Folding Polygons to Convex Polyhedra

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1 Introduction

A line of research investigation has opened in the last decade that spans two millennia of geometry: from Greek explorations of convex polyhedra to cutting-edge geometrical research today. And yet the topic is elementary and can be fruitfully explored in the classroom at a wide range of educational levels.

The main question driving this research is simply stated:

Q1. Which polygons can fold to a convex polyhedron?

Unpacking this question requires defining its four technical terms. A *polygon* P is a planar shape whose boundary is composed of straight segments. It is a single-piece shape that could be cut out from a piece of paper by straight scissors cuts. A *polyhedron* $\mathcal{Q}$ is the 3D analog of a 2D polygon. It is a solid in space whose boundary is composed of polygonal *faces*. As we are concerned mainly with this surface boundary, we will use $\mathcal{Q}$ to refer to the surface rather than the solid. A *convex* polyhedron is one without dents or indentations. Examples include the five "regular" Platonic solids (tetrahedron, octahedron, cube, dodecahedron, icosahedron), the thirteen "semi-regular" Archimedean solids (truncated icosahedron (i.e., a soccer ball), etc.), or any of an infinite variety of irregular convex polyhedra. As we will only discuss convex polyhedra in this article, the "convex" qualification will be often left implicit. Finally, to *fold* a polygon to a polyhedron means to crease the polygon and fold it into 3D so that it forms precisely the surface of the polyhedron, without any wrapping overlap, and without leaving any gaps. Another way to view this is in reverse: a polygon P can fold to a polyhedron $\mathcal{Q}$ if $\mathcal{Q}$ could be cut open and unfolded flat to P.

Two examples are shown in Figure 1. Note from (a) that creases of P, which become edges of $\mathcal{Q}$, do not necessarily begin or end at vertices (corners) of P. Note from (b) that a nonconvex polygon might fold to a convex polyhedron. In the alternative cut-open-and-unfold view, the cuts in both these examples are along polyhedron edges. We will see that in general the cuts are arbitrary surface segments.

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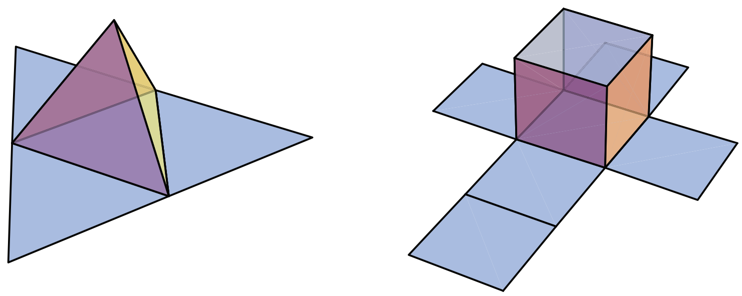

Figure 1: (a) Folding an equilateral triangle to a regular tetrahedron. (b) Folding the Latin cross to a cube.

These examples already show that at least some polygons can fold to some polyhedra. This naturally raises this question:

Q2. Does every polygon fold to some convex polyhedron?

The answer is NO, established by the polygon P_u shown in Figure 2. We now

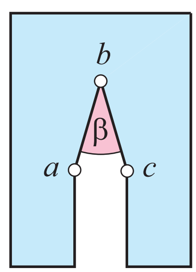

Figure 2: An un-foldable polygon P_u. [Fig. 25.1 in [DO07].]

offer an elementary proof that this P_u cannot fold to a convex polyhedron. The key observation is that the total angle surrounding any true vertex v (corner) of a convex polyhedron is less than $360°$. In technical language, the sum of the face angles *incident to* v is less than $360°$. This is one reason why there is not a 6-th Platonic solid: gluing together three equilateral triangles to a vertex produces the tetrahedron, four yields the octahedron, five the icosahedron, but six times $60° = 360°$ makes a flat region, not a true vertex. Another way to phrase this constraint, more useful for our purposes, is that the total angle surrounding any point p on a convex polyhedron (vertex or not) is $\leq 360°$. When the angle is exactly $360°$, p is not a vertex. This constraint does not hold, incidentally, for nonconvex polyhedra: there there is no a priori bound on the total face angle surrounding surface points.

The consequence of the angle constraint is that, when we *glue* the perimeter of any P to itself to form the folding, we can never glue more than $360°$ around any one point, for otherwise, $\mathcal{Q}$ would not be convex.

Now, classify the vertices of a polygon P as either *convex*, having internal angle $< 180°$, or *reflex*, having internal angle $> 180°$. The polygon P_u in Figure 2 has three consecutive reflex vertices (a, b, c), with the complementary exterior angle β at b small. All other vertices are convex, with interior angles strictly larger than β.

Now we imagine how we might glue up the perimeter in the vicinity of the problematic vertex b. There are only two options. Either we "zip" together edges ba and bc, or some other point or points of the perimeter glue into b. The first possibility forces a to glue to c, exceeding $360°$ there, and violating the angle constraint. So this is ruled out. The second possibility cannot occur with P_u, because there is no perimeter point with small enough internal angle to fit inside β at b. This rules out the second possibility, and shows that P_u cannot fold to any convex polyhedron.

So now we know that sometimes polygons can, and sometimes they cannot, fold to a polyhedron, which justifies the phrasing of **Q1**: *Which* polygons can fold to a polyhedron? Before pursuing this question further, it is natural to wonder how common *foldability* in this sense is. This would lead into a thicket of questions of how to define a "random polygon," but suffice it to say that under reasonable assumptions, the answer is that foldability is rare: if you cut out a random polygon of n sides from a piece of paper, the probability that it will fold (to a convex polyhedron) approaches zero as n gets large.

Now that we have explained **Q1** and explored a few basic issues, it is time to admit that there is as yet no satisfactory answer to the question. In particular, there is no characterization of which polygons fold and which do not, except in certain special cases, explored below. Nevertheless, there is now an algorithm, implemented in publicly available software,[1] that will take any specific P and tell you whether it can fold, and if so, give some information about the $\mathcal{Q}$ to which it can fold. Before we can explain this somewhat mysterious statement, we turn to the powerful theorem that sits at the heart of this research.

2 Alexandrov's Theorem

Alexandrov's theorem is both beautiful and elementary (in statement, not in its proof techniques), but is not taught in any Western school curriculum below specialized sporadic graduate-level courses, as far as I know. Part of the reason is language: he published his theorem in Russian in 1941 [Ale41], and included it in his 1950 masterwork *Convex Polyhedra* [Ale50], again in Russian. This was translated to German in 1958 [Ale58], but only in 2005 did an English translation of his book appear [Ale05]. Fortunately, the beauty and utility of this theorem is now more widely recognized, and is essential for our topic.

I will simplify and specialize his theorem to our needs. First, let us define an *Alexandrov gluing* of a polygon to be just what we need for a folding to a convex

[1] `http://theory.lcs.mit.edu/~edemaine/aleksandrov/cross/`.

polyhedron. There are three conditions that must be satisfied for a gluing to be Alexandrov:

(a) The gluing must entirely consume the perimeter of the polygon with matches: every point p of the perimeter must be matched with one or more points of the perimeter. Here we allow isolated points to be mateless (or to match with themselves), as we did in Figure 2 when considering "zipping" in the neighborhood of b.

(b) The gluing creates no more than 360° angle at any point. (This is our angle condition for convex polyhedra.)

(c) The gluing should result in a *topological sphere*, that is, a surface that could be deformed to a sphere. In other words, not a torus (donut), not a fundamentally twisted shape, etc., but rather what amounts to a lumpy, closed bag.

This third condition is difficult to state precisely without introducing technical language from topology.[2] In any case, I hope it is clear that if a gluing has any hope of producing a convex polyhedron, it must be an Alexandrov gluing, for the three conditions just specify what is obviously necessary—no gaps, the 360° condition, and producing a spherical shape.

Theorem (Alexandrov). Any Alexandrov gluing corresponds to a unique convex polyhedron (where a doubly covered polygon is considered a polyhedron).

Let us ignore the parenthetical caveat for a moment to emphasize what this is saying: the obvious necessary conditions for a polygon to fold to a polyhedron are also sufficient. Not only that, the resulting polyhedron is unique. This means that any time you can find an Alexandrov gluing, you have created a convex polyhedron. We will see that one catch is that Alexandrov's proof was an existence proof: so you have created a particular convex polyhedron but you don't know what it looks like!

We have already seen two Alexandrov gluings in Figure 1, but their foldings were obvious, both due to their regularity and because the crease lines are self-evident. But consider the unusual folding of an equilateral triangle in Figure 3(a). One can easily check that it is an Alexandrov gluing. Condition (a) is satisfied because no perimeter sections are left unmatched. Condition (b) is met at the gluing together of $\{x, A, B, C\}$, whose angles sum to 360°, and at the four "pinch" or *fold points* $\{a, b, c, d\}$ glued to themselves, with each angle 180° there, i.e., the flat side of the triangle. All other points glued together are $180° + 180° = 360°$. That condition (c) is satisfied is perhaps best verified by taping a folded paper triangle according to the gluing instructions and seeing that the result is a sort of bag. Alexandrov's theorem says that this bag is a particular convex polyhedron. In fact, it is the irregular tetrahedron shown

[2] The shape should be *homeomorphic* to a sphere.

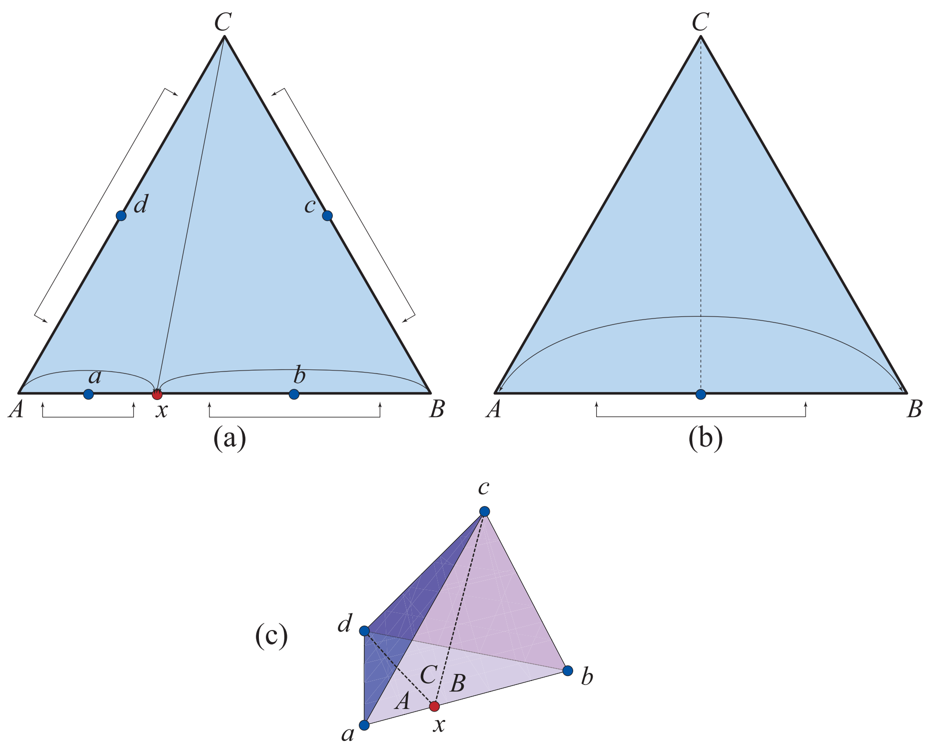

Figure 3: Alexandrov gluings of an equilateral triangle. (a) The three corners $\{A, B, C\}$ all glue to point x. The four fold points $\{a, b, c, d\}$ become the four vertices of the resulting tetrahedron in (c), two of whose faces are $\triangle abd$, and $\triangle bcd$. (b) A folding that creases the triangle down an altitude, gluing A and B together, and edge AC to BC. (c) View of the tetrahedron that results from the gluing in (a). Note the three corners $\{A, B, C\}$ "disappear," forming 360° at x on tetrahedron edge ab.

in (c) of the figure, which you might be able to coax out of your taped-triangle bag with some nudging.

The folding in Figure 3(b) is also an Alexandrov gluing, but what it produces is simply a doubly covered, flat $30°-60°-90°$ triangle. And this is the reason for the exception clause in Alexandrov's theorem: gluings might produce zero-volume flat "polyhedra."

3 Folding Convex Polygons

Although foldability in general is rare, every convex polygon folds to a polyhedron.[3] A *convex polygon* is one without dents: every vertex is convex. Not only do all these fold, they all fold to an infinite variety of polyhedra:

Theorem. Every convex polygon folds to an infinite number (a continuum) of noncongruent convex polyhedra.

The essence of this claim follows from Alexandrov's theorem. Take any convex polygon P, and mark a point x on its boundary. Walk around half the perimeter and mark an opposite point y. Now glue the perimeter half from x to y to the half from y to x; see Figure 4(a). Now we argue that this

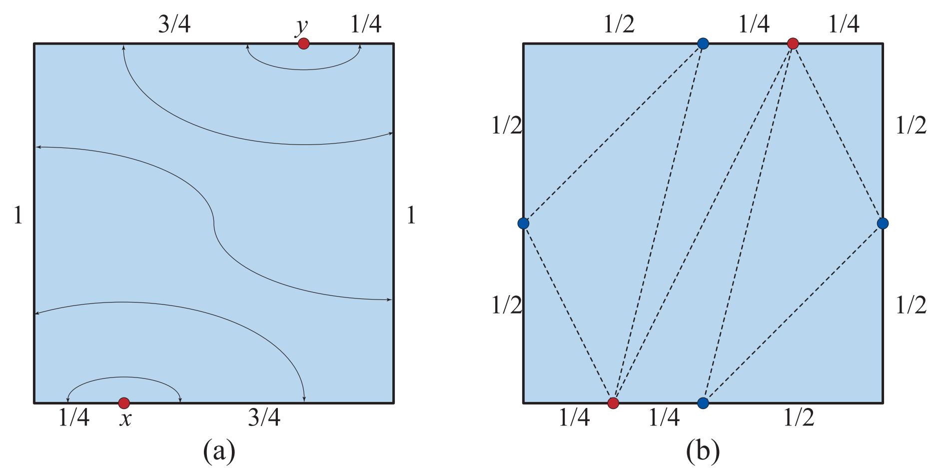

Figure 4: (a) A perimeter-halving folding of a unit square. x is $\frac{1}{4}$ from the lower left corner, and y is $\frac{1}{4}$ from the upper right corner. The length 2 perimeter half is glued symmetrically as indicated. The folding produces (non-obviously!) an octahedron. (b) Crease pattern of edges, and vertices of octahedron. As in Figure 3, the corners of the polygon "disappear" in the folding.

[3]Shephard [She75] was the first to study the connection between what he called convex "nets" (polygons) and convex polyhedra, largely from the viewpoint of unfolding rather than folding.

gluing is Alexandrov. Certainly it consumes all the perimeter (a). The key is requirement (b): no more than 360° is glued at any one point. At the fold points x and y, the amount of angle is $\leq 180°$. Any other two points glued together either sum to exactly 360°, if the points are interior to an edge of P, or to less than 360°, if one or the other is a vertex of P. It is here that convexity of P is used: any convex vertex has an interior angle $< 180°$. That the gluing is a topological sphere (c) can be seen if one views the perimeter-halving gluing as zipping up a pocketbook.

So, Alexandrov's theorem says that every perimeter halving folds to a convex polyhedron. (The folding of Figure 4(a) leads to an octahedron following the crease pattern in (b).) Sliding x around the boundary, and y correspondingly, leads to a continuum of foldings. That among these are also an infinite number of noncongruent polyhedra is not obvious, but it is so.

Figure 5 shows the continuum achieved by perimeter-halving foldings of a square [ADO03]. Starting from the doubly covered $1 \times \frac{1}{2}$ rectangle achieved by creasing down a midline (3-o'clock position in the diagram) the continuum continues clockwise to the doubly covered right triangle achieved by creasing down a diagonal (9-o'clock position). This corresponds to sliding x from the midpoint of an edge of the square to an adjacent corner. Continuing sliding x repeats the shapes in mirror image (clockwise from 9- to 3-o'clock). Incidentally, this figure represents only a portion of the polyhedra foldable from a square. See [DO07, p. 416] for the full variety.

As a practical experiment, one could cut out of paper any convex polygon, start creasing it at an arbitrary x, and "zip" up the boundary from there with tape, and eventually arriving at y; no measurement of the perimeter need be made. The result will be a handbag- or pita-like shape, which, by Alexandrov's theorem, may be coaxed (with patience!) to reveal the creases that fold it into its unique polyhedral form.

So far the "space" of all foldings of regular polygons has been explored, but there remains as yet little general understanding of the phenomenon.

4 The Foldings of the Latin Cross

We have just canvased the foldings of convex polygons. How about nonconvex polygons? Here we enter largely unknown territory. My coauthors (including five college students) and I decided to explore the foldings of the Latin cross, as a test case [DDL+99] [DO07, Sec. 25.6]. What we found, to our surprise, is that the Latin cross folds not only to the cube (Figure 1(b)), but to 22 other distinct convex polyhedra: two flat quadrilaterals, seven tetrahedra, three pentahedra, four hexahedra, and six octahedra. See Figure 6. Here there is no continuum—the nonconvexities block the sliding possible with convex polygons.

How these foldings are achieved is by no means obvious. Figure 7 illustrates just one of the 23 foldings in detail, a delicate folding to a tetrahedron. The other foldings are equally intricate.

Aside from this one detailed example, we are left largely without a general

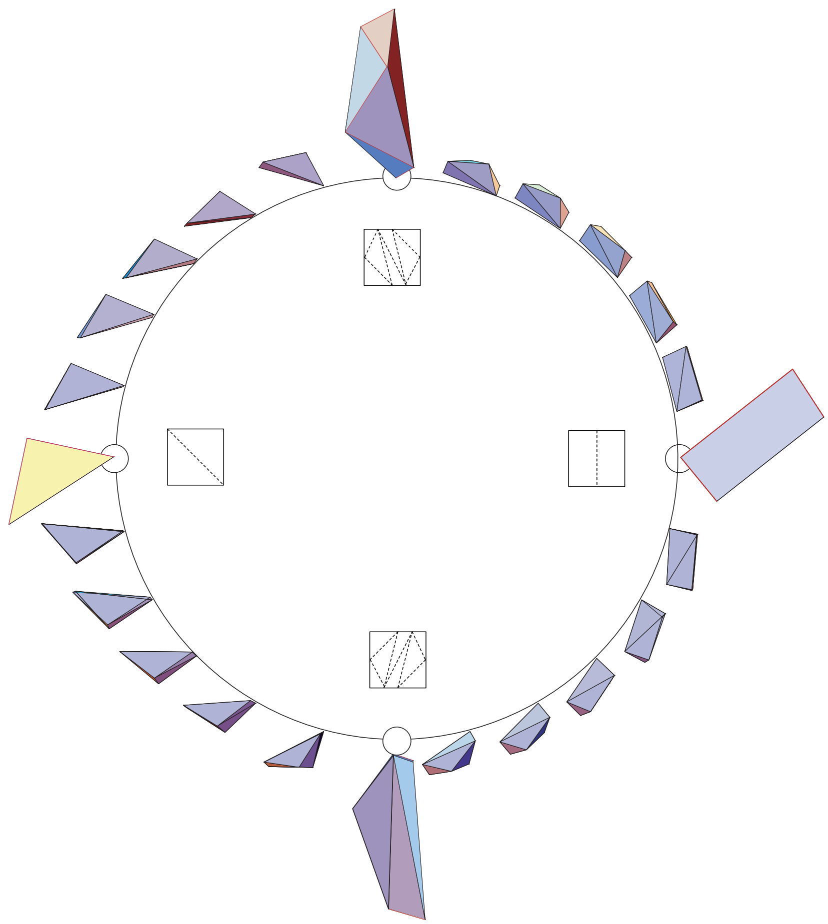

Figure 5: Continuum of perimeter-halvings of square. Four crease patterns are shown. The octahedron at the 6- and 12-o'clock positions corresponds to Figure 4(b).

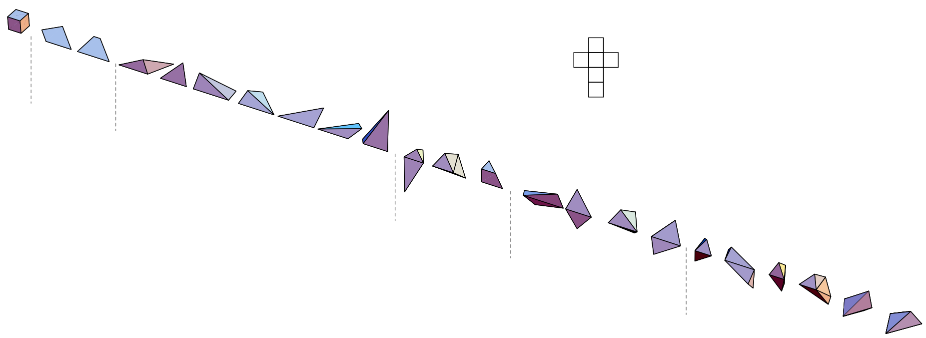

Figure 6: The 23 polyhedra foldable from the Latin cross. [Fig. 25.30 in [DO07].]

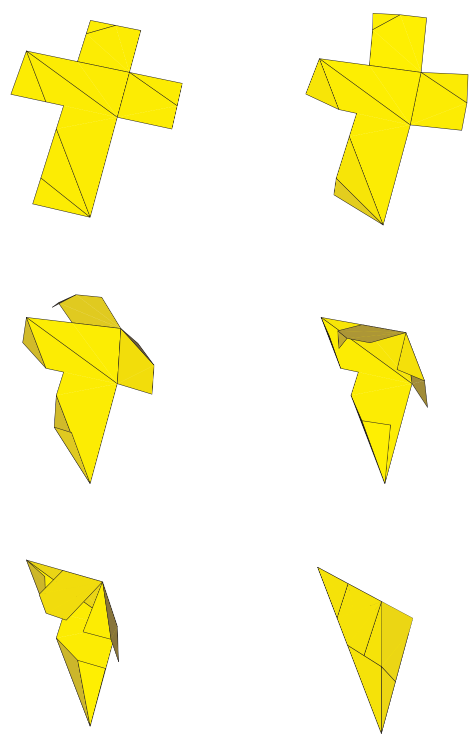

Figure 7: Folding the fourth polyhedron in Figure 6. [Fig. 25.10 in [DO07], from the video [DDL+99].]

theory encompassing the foldings of nonconvex polygons. In particular, the polyhedra achievable from the other ten hexamino unfoldings of a cube (besides the Latin cross) remain to be explored.

5 Reconstruction of 3D Polyhedra

As mentioned previously, there is an algorithm (and software) to take any given polygon P and list all the Alexandrov gluings of P. But to which (unique) polyhedra these gluings correspond is unknown. The polyhedra displayed in Figures 5 and 6 were reconstructed by laborious ad hoc techniques that cannot extend much beyond octahedra. Quite recently a group of researchers [BI06] [O'R06] discovered a way to convert Alexandrov's existence proof into a constructive proof, with the implication that solving a particular differential equation will lead to the unique 3D shape guaranteed by Alexandrov's theorem. It remains to be seen if this advance will lead to a practical numerical method of computing the 3D shape of the polyhedron guaranteed by Alexandrov's theorem.

6 Nonconvex Polyhedra

Here we reach the frontier of knowledge on this topic. Let me close with one outstanding unsolved ("open") problem, which involves nonconvexity:[4]

> **Q3.** Does every polygon fold to some (perhaps nonconvex) polyhedron?

Even restricting the fold to be perimeter-halving leaves the question unresolved. If, as I suspect, the answer to **Q3** is NO (which could be established by a single counterexample, incidentally), then immediately we enter new uncharted territory by deleting the qualifier "convex" from **Q1**: Which polygons can fold to a polyhedron? Rarely can an open problem at the frontier of mathematical research be stated so succinctly.

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