

Hamiltonian Quasigeodesics yield Nets

Joseph O’Rourke*

June 14, 2022

Abstract

This note establishes that every polyhedron that has a Hamiltonian quasigeodesic can be edge-unfolded to a net.

1 Introduction

This note establishes one result (Theorem 1) that is more an observation than a theorem, as it largely depends on definitions and a straightforward argument. Nevertheless, it may be of some interest, making connections between several different aspects of convex polyhedra.

We start with the definitions needed to describe the result. We restrict attention to convex polyhedra. An *edge-unfolding* of a convex polyhedron P is a collection of edge cuts forming a spanning tree of the vertices, which unfolds the surface of P to one piece in the plane. If the planar unfolding is a simple polygon, with no self-overlap, it is known as a *net*. It is a long unsolved problem to decide whether or not every convex polyhedron has a net. This has become known as “Dürer’s problem” [DO07] [O’R13]. The conclusion of Theorem 1 is that, under certain conditions, P has a net.

A *quasigeodesic* has at most π angle to each side at every point, in contrast to geodesics which have exactly π to each side. Quasigeodesics can pass through vertices, and are geodesic segments between vertices. Pogorelov proved that every convex polyhedron has at least three simple closed quasigeodesics [Pog49]. Here we focus on quasigeodesics that follow edges of the 1-skeleton of P . A *Hamiltonian quasigeodesic* is a simple closed quasigeodesic following edges of P , and passing through every vertex of P . Thus a Hamiltonian quasigeodesic is a Hamiltonian circuit, but with angle restrictions at each vertex. This notion was introduced and played a role in [HLM⁺22]. Earlier, Hamiltonian unfoldings without the quasigeodesic condition (also called *zipper edge-unfoldings*) were studied in [DDL⁺10].

Of course not every convex polyhedron has a Hamiltonian quasigeodesic because some polyhedra have no Hamiltonian circuit, for example, the rhombic

*Smith College, Northampton, MA, USA. jorourke@smith.edu.

dodecahedron. In [HLM⁺22] it was shown that three of the Platonic solids—tetrahedron, octahedron, cube—have a Hamiltonian quasigeodesic, but the dodecahedron and the icosahedron do not. Note that the boundary of a doubly-covered convex polygon, which is treated as a polyhedron in this literature, is a Hamiltonian quasigeodesic.

We can now state the theorem.

Theorem 1 *If a convex polyhedron P has a Hamiltonian quasigeodesic Q , then there is an edge-unfolding of P to a net.*

Proof: The proof follows from two straightforward claims.

- (1) The portion of P enclosed to the right or left of Q is isometric to a convex polygon.
- (2) Joining two convex polygons A and B along a shared edge e avoids overlap between A and B .

We now add more detail to these claims. Consider Q directed, partitioning P into two “halves” A and B , with $P = A \cup B$.

- (1) Because Q is a quasigeodesic, the angle to the left of every vertex of Q is $\leq \pi$. Because Q passes through every vertex of P , there are no vertices of P to the left, enclosed by Q , and because all curvature is concentrated at vertices, no curvature. Therefore the region A of the surface of P to the left of Q is isometric to a planar convex polygon (convex because of the $\leq \pi$ condition); and similarly for B . Let \bar{A} and \bar{B} be planar embeddings of A and B .
- (2) Although \bar{A} and \bar{B} are not necessarily congruent, their boundaries are each composed of the same edges of Q . Select any edge $e \in Q$. Then joining \bar{A} to \bar{B} to either side of and sharing e produces a non-overlapping simple polygon—a net—because they sit on opposite sides of the line containing e , and so cannot overlap one another. One can view this as unfolding after cutting all edges of Q but e , a spanning cut-path.

□

2 Examples

We present three examples. In each we label faces as F, R, K, L, T, B, for Front, Right, back, Left, Top, Bottom. The first is a pyramid, the top half of a regular octahedron: see Fig. 1. Note the angles along Q are $\pi \times \{\frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1\}$ —all $\leq \pi$. Two different ways of joining A to B are shown.

Our second example is the full regular octahedron: see Fig. 2. Here and in the next example, \bar{A} and \bar{B} are congruent. Again two unfoldings are shown.

Finally, Fig. 3 illustrates the cube with Q forming a “napkin holder,” leading to two of the 11 nets of a cube.

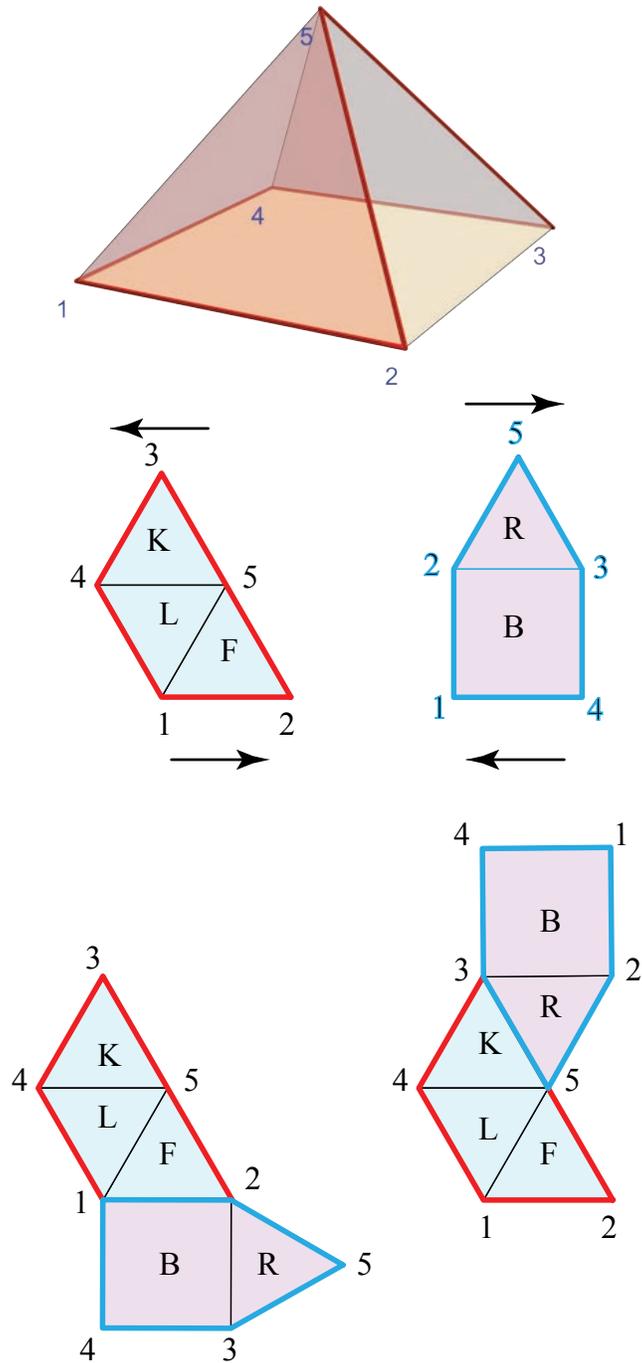


Figure 1: $Q = 12534$. Arrows indicate counterclockwise (red) ordering around A , and clockwise (blue) ordering around B .

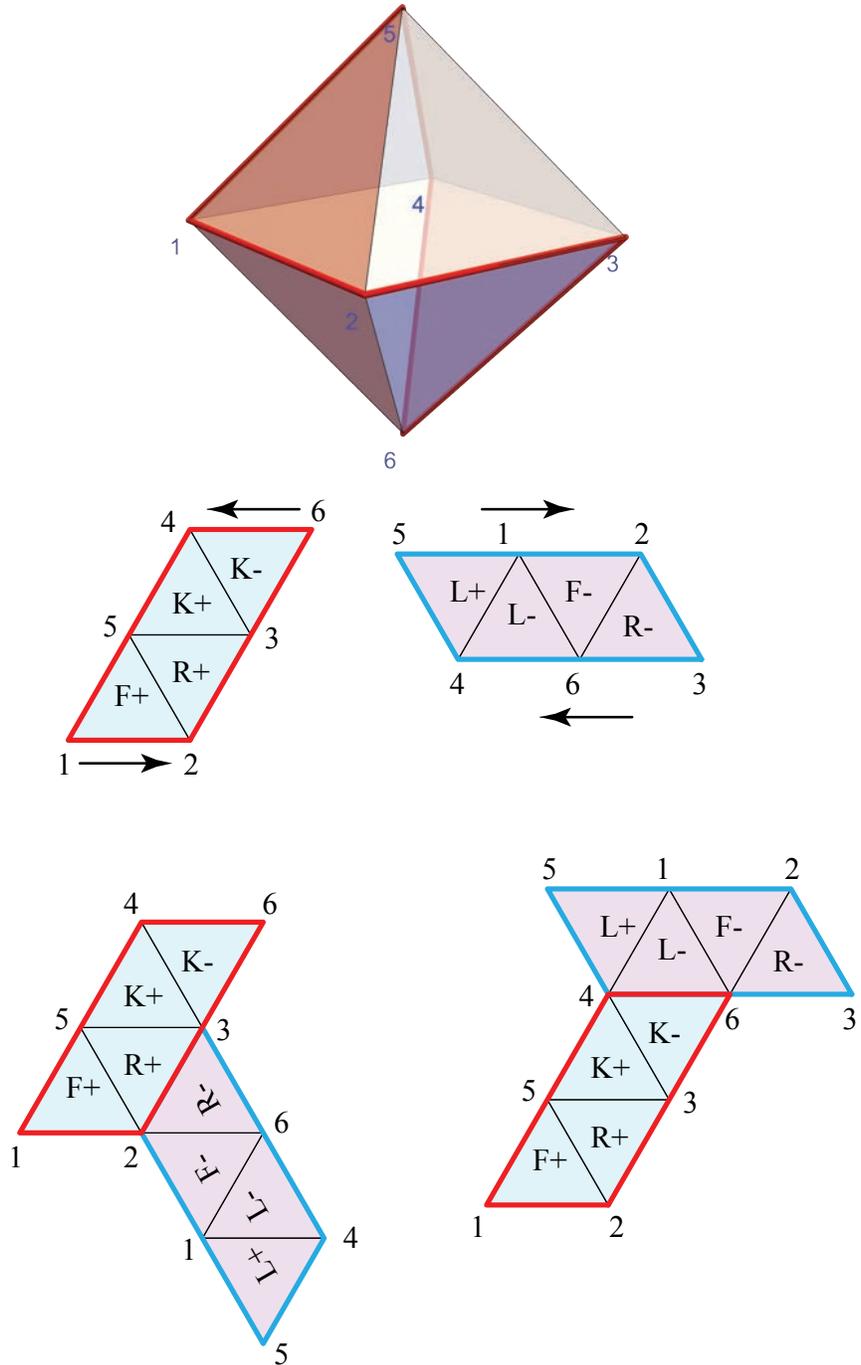


Figure 2: $Q = 123645$.

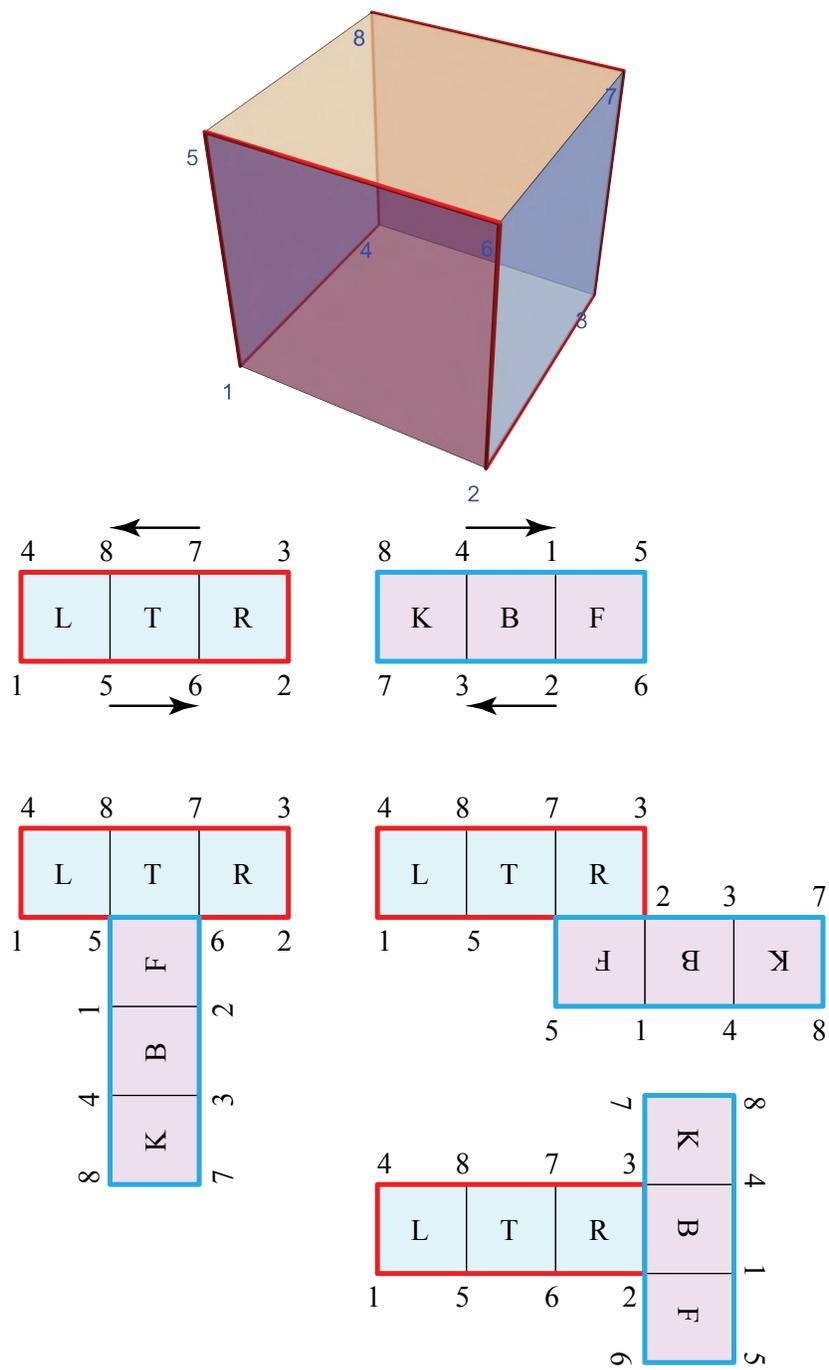


Figure 3: $Q = 15623784$.

3 Open Problems

Although simple closed quasigeodesics are always present, it appears that Hamiltonian quasigeodesics are relatively rare. It would be useful (and likely difficult) to characterize those convex polyhedra that have a Hamiltonian quasigeodesic, for we then know that each has a net.

I have not explored non-convex polyhedra P , but Theorem 1 still holds as long as P has a Hamiltonian quasigeodesic.

Acknowledgements. I thank my coauthors on [HLM⁺22] for stimulating discussions.

References

- [DDL⁺10] Erik D. Demaine, Martin L. Demaine, Anna Lubiw, Arlo Shallit, and Jonah Shallit. Zipper unfoldings of polyhedral complexes. In *Proc. 22nd Canad. Conf. Comput. Geom.*, pages 219–222, August 2010.
- [DO07] Erik D. Demaine and Joseph O’Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge University Press, 2007. <http://www.gfalop.org>.
- [HLM⁺22] Thomas Hull, Anna Lubiw, Klara Mundilova, Chie Nara, Joseph O’Rourke, Josef Tkadlec, and Ryuhei Uehara. Quasi-twisting convex polyhedra. Submitted for publication, May 2022.
- [O’R13] Joseph O’Rourke. Dürer’s problem. In Marjorie Senechal, editor, *Shaping Space: Exploring Polyhedra in Nature, Art, and the Geometrical Imagination*, pages 77–86. Springer, 2013.
- [Pog49] Aleksei V. Pogorelov. Quasi-geodesic lines on a convex surface. *Mat. Sb.*, 25(62):275–306, 1949. English transl., *Amer. Math. Soc. Transl.* 74, 1952.