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Nonorthogonal Polyhedra Built from Rectangles

Melody Donoso Joseph O'Rourke*

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Abstract

We prove that any polyhedron of genus zero or genus one built out of rectangular faces must be an orthogonal polyhedron, but that there are nonorthogonal polyhedra of genus seven all of whose faces are rectangles. This leads to a resolution of a question posed by Biedl, Lubiw, and Sun [BLS99].

1 Introduction

A paper by Biedl, Lubiw, and Sun [BLS99] raised the following intriguing question: "Can an orthogonal net ever fold to a nonorthogonal polyhedron?" We answer this question here: YES for sufficiently high genus, and NO for sufficiently small genus. First we clarify the meaning of the terms in the question.

An *orthogonal net* is an orthogonal polygon: a simple, planar polygon all of whose sides meet at right angles. See, e.g., Fig. 1. To *fold* an orthogonal net

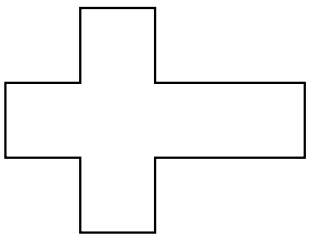

Figure 1: A simple orthogonal polygon.

means to crease it along segments parallel to the polygon's sides and close up its perimeter. Let us call this an *orthogonal folding*, to contrast it with an arbitrary folding (when the creases can be at arbitrary angles). A *polyhedron* is a surface

*Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. {mdonoso, orourke}@cs.smith.edu. Supported by NSF Distinguished Teaching Scholars award DUE-0123154.

formed of planar faces, each a simple polygon, such that any pair is either disjoint, share one vertex, or share an edge; such that each polyhedron edge is shared by exactly two faces; and such that the "link" of each vertex is a cycle. An *orthogonal polyhedron* is a polyhedron all of whose faces meet at edges with dihedral angles that are a multiple of $\pi/2$. It is known that arbitrary foldings of orthogonal polygons can lead to nonorthogonal polyhedra [LO96][DDL+99], but the question obtained by restricting to orthogonal foldings seems to be new.

We display their question, phrased in a positive sense, for later reference:

> **Question 1**. If a polyhedron is created by an orthogonal folding of an orthogonal polygon, must it be an orthogonal polyhedron?

We first study a related question, which is divorced from the concept of folding:

> **Question 2**. If a polyhedron's faces are all rectangles, must it be an orthogonal polyhedron?

We should note that the definition of a polyhedron permits coplanar rectangles, so this second question could be phrased as: "if a polyhedron's faces can be partitioned into rectangles," or, "if a polyhedron can be constructed by gluing together rectangles." Both Questions 1 and 2 can be viewed as seeking to learn whether, loosely speaking, orthogonality in $\mathbb{R}^2$ forces orthogonality in $\mathbb{R}^3$.

An orthogonal folding of an orthogonal polygon produces faces that can be partitioned into rectangles. Thus if rectangle faces force orthogonal dihedral angles, then orthogonal foldings force orthogonal dihedral angles. So we have the following implications:

- Q2: YES $\Rightarrow$ Q1: YES.
- Q1: NO $\Rightarrow$ Q2: NO.

(The second is merely the contrapositive of the first.) However, it is possible that the answer to Q2 is NO but the answer to Q1 is YES, for it could be that the nonorthogonal polyhedra made from rectangles cannot be unfolded without overlap to an orthogonal net. We first explore Q2 to remove ourselves from the little-understood area of nonoverlapping unfoldings. We will see that, nevertheless, we can ultimately prove the answer to both questions is NO for polyhedra of genus seven or above, but YES for genus zero and one.

2 A Nonorthogonal Polyhedron made with Rectangles

Although this reverses the order of our actual development, and might appear unmotivated, we first present the example that shows that the answer to Question 2 is NO. The polyhedron is shown in two views in Fig. 2. It has the structure

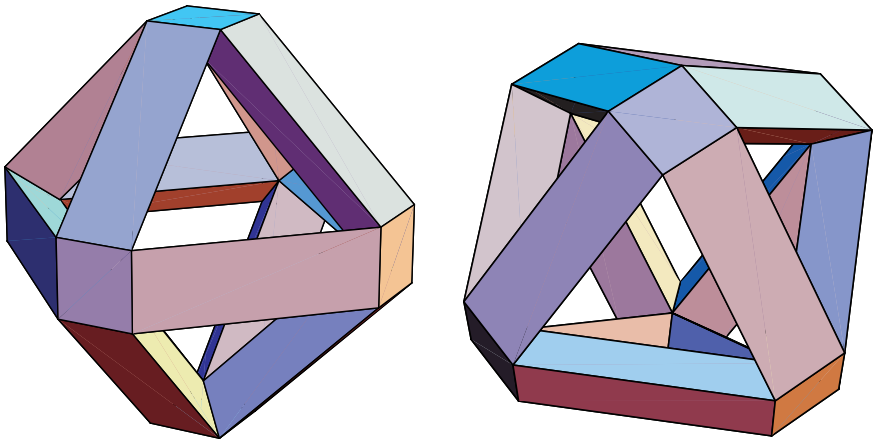

Figure 2: An nonorthogonal polyhedron composed of rectangular faces.

(and symmetry) of a regular octahedron, with each octahedron vertex replaced by a cluster of five vertices, and each octahedron edge replaced by a triangular prism. Let us fix the squares at each octahedron vertex to be unit squares. The length L of the prisms is not significant; in the figure, $L = 3$, and any length large enough to keep the interior open would suffice as well. The other side lengths of the prism are, however, crucially important. The open triangle hole at the end of each prism has side lengths 1 (to mesh with the unit square), $\sqrt{3}/2$ and $\sqrt{3}/2$; see Fig. 3.

This places the fifth vertex in the cluster displaced by 1/2 perpendicularly from the center of each square, forming a pyramid, as shown in Fig. 4.

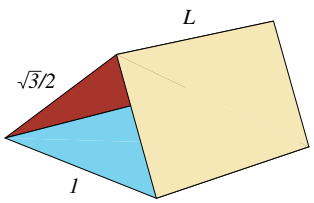

Figure 3: Right triangular prism, used twelve times in Fig. 2.

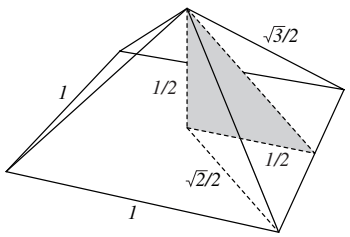

Figure 4: The cluster of five vertices replacing each octahedron vertex form a pyramid.

The central isosceles right triangle (shaded in Fig. 4) guarantees that two oppositely oriented incident triangular prisms meet at right angles, just as do the corresponding edges of an octahedron. The result is a closed polyhedron of $V = 6 \cdot 5 = 30$ vertices, $E = 84$ edges, and $F = 42$ faces. The 42 faces include 6 unit squares, 12 $L \times 1$ rectangles, and 24 rectangles of size $L \times \sqrt{3}/2$.

Clearly the polyhedron is nonorthogonal: the three long edges of each prism have dihedral angles of approximately $54.7°$ ($= \tan^{-1}\sqrt{2}$), $54.7°$, and $70.5°$ ($= \pi - 2\tan^{-1}\sqrt{2}$), two adjacent prisms meet at the same $70.5°$ angle, and the prisms meet the squares at $45°$. So not a single dihedral angle is a multiple of $\pi/2$.

By Euler's formula,

$$V - E + F = 2(1 - g) \tag{1}$$

where g is the genus. From this we can compute the genus of the surface:

$$\begin{aligned} 30 - 84 + 42 = -12 &= 2(1-g) \\ g &= 7 \end{aligned}$$

As it is easy to augment the polyhedron with attached (orthogonal) structures that increase the genus, this example establishes that the answer to Question 2 is NO:

Theorem 1 *There exist nonorthogonal polyhedra of genus $g \geq 7$ whose faces are all rectangles.*

Theorem 1 does not settle Question 1 immediately, for it could be that the polyhedron cannot be unfolded to an "orthogonal net." In fact, we were not successful in finding a nonoverlapping unfolding of its surface. However, it can be unfolded without overlap if boundary sections are permitted to touch but not cross. Such an unfolding could be cut out of a single sheet of paper, but the cuts would have to be infinitely thin for perfect fidelity. The unfolding is shown in Fig. 5. The solid edges are cuts; dashed edges are folds. The folding is somewhat intricate, hinting at by the partial labeling shown.

However, the goal usually is to obtain a simple orthogonal polygon, and such touchings violate the usual definition of a polygon. A simple modification of the example illustrated in Fig. 6 does lead to nonoverlap. Each square is replaced by the five faces of a unit cube. The polyhedron remains nonorthogonal (although now it has some $\pi/2$ dihedral angles), and it may be unfolded to the net shown in Fig. 7. The role of the cubes is to spread out the unfoldings of each prism so that they avoid overlap.

This establishes that the answer to Question 1 is also NO:

Theorem 2 *There is a simple orthogonal polygon that folds orthogonally to a nonorthogonal polyhedron.*

3 Overview of Proof

The proof that the answer to Questions 1 and 2 is YES for sufficiently small genus is long enough that a summary might be useful.

1. Sec. 4 establishes constraints on low-degree vertices, showing how orthogonal and nonorthogonal dihedral angles may mix locally at one vertex.

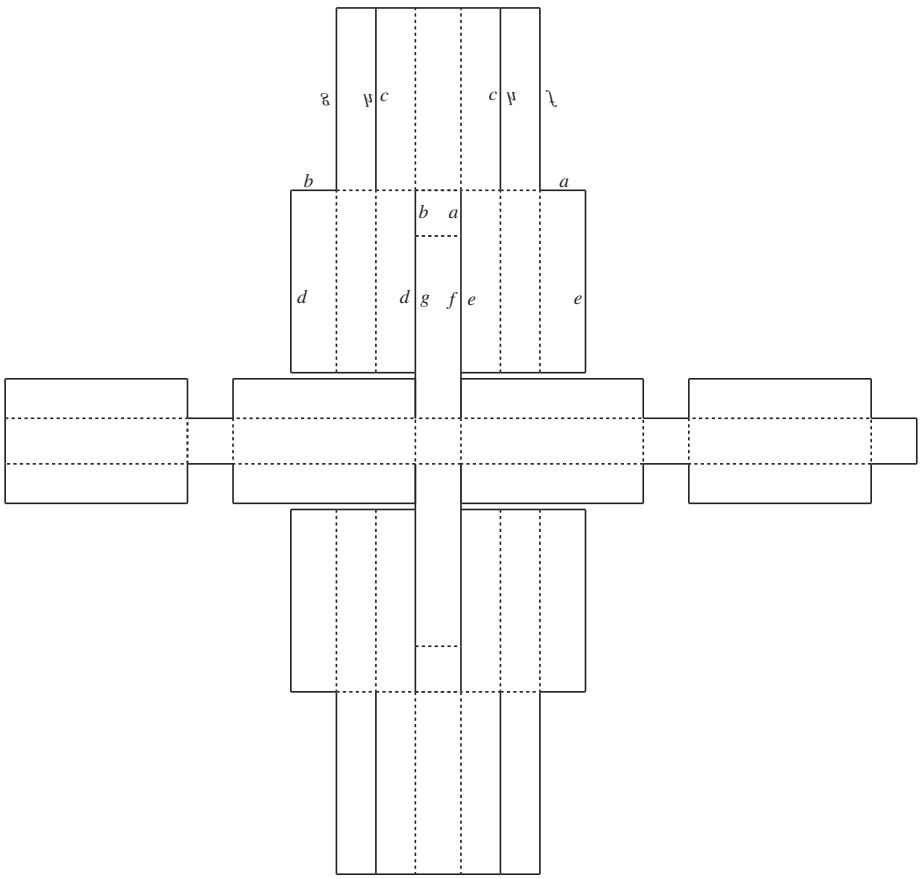

Figure 5: A nonsimple net for the polyhedron of Fig. 2. The edges with the same label are glued to each other when folded. Some labels are upside down, indicating the need for a spatial rotation to reorient the edge before gluing.

2. Sec. 5 derives a corollary to Euler's formula (Eq. (1)) relating a lower bound on the number of edges around a face to the average vertex degree.

3. Sec. 6 derives a lower bound of 4 on the number of face edges with nonorthogonal dihedral angle in a subgraph of the 1-skeleton of the polyhedron, and uses this to establish the claim by contradiction.

4 Local Vertex Constraints

We now embark on a study of the possible configurations of rectangles glued to one vertex of degree δ. Each rectangle incident to a vertex glues a face angle of $\pi/2$ there. It is possible for an arbitrary number δ of such angles to be glued to one vertex: they can squeeze together like an accordian. What is not possible

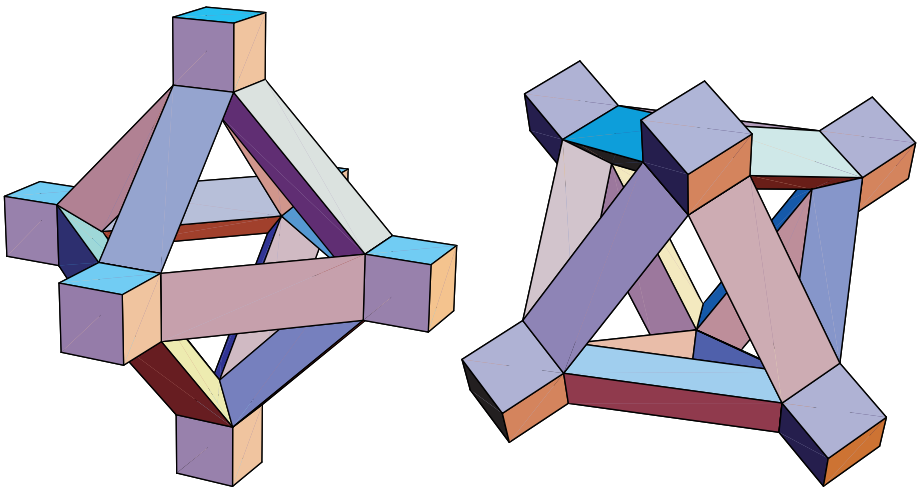

Figure 6: Adding cubes to Fig. 2.

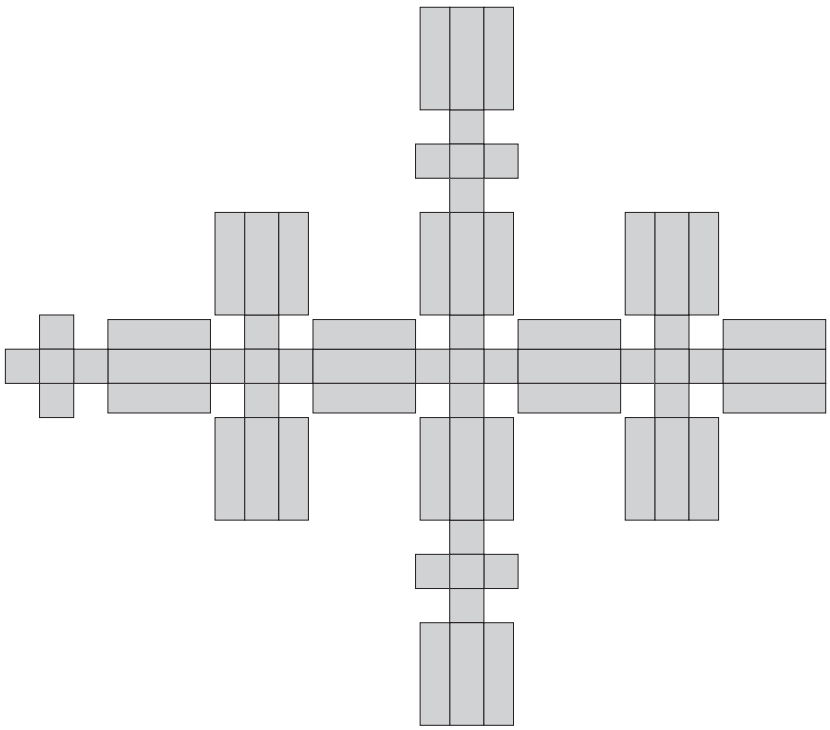

Figure 7: An orthogonal polygon that folds to the nonorthogonal polyhedron in Fig. 6.

is for there to be, in addition, an arbitrary number of dihedral angles that are multiples of $\pi/2$ incident to one vertex. It is this tradeoff we explore here.

4.1 Conventions and Notation

We need a term to indicate a dihedral angle that is a multiple of $\pi/2$: such angles we call *rectilinear*,[1] an angle $\alpha = k\pi/2$ for k an integer. For our purposes, the only rectilinear angles that will occur are $\{0, \pi/2, \pi, 3\pi/2\}$; the $\pm$ sign of an angle will not be relevant. An orthogonal polyhedron has only rectilinear dihedral angles. The issue before us is: Can there be a polyhedron, all of whose faces are rectangles, which has at least one nonrectilinear dihedral angle? It will help to think of all edges with rectilinear dihedrals as colored *green* (good), and nonrectilinear dihedrals as *red* (bad). We seek conditions under which a polyhedron may have one or more red edges.

Our proofs will analyze the geometry around a vertex v by intersecting the polyhedron P with a small sphere S_v centered on v, and examining the intersections of the faces incident to v with S_v. We normalize the radius of S_v after intersection to 1 so that an arc from pole to pole has length π. The intersection $P \cap S_v$ is a spherical polygon P_v of δ great circle arcs, each $\pi/2$ in length (because each face angle is $\pi/2$). Edges incident to v map to vertices of P_v, and the dihedral angle α at an edge maps to the polygon angle α on S_v at the corresponding vertex. P_v is simple (non-self-intersecting) because the polyhedron is simple.

With a Cartesian coordinate system centered on v, the intersection of the xy-plane with S_v is its *equator*, the points of intersection of the z-axis the *poles*, the six points at which the three axis pierce S_v *coordinate points*, and a great circle arc between coordinate points a *coordinate arc*. Two points on S_v are *antipodal* if the shortest arc between them has length π (for example, the poles are antipodal). We call two points a *quarter pair* if the shortest arc between them has length $\pi/2$. Note that neither antipodal points nor quarter pairs are necessarily coordinate points. Define the *separation* $d(p_1, p_2)$ between two points on S_v to be the length of a shortest arc between them; e.g., antipodal points have separation π.

Finally, define an *orthogonal path* to be a simple path $(p_0, p_1, \ldots, p_{n-1})$ on S_v, $n - 1 \geq 1$, such that adjacent points have separation $\pi/2$, and such that the angle at all interior points, $p_1, \ldots, p_{n-2}$, is rectilinear. Note that a $\pi/2$-arc connecting two points is considered an orthogonal path, with no interior points. A consecutive series of rectlinear dihedral angles incident to v produce an orthogonal path on S_v.

4.2 Preliminary Lemmas

Lemma 3 *An orthogonal path $\rho = (p_0, \ldots, p_{n-1})$ with one endpoint p_0 a coordinate point, and whose first arc is a coordinate arc, must have the other endpoint p_{n-1} a coordinate point.*

[1] The use of this term for this precise meaning is not standard.

Proof: The proof is by induction on n. If $n-1=1$, the claim is trivial. Otherwise, let $\rho' = (p_1, \ldots, p_{n-1})$ be the path without the first arc. ρ' starts at a coordinate point (because p_0p_1 is a coordinate arc), and its first arc is a coordinate arc (because the angle at p_1 is rectilinear). Thus the induction hypothesis applies to ρ' establishing the claim. □

Lemma 4 *If two points p_1 and p_2 are connected by an orthogonal path, they are separated by a multiple of $\pi/2$, and are therefore either antipodal or form a quarter pair.*
Proof: Rotate so that p_1 is a coordinate point, and the arc of the path incident to p_1 is a coordinate arc. Then Lem. 3 applies and shows that p_2 is a coordinate point. Two coordinate points have a separation of either π or $\pi/2$. □

Of course this lemma implies (its contrapositive) that two points p_1 and p_2 whose separation is not a multiple of $\pi/2$ cannot be connected by an orthogonal path.

Lemma 5 *Let (p_0, p_{n-1}) be a quarter pair and let $a = p_0p_{n-1}$ be the $\pi/2$-arc connecting them. Every orthogonal path ρ between p_0 and p_{n-1} forms angles with a which are both rectilinear, (perhaps different) multiples of $\pi/2$.*
Proof: If $\rho = a$ then the claim follows trivially with both angles 0. Suppose that $\rho = (p_0, p_1, \ldots, p_{n-1}) \neq a$. Rotate the coordinate system for S_v so that p_0 is the north pole, which places p_{n-1} on the equator; see Fig. 8. It must be that

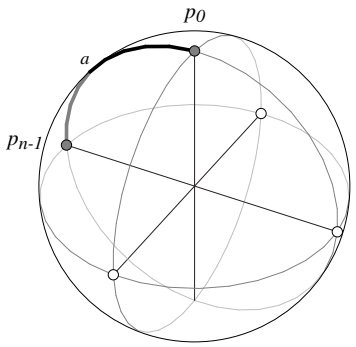

Figure 8: p_1 must lie at one of the three remaining compass points on the equator.

p_1 lies on the equator as well (because each arc of an orthogonal path is of length $\pi/2$), at a point different from p_{n-1} (because $\rho \neq a$ and the path is simple). Suppose that $d(p_1, p_{n-1})$ is not a multiple of $\pi/2$. Then the contrapositive of Lem. 4 says there can be no orthogonal path between them, contradicting the assumption that ρ is an orthogonal path. Thus $d(p_1, p_{n-1})$ must be a multiple of $\pi/2$. This means that the p_0p_1 arc makes an angle with a that is also a multiple of $\pi/2$. Repeating the argument with the roles of p_0 and p_{n-1} reversed leads to the same conclusion at the p_{n-1} end. □

4.3 Vertices with k Red Edges

We now employ the preceding lemmas to derive constraints on vertices with few incident red edges (edges at which the dihedral angle is not a multiple of $\pi/2$). We will show that it is not possible to have just one, or three incident red edges, that two and four red edges force collinearities, and that zero or five (or more) red edges are possible. A summary is listed below.

k	Constraint	Lemma
0	possible	
1	not possible	Lem. 6
2	collinear	Lem. 7
3	not possible	Lem. 8
4	'+' shape	Lem. 9
5	possible	

Throughout the lemmas below, e_i will represent the red edges, and p_i the corresponding points on S_v.

Lemma 6 *No vertex has exactly one incident red edge.*
Proof: Suppose otherwise, and let p_i be the corresponding point of P_v for red edge e_i. Rotate the coordinate system so that p_i is the north pole. Then p_{i-1} and p_{i+1} both lie on the equator. Now, p_{i-1} is connected to p_{i+1} by an orthogonal path. By Lem. 4, their separation must be a multiple of $\pi/2$. This in turn implies the angle at p_i is rectilinear, contradicting the assumption that e_i is red. □

Lemma 7 *If exactly two red edges are incident to a vertex, they are collinear in* $\mathbb{R}^3$.
Proof: Let p_i and p_j be the points on S_v corresponding to the two red edges e_i and e_j. Because p_i and p_j are connected by an orthogonal path, Lem. 4 implies they are antipodal or quarter pairs. If they are antipodal, the claim of the lemma is satisfied. And indeed it is easy to realize this; see Fig. 9. So suppose they form a quarter pair. Let $a = p_ip_j$. Then Lem. 5 implies the orthogonal path $(p_i, \ldots, p_j)$ makes a rectilinear angle at both p_i and p_j. The same holds true for the orthogonal path $(p_j, \ldots, p_i)$. Therefore, e_i and e_j are green edges, a contradiction. □

Although we will not need this fact, it is not difficult to prove that the dihedral angles at the red edges in the previous lemma are equal mod $\pi/2$, as is evident in Fig. 9.

Lemma 8 *No vertex has exactly three incident red edges.*
Proof: Let the red edges be e_i, and p_i their corresponding points on S_v, $i = 1, 2, 3$, and let ρ be the path on S_v. Each of the consecutive pairs (p_i, p_{i+1}) either represents adjacent points of the path ρ, or points connected by an orthogonal path corresponding to intervening green edges; see Fig. 10.

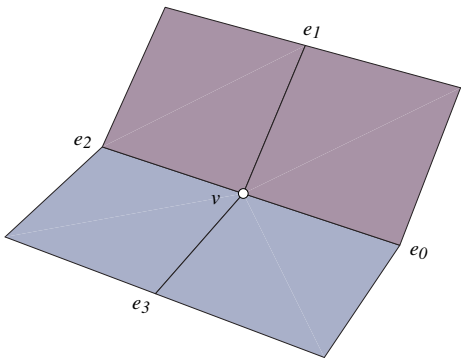

Figure 9: Two red edges e_0 and e_2 incident to v.

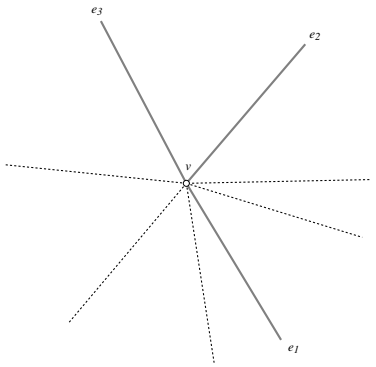

Figure 10: Red edges e_1, e_2, e_2 separated by green edges (dashed).

In the first case, we have a quarter pair; in the second, Lem. 5 implies that the pair is either antipodal or a quarter pair. We now consider cases corresponding to the number of antipodal pairs.

1. Two or more pairs are antipodal. This necessarily places two points p_i, p_j, $i \neq j$, at the same location on S_v, violating the simplicity of ρ.

2. (p_1, p_2) are antipodal, and (p_2, p_3) and (p_3, p_1) are quarter pairs. Rotate so that p_1 and p_2 are poles; then p_3 lies on the equator (because it forms a quarter pair with both p_1 and p_3); see Fig. 11. By Lem. 5 the path $(p_2, \ldots, p_3)$ forms a rectilinear angle with $a = p_2p_3$, as does $(p_3, \ldots, p_1)$ with $a' = p_1p_3$. Thus the angle of the path at p_3 must be rectilinear, and e_3 is a green edge, a contradiction.

3. All three are quarter pairs. This forces the three points p_1, p_2, p_3 to lie at the corners of a triangle with three $\pi/2$ angles; see Fig. 12. Applying Lem. 5 to each of the connecting orthogonal paths between these corners

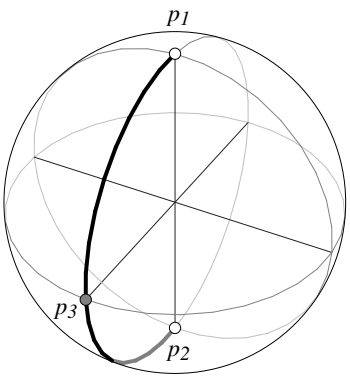

Figure 11: Case 2.

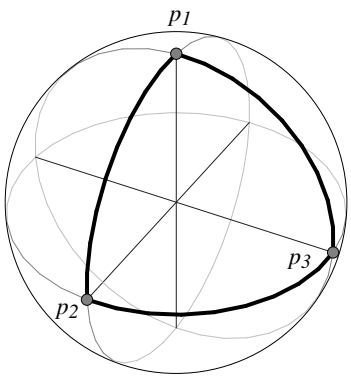

Figure 12: Case 3.

leads to a rectilinear angle at all three points. So all three edges e_1, e_2, e_3 are green, a contradiction.

□

Lemma 9 *If a vertex has exactly four incident red edges, then they must fall into two collinear pairs meeting orthogonally, forming a '+' in 3-space.*
Proof: Let the red edges be e_i, and p_i their corresponding points on S_v, $i = 1, 2, 3, 4$, and let ρ be the path on S_v. As in the proof of Lem. 8, each of the consecutive pairs (p_i, p_{i+1}) is either antipodal or a quarter pair. We consider cases corresponding to the number of antipodal pairs.

1. Three or four pair are antipodal. This necessarily places two points p_i, p_j, $i \neq j$, at the same location on S_v, violating the simplicity of ρ.

2. Two pair are antipodal. Then all four points lie on one great circle, say the equator. Because each p_i is connected to p_{i+1} by an orthogonal path, Lem. 5 implies that their separation must be a multiple of $\pi/2$. To maintain simplicity, the four points must be distributed at quarter-circle intervals. This configuration is realizable, as is shown in Figs. 13 and 14. In both figures, v is of degree 8, but the red/green pattern of incident edges is different. The claim of the lemma is satisfied in that the red edges form a '+'.

3. One pair (p_1, p_2) is antipodal, say at the poles, and all others form quarter pairs. Then p_3 and p_4 must lie on the equator, separated by $\pi/2$; see Fig. 15. The angle between the arcs $a = p_1p_4$ and $a' = p_4p_3$ (which are not necessarily part of the path P_v) is $\pi/2$. Applying Lem. 5 to the orthogonal paths connecting p_1 to p_4 and p_4 to p_3 leads to a rectilinear angle in the path at p_4, a contradiction.

4. No pair is antipodal, so all are quarter pairs. Rotate so that p_1 is the north pole; then both p_2 and p_4 must be on the equator.

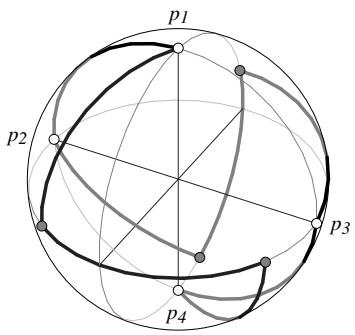

Figure 13: Case 2: $rrggrrgg$.

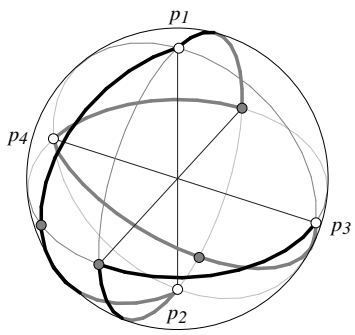

Figure 14: Case 2: $rgrgrgrg$.

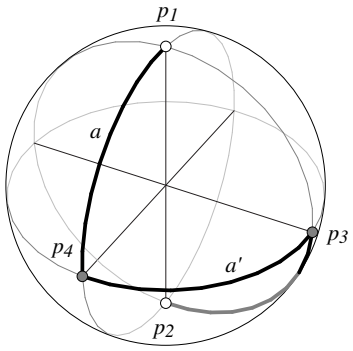

Figure 15: Case 3.

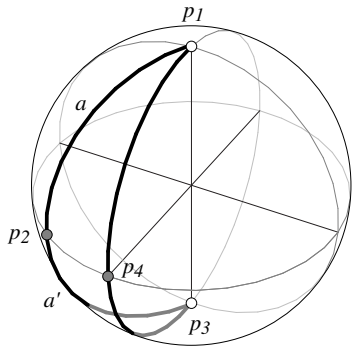

Figure 16: Case 4.

(a) $d(p_2, p_4)$ is a multiple of $\pi/2$. Then the arcs $a = p_2p_1$ and $a' = p_4p_1$ meet at this same rectilinear angle at p_1. Applying Lem. 5 to the paths $(p_1, \ldots, p_2)$ and $(p_4, \ldots, p_1)$ leads to a rectilinear path angle at p_1. Thus e_1 is green, a contradiction.

(b) $d(p_2, p_4)$ is not a multiple of $\pi/2$. Then p_3 must lie on both the circle orthogonal to p_2v and the circle orthogonal to p_4v. These circles intersect at the poles, which forces p_3 to the south pole. The arcs $a = p_1p_2$ and $a' = p_2p_3$ are necessarily collinear at p_2; see Fig. 16. Applying Lem. 5 to the paths $(p_1, \ldots, p_2)$ and $(p_2, \ldots, p_3)$ leads to a rectilinear angle at p_2, a contradiction.

□

It is possible to have five red edges incident to a vertex. Fig. 17 shows one such example, where a quarter pair (p_1, p_5) is connected by four arcs, connecting five edges none of whose dihedral angles are rectilinear. It is also possible to have more than five red edges: any larger number can accordian-fold into a small space, say connecting a quarter pair of points.

The lemmas just derived will be employed in Sec. 6.

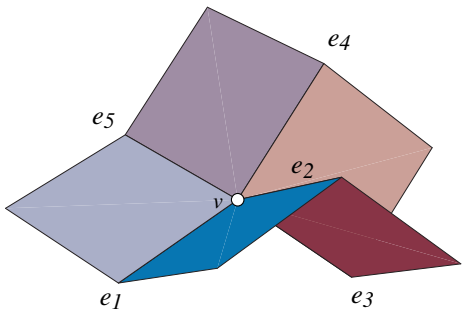

Figure 17: Five red edges $e_1, \ldots, e_5$ incident to v.

5 Euler Formula Computation

Euler's formula says that a (closed, bounded) polyhedron P of V vertices, E edges, and F faces, and of genus g, satisfies this linear relationship:

$$V - E + F = 2 - 2g \ . \tag{2}$$

The quantity $\chi = 2 - 2g$ is known as the *Euler characteristic* of the surface. Euler's formula applies to more general connected graphs: those that are *2-cell* embeddings, in that each of its faces (the regions remaining when the vertices and edges are subtracted from the surface) is a 2-*cell*, a region in which any simple closed curve may be contracted to a point [CO93, p. 274].

We need a somewhat more general form of Euler's formula, which applies to any connected graph embedded[2] in a sphere with h handles: [MT01, p. 84] show that then

$$V - E + F \geq 2 - 2h \ . \tag{3}$$

A sphere with h handles is a surface of genus $g = h$. This is more general for two reasons. First, not every such graph is the 1-skeleton of a polyhedron; for example, a plane tree is in a surface of genus 0, with one exterior 2-cell face. Second, the faces might not be 2-cells; for example, a face could include a handle.

Our goal is to establish a corollary to Euler's formula in the form of Eq. (3) expressed in terms of two quantities: d, the average degree of a vertex, and k, a lower bound on the number of edges in a boundary walk of a face. The first quantity needs no explanation; the second is straightforward for polyhedral graphs but needs a definition for arbitrary graph embeddings. A *facial walk* [MT01, p. 100] visits all the edges bounding a face in a complete traversal. The same edge might be visited twice in one facial walk, as illustrated in Fig. 18. If so, that edge is counted twice in its face-walk count. The quantity k reflects this possible double counting. Each edge of G either gets visited twice by one facial walk, or it appears in exactly two facial walks. Each face of G

[2] To be *embedded* means to be drawn without crossings.

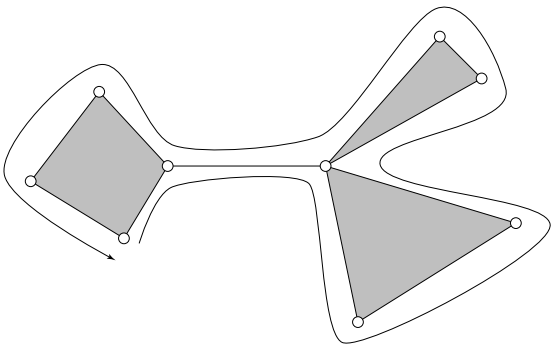

Figure 18: The exterior face has 12 edges in a boundary walk: the 10 surrounding the two triangles and quadrilateral, and the central bridge edge counted twice.

either contains one or more repeated vertices or edges in its facial walk, or it is a cycle of G.

Lemma 10 *Any embedding of a connected graph of F faces on a surface of genus g, whose average vertex degree is d and whose facial walks each include k or more edges, satisfies*

$$F[k - d(k-2)/2] \geq d\chi \ . \tag{4}$$

Proof: The proof is an elementary counting argument, which, due to its unfamiliar form, we present in perhaps more detail than it deserves. We start from the form of Euler's formula in Eq. (3):

$$\begin{aligned} V - E + F &\geq \chi \\ V + F - \chi &\geq E \end{aligned} \tag{5}$$

Because every edge is incident to two vertices, $dV = 2E$, and so:

$$V = \frac{2}{d}E \ . \tag{6}$$

Because every edge is counted exactly twice in facial walks, and because k is a lower bound on the number of edges in any walk, we have $kF \leq 2E$, or

$$\frac{k}{2}F \leq E \ . \tag{7}$$

Now we substitute Eq. (6) into Eq. (5) to eliminate V:

$$\frac{2}{d}E + F - \chi \quad \geq \quad E$$

$$\begin{aligned} F - \chi &\geq E(1 - 2/d) \\ \frac{F - \chi}{1 - 2/d} &\geq E \ . \end{aligned} \tag{8}$$

Putting this together with Eq. (7) eliminates E:

$$\begin{aligned} \frac{F - \chi}{1 - 2/d} &\geq \frac{k}{2} F \\ \frac{d(F - \chi)}{d - 2} &\geq \frac{k}{2} F \\ dF - d\chi &\geq (dk/2 - k)F \\ F[d(1 - k/2) + k] &\geq d\chi \\ F[k - d(k-2)/2] &\geq d\chi \ . \end{aligned} \tag{9}$$

□

To illustrate the import of this result, consider a polyhedron of genus zero: $k = 3$, because every face must have at least three edges, and $\chi = 2 - 2g = 2$. Then Eq. (9) becomes

$$F[3 - d/2] \geq 2d \ . \tag{10}$$

This requires the factor multiplying F to be strictly positive: $3 - d/2 > 0$, which implies that $d < 6$. This follows from the familiar fact that the average vertex degree of a simple planar graph is less than six.[3]

In the next section we will apply Lem. 10 in two circumstances: genus zero and one, both with the lower bound $k = 4$.

Corollary 11 *For connected graphs embedded on a surface of genus zero, if $k = 4$, then $d < 4$.*

Proof: Substituting $\chi = 2$ and $k = 4$ into Eq. (4) gives

$$F[4 - d] \geq 2d \tag{11}$$

which, in order for the factor of F to be positive, implies that $d < 4$. □

For example, a cube has $k = 4$ and all vertices have degree 3, so $d = 3 < 4$. A more complex example is the "trapezoidal hexacontahedron,"[4] an Archimedian compound all of whose faces are quadrilaterals, and so $k = 4$. It has 12 degree-5 vertices, 30 degree-4 vertices, and 20 degree-3 vertices. Thus its average vertex degree d is

$$\frac{5 \cdot 12 + 4 \cdot 30 + 3 \cdot 20}{12 + 30 + 20} = \frac{240}{62} \approx 3.87 \ .$$

Corollary 12 *For connected graphs embedded on a surface of genus one, if $k = 4$, then $d \leq 4$.*

[3] E.g., [Har72, p. 104], Cor. 11.1(e), or [AH88, p. 446], Exer. 21.

[4] See, e.g., `www.georgehart.com` .

Proof: Substituting $\chi = 0$ and $k = 4$ into Eq. (4) gives

$$F[4 - d] \geq 0 \tag{12}$$

Now it could be that the factor of F is zero; so we must have $4 - d \geq 0$, i.e., $d \leq 4$. □

For example, a cube with a rectangular hole connecting top and bottom faces leads to $k = 4$ when the punctured top and bottom faces are partitioned into four quadrilaterals each. All vertices then have degree 4, so $d = 4$.

6 Orthogonality Forced for Genus Zero and One

We establish in this section that the answer to Question 2 is YES for polyhedra of genus zero and one. We first define a red subgraph, then prove that $k = 4$ for it, and, finally, exploit the two corollaries above.

6.1 Red Subgraph G_r

Starting with the 1-skeleton G of the polyhedron, with its edges colored green or red depending on whether the dihedral angle is rectilinear or not, we perform the following operations to reach G_r:

1. Remove all green edges from G, retaining only red edges.
2. Merge edges meeting at a degree-2 vertex: if a node y is of degree 2, with incident edges (x, y) and (y, z), delete y and those edges and replace with (x, z).
3. Select one component of the resulting graph and call it G_r, the *red subgraph*.

If G contains any red edge, then there is a nonempty G_r. Note that G_r is realized in $\mathbb{R}^3$ with straight segments for each edge, because Lem. 7 guarantees that the "erasing" of degree-2 vertices merges collinear polyhedron edges.

6.2 Facial Walks in G_r

Lemma 13 *Every face walk in G_r contains a cycle.*

Proof: Suppose a walk $W = (v_0, v_1, \ldots, v_{m-1})$ of some face is not itself a cycle. Then it must repeat one or more vertices (to violate the definition of a cycle). Let $v_i = v_j$, $i < j$, be the first vertex repeated starting from v_0.

1. It cannot be that $j = i + 1$, for then the edge (v_i, v_j) is a loop, and G_r is loopless.
2. Suppose $j = i + 2$. Then v_i must be degree 1 in G_r, in contradiction to Lem. 6.
3. So $j \geq i+3$. Then $(v_i, v_{i+1}, \ldots, v_j)$ is a cycle contained in W, establishing the lemma.

□

Lemma 14 *Every cycle of G_r contains at least four edges.*
Proof: Let $C = (e_1, e_2, \ldots, e_m)$ be a cycle that contains no other cycle in its (counterclockwise) interior. If G_r contains any cycle, then it contains such an empty one. Because G_r includes no degree-1 vertices by Lem. 6, C contains no (red) edges at all. Thus, in G, e_i and e_{i+1} are separated by green edges (or no edges). Let v be the vertex shared by e_i and e_{i+1}. Then p_i and p_{i+1} are connected by an orthogonal path on S_v. Lem. 4 then says their separation on S_v is a multiple of $\pi/2$. Thus, in $\mathbb{R}^3$, the geometric angle between e_i and e_{i+1} is $\pm\pi/2$, or π–i.e., they are collinear. The claim follows because one cannot form a triangle in $\mathbb{R}^3$ from segments that meet at these angles. Four edges are needed to close a cycle, e.g., Fig. 19. □

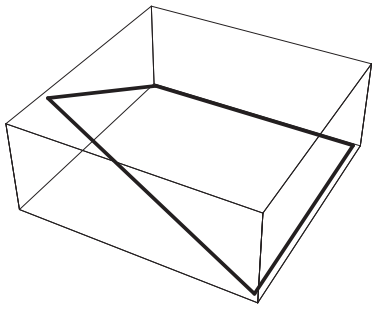

Figure 19: A skew quadrilateral in space; all turns are $\pi/2$.

6.3 Concluding Theorem

Theorem 15 *Any polyhedron of genus zero or one, all of whose faces are rectangles, must be an orthogonal polyhedron: all of its dihedral angles are multiples of $\pi/2$.*
Proof: By Lem. 14 we know that every cycle in G_r contains at least four edges, and by Lem. 13 we know that every face walk contains a cycle. Thus every face walk for an embedding of G_r contains at least four edges. Thus $k = 4$ in the notation of Lem. 10.

$g = 0$ Applying Cor. 11 leads to the conclusion that d must be strictly less than 4, which implies that G_r must include at least one vertex of degree 3. We defined G_r to merge all degree-2 nodes, so it has none of those. Lem. 6 and 8 show that it can have no nodes of degree 1 or 3 respectively. Thus the minimum degree of a node of G_r is 4, and an average degree $d < 4$ is impossible.

$g = 1$ Applying Cor. 12 leads to the conclusion that $d \leq 4$. As above, the minimum degree of a node of G_r is 4. So then we must have every node of

G_r exactly degree 4. (This is precisely what is achieved in the cube-with-a-hole example, incidentally.) Now Lem. 9 says that the edges incident to a degree-4 vertex of G_r come in two collinear pairs. We now argue that this implies that the surface contains an infinite line, a contradiction to the fact that a polyhedron is bounded.

Let e_1 be an edge incident to a vertex v_1 of G_r. v_1 must be of degree 4 as above, and therefore Lem. 9 provides an e_2 collinear with e_1. Call the other endpoint of e_2 vertex v_2. Repeating the argument leads to an edge e_3 collinear with e_2. In this way we produce a sequence of collinear edges, $e_1, e_2, \ldots$. There is no end to this sequence, providing a contradiction.

□

One way to view the above proof is that a red cycle cannot turn in $\mathbb{R}^3$ with only vertices of red-degree no more than 4. It requires degree-5 vertices, or higher degree vertices, to permit a cycle to form. This is exactly how the polyhedron described in Sec. 2 is constructed.[5] The four corners of each square in Fig. 2 are red-degree 5 vertices, and the fifth vertex $\frac{1}{2}$ offset from the center of each square is red-degree 8, as illustrated in Fig. 20.

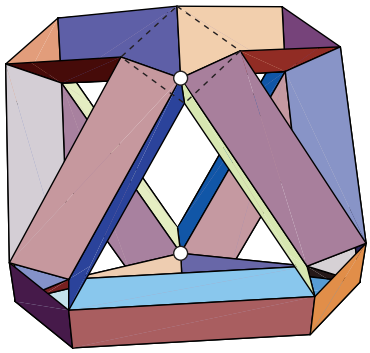

Figure 20: The polyhedron in Fig. 2 with five top faces removed. The indicated vertices have degree 8 in G_r: 8 incident nonrectilinear edges.

6.4 Genus Two and Higher

It seems the computations used in the previous theorem provide no useful constraint when $g \geq 2$. Then $\chi \leq -2$, and Eq. (4) only yields (for $k = 4$)

$$\begin{aligned} F(4-d) &\geq -2 \\ F &\geq \frac{2}{d-4} \end{aligned}$$

which can be satisfied for all d. So there is effectively no constraint on d for $g \geq 2$. This seems to reveal the limit of this proof technique.

[5] This is, in fact, how it was discovered.

7 Discussion

The obvious open problem is to close the gap—between Theorem 15, $g \leq 1$, when orthogonality in $\mathbb{R}^3$ is forced, and Theorem 1, $g \geq 7$, when it is not forced. Is there a nonorthogonal polyhedron of genus g, $2 \leq g \leq 6$, constructed entirely from rectangles?

Perhaps a more interesting problem is to extend these results to other planar constraints, and ask if they imply a restriction in $\mathbb{R}^3$. For example, if a polyhedron is constructed out of convex polygons whose angles are all multiples of π/k, $k > 2$, does this imply any restriction on the realizable dihedral angles?

Finally, because so little is known about nonoverlapping (simple) edge-unfoldings, perhaps just-barely nonsimple unfoldings, of the type exemplified by Fig. 5, should be considered.

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