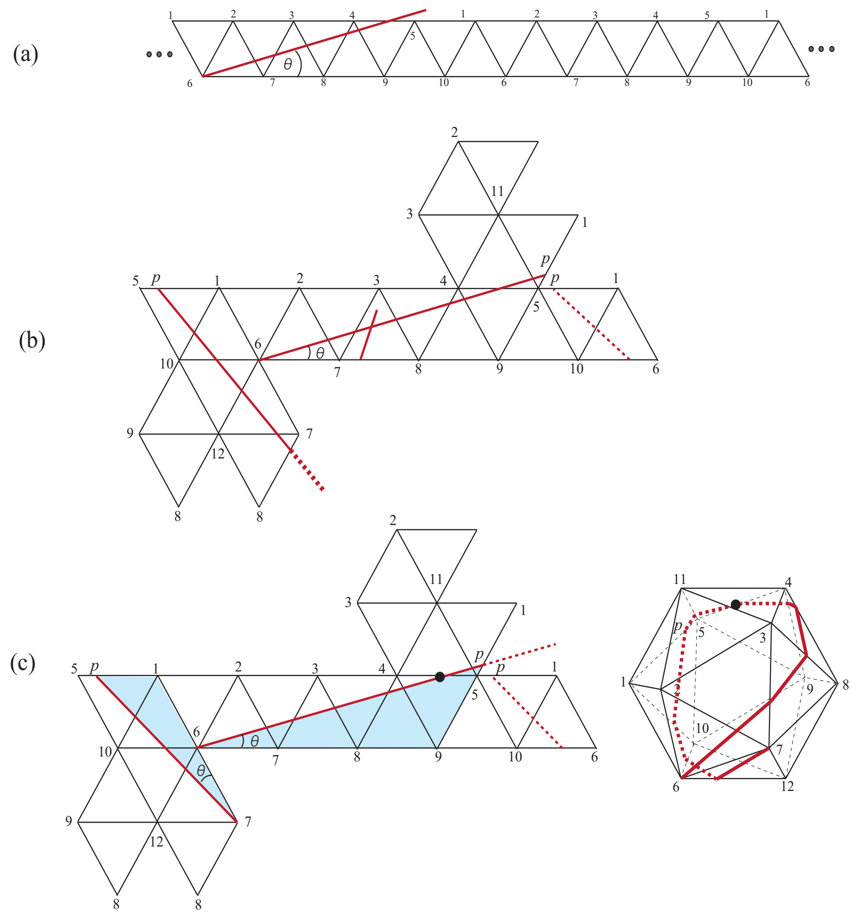

Fig. 6. Suppose there is a v_6-geodesic segments with $\arctan(\sqrt{3}/7) < \theta < \arctan(\sqrt{3}/5)$: (a) Depicted in the unbounded parallelogram; (b) The corresponding geodesic segment on P which self-crosses; (c) The v_6-v_7 geodesic segment where the ratio of the distances $|v_5p|$ and $|pv_1|$ is $1/5$, and the segment passes through the midpoint of v_4v_5.

v_7v_1p, we have

$$(2+x)/4 = (4-x)/4,$$

and then $x = 1$, as shown in Fig. 8 (b). Hence p is the midpoint of v_4v_5. This geodesic segment cannot be a part of a simple closed quasigeodesic, which can be proved similarly to the geodesic segment obtained in Subsection 3.2 and shown in Fig. 6 (c).

Case 3: v_4 is the other endvertex of the v_6-geodesic segment. Let x and y denote the lengths of the line segments v_4p and v_6v_7, respectively, as shown in

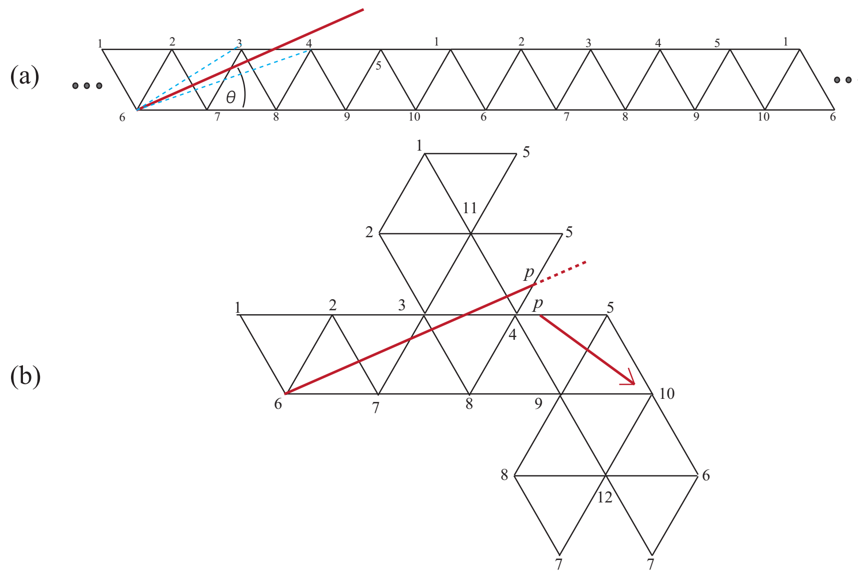

Fig. 7. Suppose there is a v_6-geodesic segment with $\arctan(\sqrt{3}/5) < \theta < \arctan(\sqrt{3}/3)$: (a) Depicted in the unbounded parallelogram; (b) An example of a portion of the geodesic segment on P if it exits.

Fig. 9. By the similarity of the three triangles v_6v_8p, qv_1p, and qv_8v_4, we have

$$(2+x)/4 = (4-x)/(2+y) = 2/(4-y),$$

which implies $x = 6/5$ and $y = 3/2$. The segment passes through the midpoint of the edge v_5v_{10}. However, this geodesic segment cannot be a part of any simple closed quasigeodesic due to the non-crossing condition and Fact 1.

Now, we have the following additional facts.

Fact 2 ([FF16]). There is no simple geodesic segment that begins and ends at the same vertex, that is, there is no simple closed quasigeodesic including exactly one vertex of P.

Fact 3. For an arbitrary simple closed quasigeodesic Q on P, excluding geodesics and excluding the one-edge quasigeodesic, each geodesic segment is included in a belt.

4 Proof of Lemma 22

Let Q be a simple closed quasigeodesic, but neither a geodesic nor the one-edge quasigeodesic. Then Q includes at least two vertices in P by Fact 2 and each geodesic segment is included in a belt by Fact 3, e.g. in the belt $B(v_{11}, v_{12})$. Denote the edge graph of P by $G(P)$, and the edge-distance in $G(P)$ by $d_{G(P)}(u, v)$

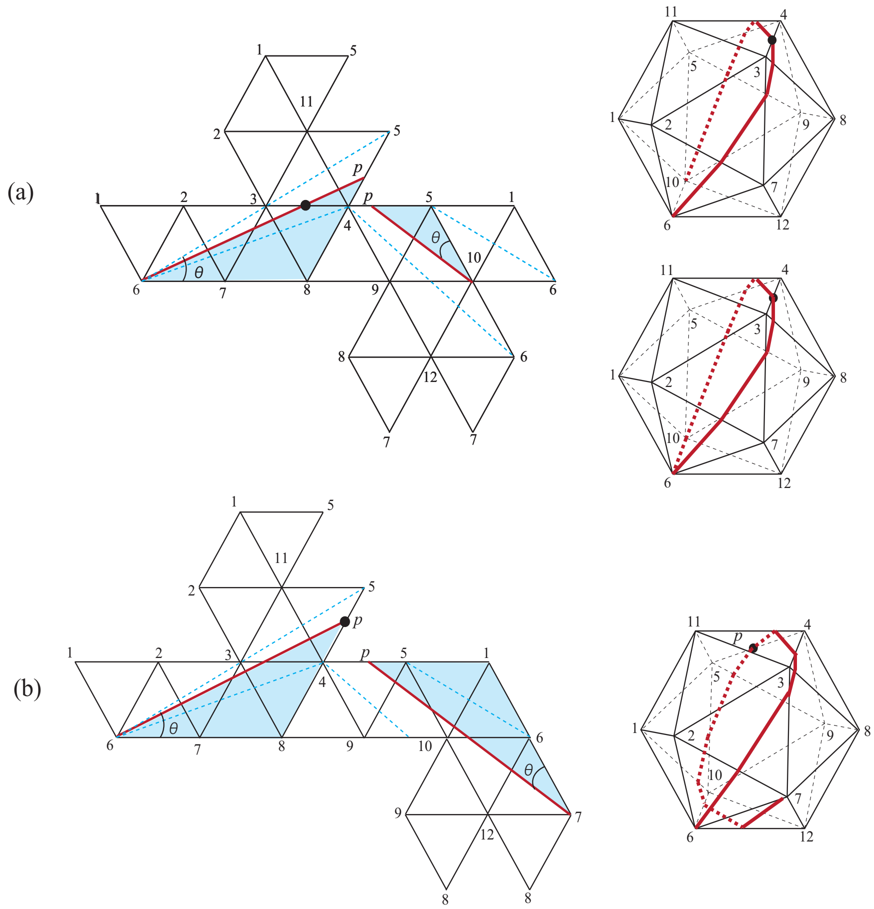

Fig. 8. Suppose there is a v_6-geodesic segment with $\arctan(\sqrt{3}/5) < \theta < \arctan(\sqrt{3}/3)$: (a) v_6-v_{10}-geodesic segment passing the point p which divides v_4v_5 into the ratio 1 to 2, and the midpoint of v_3v_4, and the simple quasigeodesic formed with the edge v_6v_{10}, the one-edge quasigeodesic; (b) v_6-v_7-geodesic segment passing through the midpoint of the edge v_4v_5.

for each pair of vertices u and v in P. Then, $d_{G(P)}(u, v) = 1, 2$ or 3. Let u and v be two vertices adjacent on Q.

Case 1: $d_{G(P)}(u, v) = 1$. We can take $u = v_1$ and $v = v_6$ without loss of generality (wlog).

Assume $Q(v_1, v_6) \neq v_1v_6$. We will show that this leads to a contradiction to Q being a simple closed quasigeodesic, as follows.

There are infinitely many v_1-to-v_6 geodesic segments. First, suppose $Q(u, v)$ is the second shortest one among them (with edge v_1v_6 the shortest). Then $Q(u, v)$,

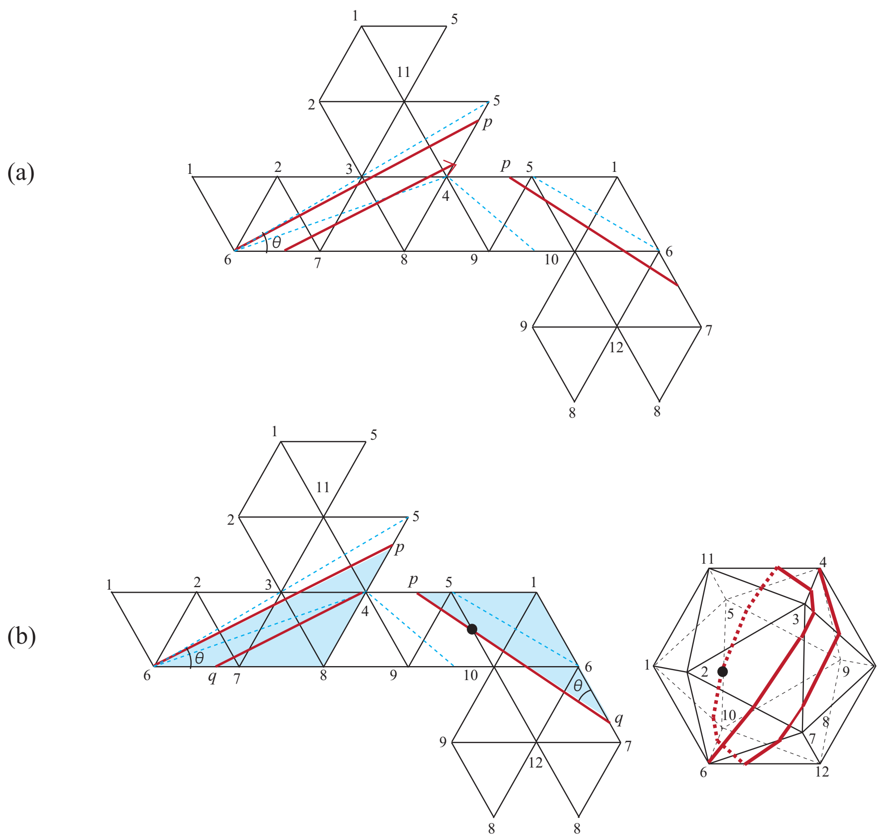

Fig. 9. Suppose there is a v_6-geodesic segment with $\arctan(\sqrt{3}/5) < \theta < \arctan(\sqrt{3}/3)$: (a) An example of the angle θ; (b) v_6-v_4-geodesic segment passing through the point p which divides v_4v_5 into the ratio 3 to 2, and the midpoint of v_5v_{10}.

length $2\sqrt{21}$, crosses 7 edges $\{v_2v_7, v_7v_3, v_3v_8, v_8v_4, v_4v_9, v_9v_5, v_5v_{10}\}$ and 8 triangular faces exactly once, as shown in Fig. 10(a).

In this case, to extend $Q(v_1, v_6)$ beyond v_1 as a part of Q, Q must include v_2 and the edge v_1v_2 by Fact 1, because if the next segment from v_1 lies strictly below v_1v_2, then it crosses itself in the belt, and if the next segment lies strictly above v_1v_2, then it curls around downward into the belt and again crosses itself.

Therefore, Q includes $Q(v_1, v_6) \cup v_1v_2$. By a similar argument to the above, the quasigeodesic $Q(v_1, v_6) \cup v_1v_2$ can be extended at v_2 to $Q(v_1, v_6) \cup v_1v_2 \cup v_2v_3$ uniquely as a part of Q, and hence Q includes $Q(v_1, v_6) \cup v_1v_2 \cup v_2v_3$. Continue this process until Q includes $Q(v_1, v_6) \cup v_1v_2 \cup v_2v_3 \cup v_3v_4 \cup v_4v_5$. Now, we cannot extend the segment anymore by Fact 1. This contradicts the existence of Q as a simple closed quasigeodesic.

Second, suppose $Q(u, v)$ is any of the remaining v_1-to-v_6 geodesic segments. Then $Q(u, v)$ crosses at least seven edges in $int\, B(v_{11}, v_{12})$ (the interior of $B(v_{11}, v_{12})$) and may cross some of those edges more than once; see Fig. 10(b). By an argument similar to the first case, we reach a contradiction extending $Q(u, v)$, by the assumption $Q(v_1, v_6) \neq v_1v_6$ and the existence of Q. Again we omit the details. So $Q(v_1, v_6) = v_1v_6$, an edge of P.

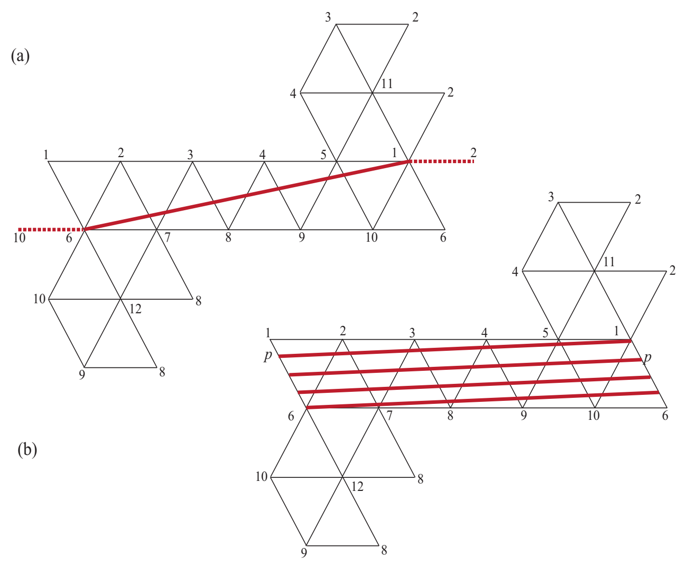

Fig. 10. (a)The second shortest v_1-to-v_6 geodesic segment, which is extended to $Q(v_1, v_6) \cup v_1v_2 \cup v_6v_{10}$ uniquely as a part of Q; (b) An example of v_1-to-v_6 geodesic segment crossing all edges in $intB(v_{11}, v_{12})$.

Case 2: $d_{G(P)}(u, v) = 2$. We can take $u = v_1$ and $v = v_7$ wlog. Suppose that $Q(v_1, v_7)$ is not the shortest v_1-to-v_7 geodesic segment but the second shortest one. Then $Q(v_1, v_7)$ is extended to $Q(v_1, v_7) \cup v_1v_2 \cup v_7v_6$ uniquely as a part of Q. We reach a contradiction by Fact 1 similarly to Case 1, as shown in Fig. 11. We again omit the details.

Case 3: $d_{G(P)}(u, v) = 3$. Then the pair $\{u, v\}$ are antipodal, and there are exactly ten shortest u-to-v geodesic segments, each of which crosses at the midpoint of an edge in $intB(u, v)$, and these ten geodesics are congruent to each other. The remainig u-to-v geodesic segments cross all edges in the interior of some belt, which cannot be a part of Q since Q is simple and closed. Hence Q includes at least one of those ten shortest u-to-v geodesic segments.

This completes the proof of Lemma 22. We now turn to proving Theorem 21.

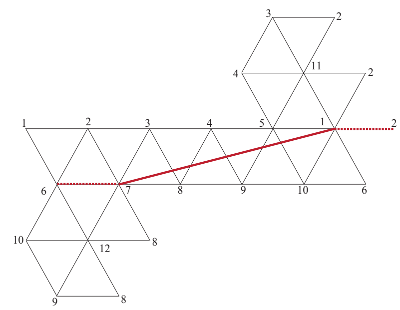

Fig. 11. The second shortest v_1-to-v_7 geodesic segment, which is extended to $Q(v_1, v_7) \cup v_1v_2 \cup v_7v_6$ uniquely as a part of Q.

5 Inventory of Simple Closed Quasigeodesics

In the remainder of the paper, we assume that Q is an arbitrary given quasigeodesic but neither geodesic on P nor the one-edge quasigeodesic Q_1 shown in Fig. 8(a) and Fig. 16. Then Q includes at least two vertices in P by Fact 2 and each geodesic segment is included in a belt as a shortest path by Lemma 22. Now we focus on how the geodesic segments can fit together to form Q. Let Q consist of $n \geq 2$ vertices $w_1, w_2, ..., w_n$ in cyclic order. We assign Q to a cycle graph ***G(Q)*** of n vertices with n labeled edges, where the label of each edge corresponds to the edge-distance in $G(P)$, and we assign $G(Q)$ to

$$[l_1, l_2, ..., l_n],$$

where $l_i = d_{G(P)}(w_i, w_{i+1})$ for $1 = 1, 2, ..., n-1$ and $l_n = d_{G(P)}(w_n, w_1)$. We ignore rotations and symmetric transpositions which leave the graph congruent.

For example, the quasigeodesic on P consists of the shortest v_1-to-v_7 geodesic segment, the edge v_7v_8, and the shortest v_8-to-v_1 geodesic segment that crosses the edge v_5v_9 at its midpoint, is assigned the label $[2, 1, 3]$. So, we identify $[1, 3, 2]$, $[3, 2, 1]$, $[3, 1, 2]$, $[1, 2, 3]$ and $[2, 3, 1]$ with $[2, 1, 3]$. Note that some simple closed quasigeodisics on P may be assigned to the same graph with labeled edges. We will see an example in Lemma 51.

Lemma 51 *All simple closed quasigeodesics on P, excluding geodesics and the one-edge quasigeodesic Q_1 shown in Fig. 16, are depicted in nine cycle graphs with labeled-edges:*

$[3, 3]$, $[3, 2, 1]$, $[3, 1, 1, 1]$, $[2, 2, 2]$, $[2, 2, 1, 1]$, $[2, 1, 2, 1]$, $[2, 1, 1, 1, 1]$, $[1, 1, 1, 1, 1]$, *and* $[1, 1, 1, 1, 1, 1]$*, as shown in Fig. 12.*

Moreover, the cycle graph $[3, 3]$ *has two distinct quaigeodesics, one of which divides the surface angle at each vertex into* $2\pi/3$ *and* π*, and the other divides it into both greater than* $2\pi/3$ *(see Figs. 13(a)(b)). Each of the other graphs corresponds to exactly one simple closed quasigeodesic up to congruence (see Figs. 14 and 15).*

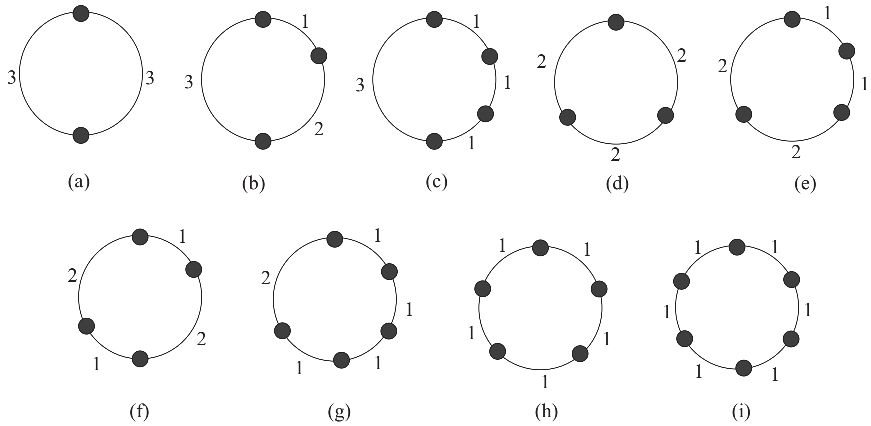

Fig. 12. Nine graphs with labeled-edges assigned for all simple closed quasigeodesics of P except geodesics: (a) $[3, 3]$; (b) $[3, 2, 1]$; (c) $[3, 1, 1, 1]$; (d) $[2, 2, 2]$; (e) $[2, 2, 1, 1]$; (f) $[2, 1, 2, 1]$; (g) $[2, 1, 1, 1, 1]$; (h) $[1, 1, 1, 1, 1]$; (i) $[1, 1, 1, 1, 1, 1]$.

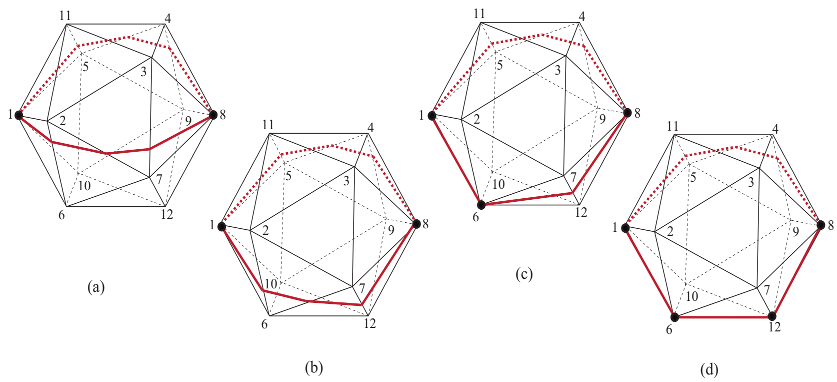

Fig. 13. Quasigeodesics with $\max_d(Q) = 3$: (a) $[3, 3]$) with division of the surface angles at vertices into $(2/3)\pi$ and π; (b) $[3, 3]$ with division of the surface angle at vertices into both greater than $(2/3)\pi$; (c) $[3, 2, 1]$; (d) $[3, 1, 1, 1]$.

Proof. We divide all simple closed quasigeodesics on P, except geodesics and the one-edge quasigeodesic Q_1 (Fig 16), according to the maximum of edge-distances in $G(P)$ of two vertices adjacent in Q, and denote it as $\max_d(Q)$. Then $\max_d(Q) = 1, 2$ or 3.

Case 1 : $\max_d(Q) = 3$. Then Q has a pair $\{u, v\}$ of antipodal vertices. If there is no other vertex in Q, then Q is type $[3, 3]$. This type has two distinct simple closed quasigeodesics due to how the surface angle is divided at each vertex. More precisely, one divides it into $(2/3)\pi$ and π (Fig. 13(a)), and the other into both greater than $(2/3)\pi$ (Fig. 13(b)). These are Q_2 and Q_3 in Fig. 16.

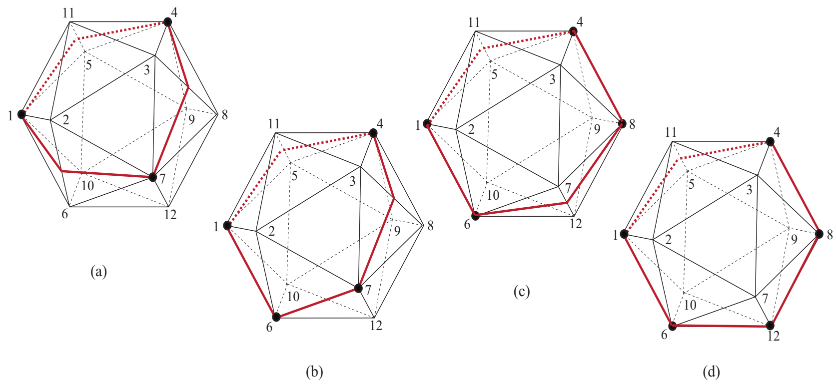

Fig. 14. Quasigeodesics with $\max_d(Q) = 2$: (a) $[2, 2, 2]$; (b) $[2, 2, 1, 1]$; (c) $[2, 1, 2, 1]$; (d) $[2, 1, 1, 1, 1]$.

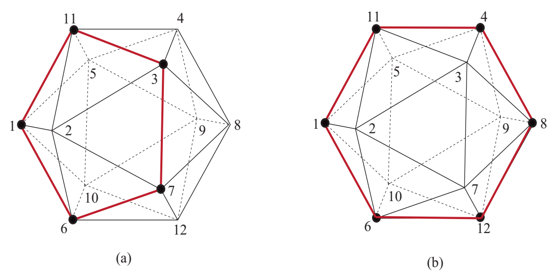

Fig. 15. Quasigeodesics with $\max_d(Q) = 1$: (a) $[1, 1, 1, 1, 1]$; (b) $[1, 1, 1, 1, 1, 1]$.

On the other hand, if Q includes another vertex, there are two possibles $[3, 2, 1]$ and $[3, 1, 1, 1]$ by Fact 1 (up to congruence as a graph with labeled-edges), as shown in Fig. 13(c)(d). Each of these two types corresponds to exactly one simple closed quasigeodesic on P. These are Q_4 and Q_5 in Fig. 16.

Case 2 : $\max_d(Q) = 2$. Then we get four types $[2, 2, 2]$, $[2, 2, 1, 1]$, $[2, 1, 2, 1]$, and $[2, 1, 1, 1, 1]$ by checking the possibilities under Fact 3, and each of them corresponds to exactly one simple closed quasigeodesic (see Fig. 14). These are Q_6, Q_7, Q_8, and Q_9 in Fig. 16.

Case 3 : $\max_d(Q) = 1$.7 Similarly, we get two types $[1, 1, 1, 1, 1]$, and $[1, 1, 1, 1, 1, 1]$ by Fact 3, each of which corresponds to exactly one simple closed quasigeodesic (see Fig. 15). These are Q_{10} and Q_{11} in Fig. 16.

Thus we have completed the proof of Lemma 51. □

6 Proof of Theorem 21

Two simple closed quasigeodesics are of type $[3, 3]$, and eight of them are exactly one for each type in $[3, 2, 1]$, $[3, 1, 1, 1]$, $[2, 2, 2]$, $[2, 2, 1, 1]$, $[2, 1, 2, 1]$, $[2, 1, 1, 1, 1]$,

$[1,1,1,1,1]$, and $[1,1,1,1,1,1]$. There is one more quasigeodesic, the one-edge quasigeodesic Q_1 (Fig. 16). Therefore, there are exactly eleven distinct simple closed quasigeodesics (excluding geodesics) on the regular icosahedron, as shown in Fig. 16.

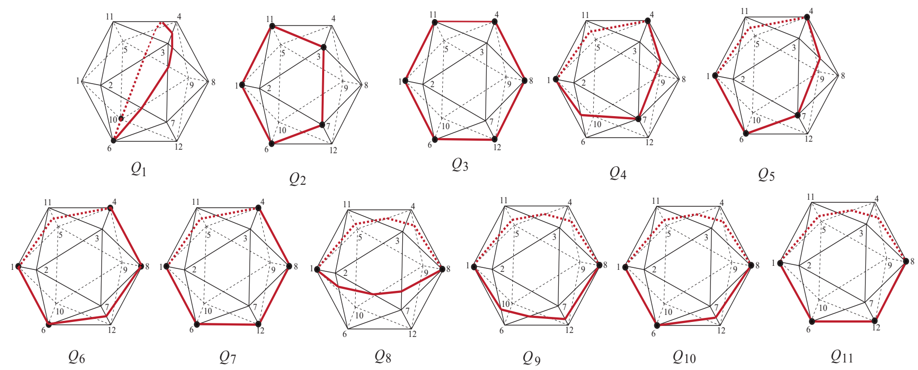

Fig. 16. Eleven quasigeodesics: $\max_d(Q_1) = 1$ with two vertices, the one-edge quasigeodesic; $\max_d(Q_i) = 1$ $(i = 2, 3)$ with more than two vertices: $\max_d(Q_i) = 2\,(i = 4, 5, 6, 7)$; $\max_d(Q_i) = 3$ $(i = 8, 9, 10, 11)$.

7 Summary

We proved that there are exactly eleven simple closed quasigeodesics on the regular icosahedron, excluding geodesics, shown in Fig. 16. It would be an interesting problem to find a similar result for the regular dodecahedron.

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