Unfolding Polyhedra

Joseph O'Rourke[*]

July 10, 2008

Contents

---

[*]Dept. Comput. Sci., Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.

1   Introduction

Imagine a polyhedron surface $\mathcal{P}$ made from paper. We would like to know when $\mathcal{P}$ can be cut and unfolded flat into the plane to a single, nonoverlapping piece. Consider the polyhedron shown in Figure 1(a), a collection of rectangular towers rising from a rectangular base. Polyhedra in this class are known as *orthogonal terrains*, orthogonal because all edges are parallel to orthogonal Cartesian axes, and terrain because the upper surface can be described by a height field. Cutting the tops of the right, left, and back vertical sides permits unfolding those together with the base as illustrated in (b,c) of the figure. Then
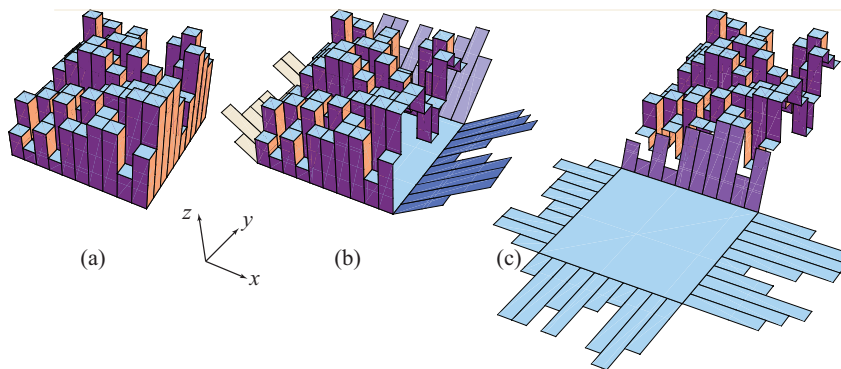

Figure 1: Unfolding the sides and base of $\mathcal{P}$.

cutting the terrain top into x-strips, but leaving one "bridge" rectangle face connecting y-adjacent x-strips, permits flattening the top as shown in Figure 2. The result (c) is a planar shape that is the union of the faces of $\mathcal{P}$, and does not self-overlap except possibly along its boundary. Thus one could cut out the shape in Figure 2(c) from paper with scissors and reverse the unfolding process to form $\mathcal{P}$ in $\mathbb{R}^3$. This algorithm works for all orthogonal terrains [O'R07].

2   Edge Unfolding

What we just described is called an *edge unfolding* because the surface is cut only along edges of $\mathcal{P}$. This is the ideal, the holy grail of this line of research. Alas, it is not always possible.

2.1   Nonconvex: Not Always Possible

Figure 3(a) shows an orthogonal polyhedron with no edge unfolding: it is *edge-ununfoldable*. Here is why. There are six congruent larger faces, and in order for the unfolding to be a single piece, some of them must be connected together. Suppose the Front and Top faces F and T remain connected in the unfolding. Now consider the "notch" of four small square faces in the edge shared between
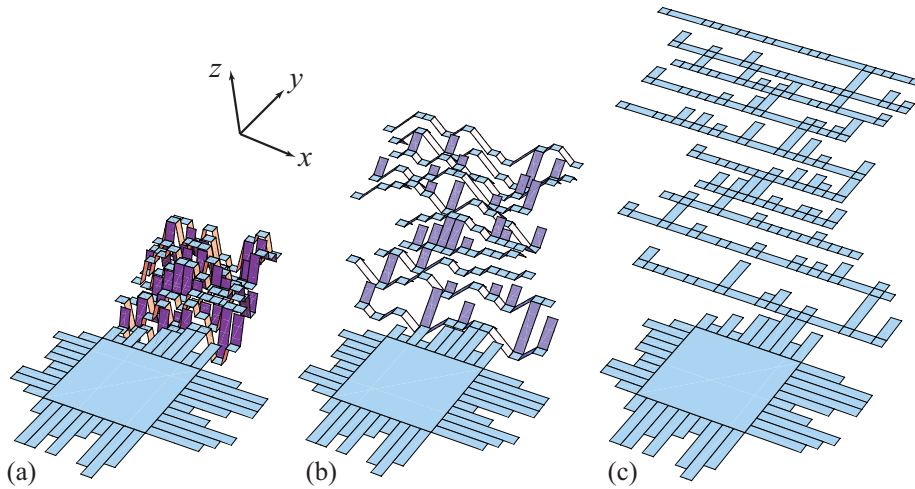
Figure 2: Unfolding the top faces of $\mathcal{P}$ into x-strips connected by y-bridges.

F and T. These small squares must fit inside a hole (the flattened notch hole) that can only accommodate two such squares. And this is impossible.
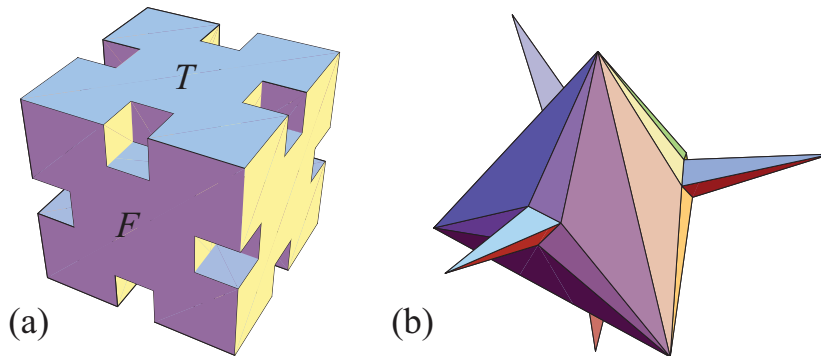

Figure 3: (a) Edge-ununfoldable orthogonal polyhedron. (b) Edge-ununfoldable triangulated polyhedron.

This impossibility result relies on the nonconvexity of the faces of this polyhedron. In contrast, all the faces of the edge-unfoldable polyhedron in Figure 1(a) are convex. What about polyhedra all of whose faces are convex? Several researchers answered this question independently at about the same time with edge-ununfoldable examples [Tar99] [BDEK99] [Grü02]. Figure 3(b) shows one, a "spiked tetrahedron" [BDE+03], which is especially interesting it that all of its (36) faces are triangles. The reason this is edge-ununfoldable is not so straightforward. Suffice it to say that the four open "hats" that sit on the four faces of the inscribed tetrahedron are each individually edge-ununfoldable, and there

is no way to keep a complete unfolding connected without unfolding one of the hats to self-overlap.

So now we know that not all polyhedra, even triangulated polyhedra, have an edge unfolding. But what about convex polyhedra?

2.2 Convex: Open

Whether or not all convex polyhedra have an edge unfolding is an open problem, unresolved since it was explicitly posed by Shephard in 1975 [She75]. The problem has been implicit in some sense since the time of Dürer, whose 1525 book [Dür25] described many convex polyhedra by presenting them as edge unfoldings (or "nets"). See, e.g., Figure 4. Over the years, this has become a
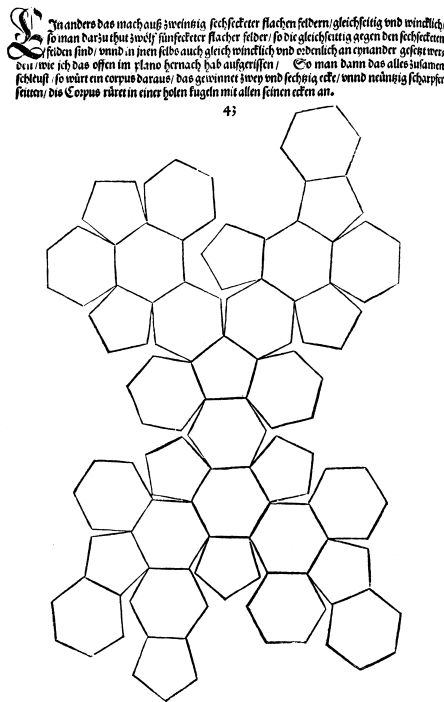

Figure 4: Dürer's edge unfolding of a truncated icosahedron.

standard presentation for convex polyhedra, and no one has yet found an example that cannot be edge unfolding. The lack of a counterexample led Grünbaum to conjecture that indeed all convex polyhedra can be edge unfolded [Grü91], but it has to be admitted that the positive evidence is slim. Only the most narrow classes of polyhedra are known to be edge-unfoldable.

Prismoids. A *prismatoid* is the convex hull of parallel convex polygons A and B. To my knowledge, there is no proof that prismatoids are edge-unfoldable,

despite their apparent simplicity. Only an even more specialized subclass is settled: prismoids. A *prismoid* is a prismatoid with A and B equiangular and oriented so that corresponding edges are parallel. Thus all lateral faces of a prismoid are trapezoids, whereas the lateral faces of a prismatoid are triangles or trapezoids. This makes it easier to control the unfolding, and indeed a simple "volcano" unfolding suffices, as illustrated in Figure 5. This unfolding splays the
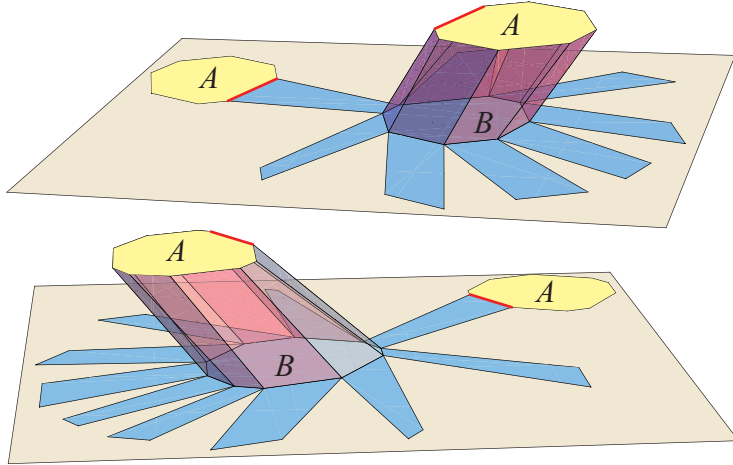

Figure 5: Edge unfoldings of two different primsoids. The top A is attached to the marked edge in both instances.

lateral faces around the base B, and attaches the top A to a carefully selected side face (not every possible attachment always avoids overlap) [O'R01].

Domes. Another narrow class of shapes that are known to be edge-unfoldable is the "domes." A *dome* is a polyhedron with a distinguished *base* face B, and the property that every nonbase face shares an edge with B. Again a volcano unfolding works, as illustrated in Figure 6. This time there is no issue of where to place the nonexistent top, which makes it easier than prismoids, but the side faces are arbitrary convex polygons rather than trapezoids, which makes it more difficult.

There are now three proofs that this is a non-overlapping unfolding [DO07] [BO07] [Pin07]. We'll present the latter, which leads to a stronger result that will be explained afterward.

Let F_1 and F_2 be two faces of the dome, incident to the base B at edges e_1 and e_2 respectively. Let Π_B be the base plane containing B. If F_1 and F_2 are adjacent along ∂B at a vertex v, then it is clear that they unfold without overlapping one another, because there is positive curvature at every vertex of a convex polyhedron, and so a positive "angle gap" at v. So, for example, a ray in Π_B bisecting this angle gap separates the unfoldings of F_1 and F_2. So assume that F_1 and F_2 are not adjacent along ∂B. If e_1 is parallel to e_2, then
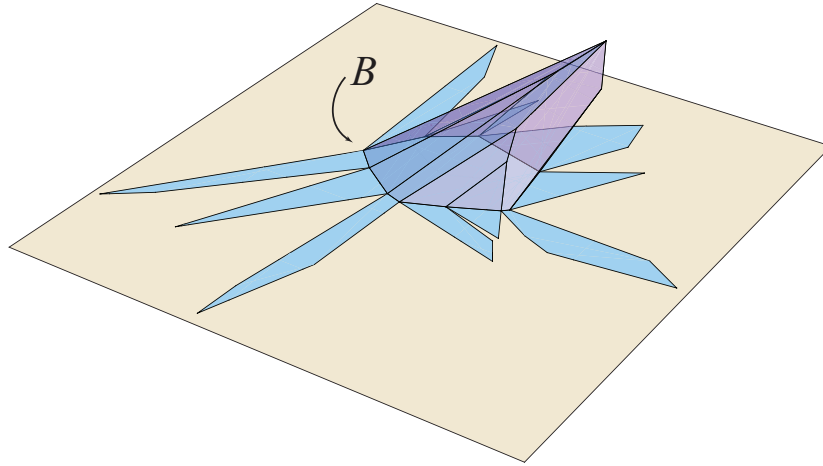
Figure 6: Unfolding of a dome.

the unfolding of F_1 is separated by the line containing e_1 from the unfolding of F_2. So assume e_1 and e_2 are not parallel. Extend F_1 to a plane Π_1 and extend F_2 to a plane Π_2. The three planes Π_1, Π_2, Π_B meet at a point a that is the intersection of the lines containing e_1 and e_2. Let b be another point on the line $\Pi_1 \cap \Pi_2$, with b above Π_B. See Figure 7. Choose b so that F_1 and F_2 are both
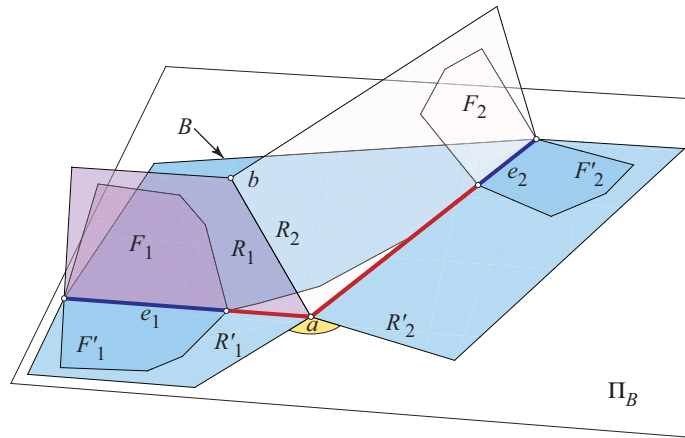

Figure 7: For $i = 1, 2$, $F_i \subseteq R_i \subset \Pi_i$, and $F_i' \subseteq R_i' \subset \Pi_B$.

nested in regions of $R_1 \subset \Pi_1$ and $R_2 \subset \Pi_2$ bounded by the shared segment ab. (This choice is always possible by the convexity of the dome.) Now the point a has positive curvature when viewed as a vertex of an enclosing convex polyhedron bounded by Π_B, Π_1, and Π_2. Thus unfolding R_1 and R_2 to Π_B leaves an angle gap at a, so those unfoldings R_1' and R_2' do not overlap. And

5

because $F_1 \subseteq R_1$ and $F_2 \subseteq R_2$, their unfoldings F_1' and F_2' do not overlap either.

Fewest Nets. Given the lack of progress on settling Grünbaum's conjecture that every convex polyhedron has an edge unfolding, I posed the "Fewest Nets" problem [DO04]: If a convex polyhedron has F faces, what is the fewest number of connected, flat, non-overlapping pieces into which it may be cut by slicing along edges? Although the answer may be 1, it is not obvious how to improve on the trivial bound F, obtained by cutting out each face individually. Upper bounds of $\frac{2}{3}F$ and then $\frac{1}{2}F$ were obtained before Pincu proved, using the dome proof, that $\frac{3}{8}F$ is an upper bound. One key observation is that the above proof works for more than a dome: it shows that any face B of a convex polyhedron $\mathcal{P}$, together with all the faces of $\mathcal{P}$ incident to ∂B, may be cut out of $\mathcal{P}$ and flattened without overlap. The proof only uses convexity and adjacency to B, not closing to a dome. This result plus a nontrivial graph domination argument lead to the $\frac{3}{8}F$ bound [Pin07]. The gap between 1 and $\frac{3}{8}F$ remains at this writing.

3 General Unfolding

The restriction to cutting along the edges of the polyhedron is natural in terms of physical models, but unnatural in terms of the intrinsic metric on the surface (for example, points on the interior of edges have no curvature). This suggests permitting arbitrary cuts to produce an unfolding. The only condition on the cuts are the necessary ones: they must form a tree on the surface of $\mathcal{P}$ (a tree implies a single-piece unfolding), and the tree must span the vertices (so that all curvature is "resolved" and the resulting piece can be flattened). For lack of a better term, we call these *general unfoldings*. Does every polyhedron have a general unfolding to a single non-overlapping piece? For convex polyhedra, the answer is: YES.

3.1 Convex: Star & Source Unfoldings

Indeed there are two general methods to unfold any convex polyhedron, complements of a sort. One is easy to explain but hard to prove avoids overlap, the other is easy to prove non-overlapping but less intuitive perhaps. We start with the first, the so-called *star unfolding*.

Let $x \in \mathcal{P}$ be a "generic" *source point* on the surface of a convex polyhedron $\mathcal{P}$. Draw the shortest path $\sigma(x, v_i)$ from x to each vertex $v_i \in P$. It is not difficult to show that x can be chosen so that $\sigma(x, v_i)$ is unique; this is the sense in which x should be generic. For example, Figure 8(a) shows the shortest paths between the midpoint x of the bottom face and the 8 vertices of a $2 \times 1 \times 1$ rectangular box. Now, cut all these shortest paths and unfold to produce the star unfolding $U^*(x)$. Note that all vertices have an incident cut, so indeed this is a spanning tree and can be flattened, as shown in (b) of the figure. What is not so evident is that the unfolding avoids overlap. The concept of the star

unfolding was introduced by Alexandrov in 1948 [Ale50, p. 181][Ale05, p. 195][1] but only proved to avoid overlap more recently [AO92].

If $\mathcal{P}$ has n vertices, the unfolding has $2n$ vertices, n of which are images of x, which alternate with the n images of the vertices of $\mathcal{P}$. Because x can be any generic point on the surface (and there is only a finite network of nongeneric points to avoid), the star unfolding provides an entire class of unfoldings for a given $\mathcal{P}$.
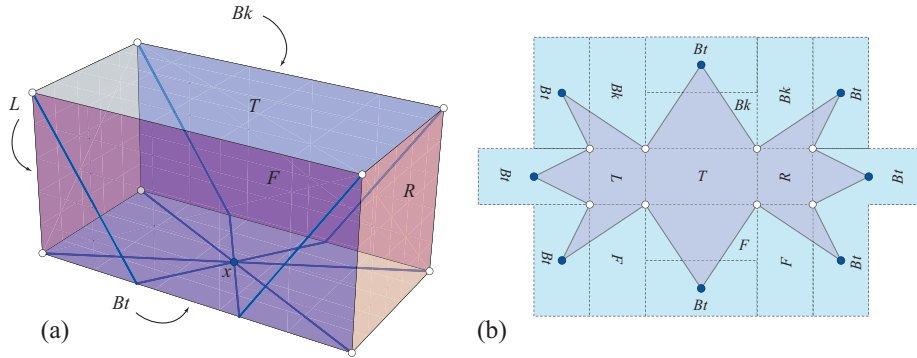

Figure 8: (a) $2\times1\times1$ box. Box faces are labeled: Bt,F,T,R,L,Bk for Bottom, Front, Top, Right, Left, and Back respectively. (b) Star unfolding with respect to x.

The second general unfolding for a convex polyhedron is the *source unfolding*. Again we start with a source point $x \in \mathcal{P}$, but this time we follow shortest paths $\sigma(x, y)$ from x to every point $y \in \mathcal{P}$. The closure of the set of points y such that $\sigma(x, y)$ is not unique forms the *cut locus* $C(x) \subset \mathcal{P}$ of x. The notion of cut locus was introduced by Poincaré in 1905 [Poi05], and since then has become a central concept in global Riemannian geometry. Its name reflects the fact that shortest paths are "cut" or terminated when they reach the cut locus. The cut locus for the box example is shown in Figure 9(a). Notice that the cut locus is indeed a spanning tree of the vertices of $\mathcal{P}$ (this the reason for the closure in the definition). So cutting $C(x)$ will enable flattening the surface. The resulting source unfolding for the box example is shown in (b) of the figure. That this does not overlap is clear, because one can view it as composed of straight-segment "spokes" of length $\sigma(x, y)$ for each $y \in C(x)$, emanating around x at every angle.

Returning to the star unfolding, the cut locus $C(x)$ unfolds to a tree in $U^*(x)$ that spans the n vertices of $U^*(x)$ which are the images of the vertices of $\mathcal{P}$.

3.2 Nonconvex

Now that we have seen that all convex polyhedra have (many) general unfoldings, it is natural to ask whether nonconvex polyhedra do also. Here again the

---
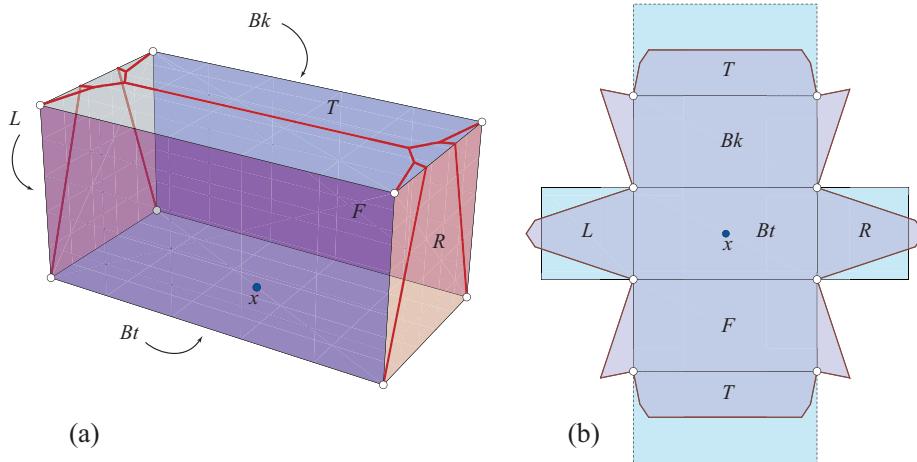
[1] And so sometimes called an "Alexandrov unfolding" [MP08].

7

Figure 9: (a) 2×1×1 box, with cut locus $C(x)$ marked. (b) Source unfolding with respect to x.

answer is unknown: there is neither a counterexample, nor a general algorithm. Progress has been made recently on orthogonal polyhedra.

3.2.1 Orthogonal Polyhedra

We saw one special class of orthogonal polyhedra that can be edge unfolded, and one example (Figure 3(b)) of an orthogonal polyhedron that cannot be edge unfolded. However, if we permit ourselves arbitrary cuts, it is not difficult to unfold this edge-ununfoldable example into a number of thin, connected strips. See Figure 10 for one way, the result of applying a variation on the algorithm from Section 1 for orthogonal terrains.

The idea of slicing an orthogonal polyhedron into strips was explored in a series of papers handling special classes (summarized in [O'R08]), finally culminating in an algorithm that unfolds any orthogonal polyhedron $\mathcal{P}$ (of genus zero) into a single, non-overlapping piece [DFO07]. This algorithm "peels" the surface into a thin strip, following a recursively-nested helical path on the surface of $\mathcal{P}$. Although the cuts are arbitrary, they are parallel to polyhedron edges, which is natural in this context. Unfortunately, the resulting unfolding can be exponentially thin and exponentially long: if $\mathcal{P}$ has n vertices and has longest dimension 1, strips might have width $1/2^{O(n)}$ and stretch out to length $2^{O(n)}$.

4 Summary & Prospects

Table 1 summarizes the status of the main questions on unfolding.

Of course there are many topics we have not discussed. For example, the source and star unfoldings have been generalized to "quasigeodesic"
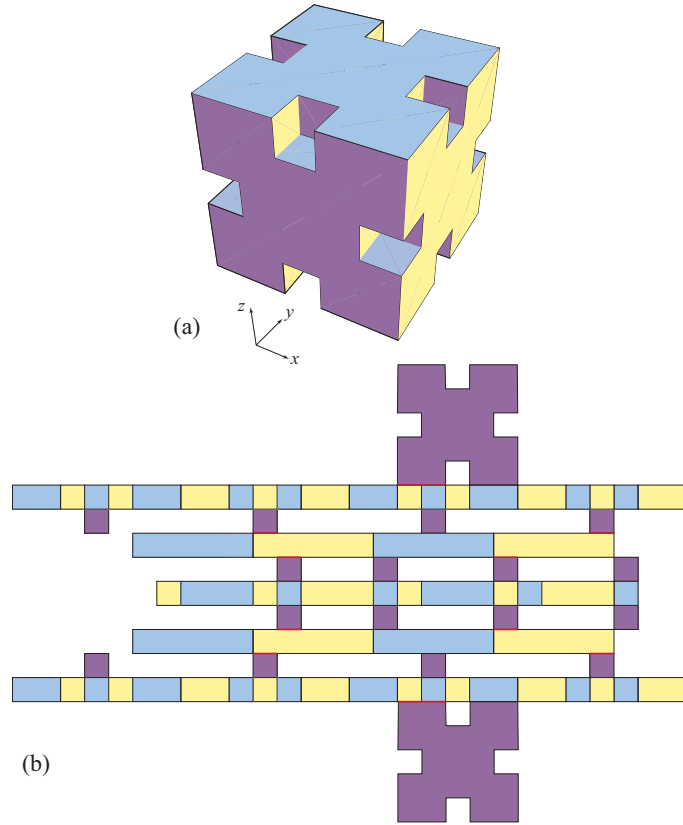
(a)

(b)

Figure 10: (a) Fig. 3(b) repeated; (b) General unfolding. Red segments indicate cuts.

Shapes	Edge Unfolding?	General Unfolding?
convex polyhedra	Open	YES
nonconvex polyhedra	NO	Open

Table 1: Status of main questions concerning nonoverlapping unfoldings.

sources [IOV07]. They have also been generalized to higher dimensions: the source unfolding of a convex polytope exists and produces a non-overlapping unfolding in one lower dimension, but the star unfolding does not generalize [MP08].

There are also many more open problems than the central ones which have been the focus of this article; see [DO07] for a sampling. One particularly intriguing one was posed by Connelly: When is there a "continuous blossoming," an unfolding that not only results in a non-overlapping planar piece, but also avoids self-intersection throughout a continuous unfolding processes from start to finish? This is achieved, for example, by the orthogonal terrain unfoldings (Figures 1 and 2), but has not been explored even for the source and star unfoldings of convex polyhedra.

Finally, let me end with two recent developments related to the open question on edge unfolding convex polyhedra. First, classes of polyhedra are known where almost all unfoldings overlap [BO08]: the percentage of the spanning cut trees that lead to non-overlap goes to zero as the number of vertices goes to infinity. Second, Tarasov constructed an intricate example of a convex polyhedron $\mathcal{P}$ whose surface may be partitioned into convex geodesic polygons in such a way that $\mathcal{P}$ cannot be "edge unfolded" along the edges of this partition [Tar08]. A *geodesic polygon* is a closed region on the surface bounded by a finite number of geodesics and enclosing no vertices. So each is intrinsically flat, but may cross edges of the polyhedron. Tarasov's result shows that, in a sense, the edge-unfolding conjecture is false from an intrinsic viewpoint. Whether this will help resolve that corner of Table 1 remains to be seen.

References

[Ale50]   Aleksandr D. Alexandrov. *Vupyklue Mnogogranniki.* Gosydarstven-noe Izdatelstvo Tehno-Teoreticheskoi Literaturu, 1950. In Russian. See [Ale58] for German translation, and [Ale05] for English translation.

[Ale58]   Aleksandr D. Alexandrov. *Konvexe Polyeder.* Akademie Verlag, Berlin, 1958. Math. Lehrbucher und Monographien. Translation of the 1950 Russian edition.

[Ale05]   Aleksandr D. Alexandrov. *Convex Polyhedra.* Springer-Verlag, Berlin, 2005. Monographs in Mathematics. Translation of the 1950 Russian edition by N. S. Dairbekov, S. S. Kutateladze, and A. B. Sossinsky.

[AO92]   Boris Aronov and Joseph O'Rourke. Nonoverlap of the star unfolding. *Discrete Comput. Geom.*, 8:219–250, 1992.

[BDE+03]   Marshall Bern, Erik D. Demaine, David Eppstein, Eric Kuo, Andrea Mantler, and Jack Snoeyink. Ununfoldable polyhedra with convex faces. *Comput. Geom. Theory Appl.*, 24(2):51–62, 2003.

[BDEK99] Marshall Bern, Erik D. Demaine, David Eppstein, and Eric Kuo. Un-unfoldable polyhedra. In *Proc. 11th Canad. Conf. Comput. Geom.*, pages 13–16, 1999. See [BDE⁺03].

[BO07]   Alex Benton and Joseph O'Rourke. Unfolding polyhedra via cut-tree truncation. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 77–80, 2007.

[BO08]   Alex Benton and Joseph O'Rourke. A class of convex polyhedra with few edge unfoldings. In *Proc. 20th Canad. Conf. Comput. Geom.*, August 2008.

[DFO07]   Mirela Damian, Robin Flatland, and Joseph O'Rourke. Epsilon-unfolding orthogonal polyhedra. *Graphs and Combinatorics*, 23[Suppl]:179–194, 2007. Akiyama-Chvátal Festschrift.

[DO04]   Erik D. Demaine and Joseph O'Rourke. Open problems from CCCG 2003. In *Proc. 16th Canad. Conf. Comput. Geom.*, pages 209–211, 2004.

[DO07]   Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, July 2007. http://www.gfalop.org.

[Dür25]   Albrecht Dürer. *The painter's manual: A manual of measurement of lines, areas, and solids by means of compass and ruler assembled by Albrecht Dürer for the use of all lovers of art with appropriate illustrations arranged to be printed in the year MDXXV*. New York: Abaris Books, 1977, 1525. English translation by Walter L. Strauss of 'Unterweysung der Messung mit dem Zirkel un Richtscheyt in Linien Ebnen uhnd Gantzen Corporen'.

[Grü91]   Branko Grünbaum. Nets of polyhedra II. *Geombinatorics*, 1:5–10, 1991.

[Grü02]   Branko Grünbaum. No-net polyhedra. *Geombinatorics*, 12:111–114, 2002.

[IOV07]   Jin-ichi Itoh, Joseph O'Rourke, and Costin Vîlcu. Unfolding convex polyhedra via quasigeodesics. Technical Report 085, Smith College, July 2007. arXiv:0707.4258v2 [cs.CG].

[MP08]   Ezra Miller and Igor Pak. Metric combinatorics of convex polyhedra: Cut loci and nonoverlapping unfoldings. *Discrete Comput. Geom.*, 39:339–388, 2008.

[O'R01]   Joseph O'Rourke. Unfolding prismoids without overlap. Unpublished manuscript., May 2001.

[O'R07]   Joseph O'Rourke. Unfolding orthogonal terrains. Technical Report 084, Smith College, July 2007. arXiv:0707.0610v4 [cs.CG].

[O'R08]   Joseph O'Rourke. Unfolding orthogonal polyhedra. In J.E. Goodman, J. Pach, and R. Pollack, editors, *Proc. Snowbird Conference Discrete and Computational Geometry: Twenty Years Later.* American Mathematical Society, 2008. To appear.

[Pin07]   Val Pincu. On the fewest nets problem for convex polyhedra. In *Proc. 19th Canad. Conf. Comput. Geom.*, pages 21–24, 2007.

[Poi05]   Henri Poincaré. Sur les lignes géodésiques des surfaces convexes. *Trans. Amer. Math. Soc.*, 6:237–274, 1905.

[She75]   Geoffrey C. Shephard. Convex polytopes with convex nets. *Math. Proc. Camb. Phil. Soc.*, 78:389–403, 1975.

[Tar99]   Alexey S. Tarasov. Polyhedra with no natural unfoldings. *Russian Math. Surv.*, 54(3):656–657, 1999.

[Tar08]   Alexey S. Tarasov. Existence of a polyhedron which does not have a non-overlapping pseudo-edge unfolding. `http://arxiv.org/abs/0806.2360`, 2008.