Super Guarding and Dark Rays in Art Galleries

MIT CompGeom Group* Hugo A. Akitaya† Erik D. Demaine‡ Adam Hesterberg§ Anna Lubiw¶
Jayson Lynch‖ Joseph O'Rourke** Frederick Stock††

Abstract

We explore an Art Gallery variant where each point of a polygon must be seen by k guards, and guards cannot see through other guards. Surprisingly, even covering convex polygons under this variant is not straightforward. For example, covering every point in a triangle $k=4$ times (a ***4-cover***) requires 5 guards, and achieving a 10-cover requires 12 guards. Our main result is tight bounds on k-covering a convex polygon of n vertices, for all k and n. The proofs of both upper and lower bounds are nontrivial. We also obtain bounds for simple polygons, leaving tight bounds an open problem.

1 Introduction

The original Art Gallery Theorem showed that $\lfloor n/3 \rfloor$ guards are sometimes necessary and always sufficient to guard a simple polygon P of n vertices [O'R87]. (Throughout, P includes its boundary ∂P.) There have been many interesting variants explored since then. In this paper we explore two variants that are interesting in combination, although not individually.

(1) *Guards blocking guards*: Suppose guards cannot see through other guards.[1] More precisely, if g_1 and g_2 are guards, and g_1, g_2, p are on a line in that order, then point p is not visible from g_1. Still the original bound $\lfloor n/3 \rfloor$ holds, because g_2 can continue g_1's line-of-sight to p, picking it up where that line-of-sight terminates at g_2.

(2) *Multiple coverage*: Suppose every point in the closed polygon must be seen by k guards i.e., the guards must ***k-cover*** the polygon. The problem of k-guarding has been explored under various restrictions on guard location [BBC+94, Sal09, BEK13]. If multiple guards can be co-located at the same point, then this is trivial. If co-location is disallowed, but guards can see through other guards, then this still reduces to the case $k = 1$ since we can replace a single guard by a cluster of k guards. (We detail the argument in Section 4.)

So neither of these variations is "interesting" by itself in the sense that easy arguments lead to $\lfloor n/3 \rfloor$ bounds. However, consider now mixing these two variants:

> **Q**: How many guards are necessary and sufficient to cover a simple polygon P of n vertices so that every point of P is seen by at least k guards, where guards cannot be co-located, and each guard blocks lines-of-sight through it?

To our surprise, answering **Q** is not straightforward, even for convex polygons, even for triangles. For example, to cover a triangle to depth $k = 3$, one guard at each vertex suffices. Note here we consider a guard to see itself. But to cover to depth $k = 4$ requires $g = 5$ guards; see Fig. 9. And covering to depth $k = 10$ requires $g = 12$ guards.

The main result of this paper is the following theorem. We use n for the number of vertices, k for the depth of cover, and g for the number of guards.

Theorem 1 *For a closed convex n-gon, coverage to depth k requires $g \in \{k, k+1, k+2\}$ guards:*

(1) *For $k \le n$: $g = k$ guards are necessary and sufficient.*

(2) *For $n < k < 4n-2$: $g = k+1$ guards are necessary and sufficient.*

(3) *For $4n - 2 \le k$: $g = k + 2$ guards are necessary and sufficient.*

Thus there are three regimes depending on the relationship between n and k. For triangles, $n = 3$, the following table details those regimes:

k	1	2	3	4	5	6	7	8	9	10	11	$\cdots$
g	1	2	3	5	6	7	8	9	10	12	13	$\cdots$

*Artificial first author to highlight that the other authors (in alphabetical order) worked as an equal group. Please include all authors (including this one) in your bibliography, and refer to the authors as "MIT CompGeom Group" (without "et al.").

†U. Mass. Lowell, hugo_akitaya@uml.edu
‡MIT, edemaine@mit.edu
§Harvard U., achesterberg@gmail.com
¶U. Waterloo, alubiw@uwaterloo.ca
‖MIT, jaysonl@mit.edu
**Smith College, jorourke@smith.edu
††U. Mass. Lowell, fbs9594@rit.edu

[1] This was posed as an exercise in [DO11], Exercise 1.28, p. 14.

Another example: For $n = 4$, $g = 14$ guards 13-cover, but a 14-cover requires $g = 16$ guards. See ahead to Fig. 10.

Our primary focus is proving Theorem 1. We also obtain in Lemma 8 tight bounds for a convex wedge, which can be viewed as a 2-sided unbounded convex polygon. Finally, we briefly address simple polygons in Theorem 7, which we do not consider as natural a fit as the question for convex polygons.

1.1 Dark Rays and Dark Points

With some abuse of notation, we will identify both a guard and that guard's location as g_i. Let g_1 and g_2 be two guards visible to one another. We say that g_2 ***generates*** a ***dark ray at*** g_1, which is a ray contained in the line through g_1 and g_2, incident to and rooted at g_1 and invisible to g_2. And similarly, g_1 generates a dark ray at g_2.

A point is called ***dark*** if it is contained in a dark ray, and ***d-dark*** if it is contained in at least d dark rays.

Because a d-dark point is hidden from d guards, we obtain an immediate relationship between dark rays and multiple guarding for convex polygons.

Observation 1

(1) *k-guarding with $g = k$ guards is possible if and only if there is no dark point inside P, i.e., all dark rays are strictly exterior to P.*

(2) *k-guarding with $g = k + 1$ guards is possible if and only if there is no 2-dark point inside P.*

(3) *k-guarding with $g = k + 2$ guards is always possible because we can perturb the guards to avoid 3-dark points, as justified in Appendix A.4.*

1.2 Outline of Proof of Theorem 1

Most steps of the proof follow directly from Observation 1, with the exception of the following non-trivial result.

Theorem 2 *The maximum number of guards that can be placed in a convex n-gon P without creating 2-dark points in P is $4n - 2$.*

We prove the upper bound (at most $4n - 2$ guards) in Section 2 and the lower bound ($4n - 2$ is possible) by a direct construction in Section 3. Both directions are non-trivial, and their proofs constitute the main focus of the paper. Assuming these results, the proof of Theorem 1 proceeds as follows:

To k-cover when $k \leq n$ (regime (1)) it is clear that k guards are necessary. For sufficiency, place k guards at vertices of polygon P. Then all dark rays are exterior to P, so by Observation 1(1), this is a k-cover.

To k-cover when $n < k < 4n - 2$ (regime (2)) the necessity of $k + 1$ guards follows from Lemma 9 (Appendix A.2) where we show that any placement of $n+1$ guards in a convex P results in a dark point inside P. Sufficiency is proved by Observation 1(2) (that we only need to avoid 2-dark points) and the lower bound of Theorem 2 (that we can place $k + 1$ points without creating 2-dark points), since $k + 1 \leq 4n - 2$.

To k-cover when $4n - 2 \leq k$ (regime (3)) the sufficiency of $k + 2$ guards follows from Observation 1(3). Necessity is proved by the upper bound of Theorem 2.

2 $4n - 2$ Upper Bound

In this section we prove that at most $4n - 2$ guards can be placed in a convex n-gon P without creating 2-dark points in P.

2.1 Triangle Lemma

The following lemma is a key tool in the proof of the upper bound. It establishes that, excluding the exceptional case, any triangle of guards in P may only contain one additional guard if we are to avoid 2-dark points in T.

Lemma 3 (Triangle) *Suppose some guards are placed in P without creating 2-dark points. Let T be a closed triangle in P with guards g_1, g_2, g_3 at its corners. Then, with one exception, T contains at most one more guard.*

The exceptional case allows two guards, g_4, g_5, in T when (up to relabelling) g_1g_3 is an edge of P, g_4 lies on that edge, and g_2, g_5, g_4 are collinear.

Proof. Refer to Fig. 1(a,b) throughout. We first discuss the non-exceptional case. First suppose that there is an extra guard g_4 strictly interior to T. Then g_1, g_2, g_3 generate 3 dark rays at g_4, each of which crosses a different edge of T. The same would be true for a second strictly interior guard g_5. So a dark ray at g_5 must cross a dark ray at g_4 to reach an edge of T. The result is a 2-dark point, marked x in (a) of the figure. Since we assumed no 2-dark points in P, there cannot be two extra guards interior to T.

Suppose now that g_4 lies on edge $e = g_1g_3$ of T. Then left and right of g_4 on e are dark rays generated by g_1 and g_3. Placing g_5 at any point not collinear with g_4 and g_2 leads to a dark ray at g_5, generated by g_2, crossing e to form a 2-dark point there.

We are left with the exceptional case, illustrated in (b) of the figure: g_4 lies on an edge of T, and g_5 is collinear with g_4 and the opposite corner of the triangle, g_2 in the case illustrated. There are no 2-dark points inside T. The dark ray at g_5 generated by g_2 contains the dark ray at g_4 generated by g_5 so, to avoid 2-dark points inside P, g_4 must be on the boundary of P. By the same argument, g_1 and g_3 must be vertices of P. $\square$

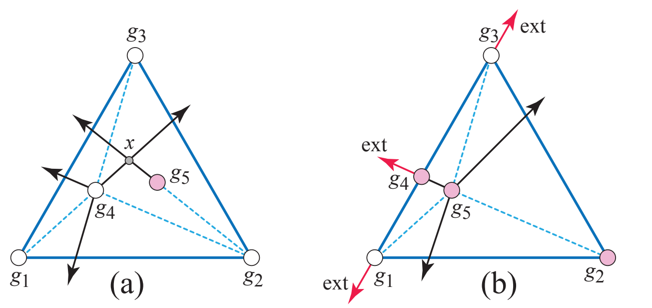

Figure 1: In this and following figures, guards are indicated by hollow circles. (a) Generic placements of g_4, g_5 produce a 2-dark point x. (b) The exceptional case, with dark rays exterior to P.

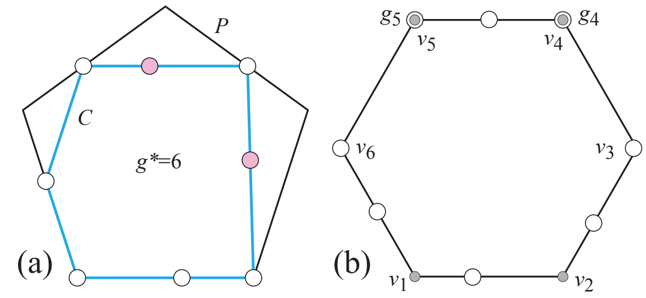

Figure 2: (a) The two pink guards are not included in $g^* = |G^*|$. (b) v_1, v_2 are darkened but have no guard; g_4, g_5 are both guards and darkened vertices. So $d = 4$ and $g^P = n + \frac{1}{2}d = 8$.

We now sketch the main idea of the $4n-2$ upper bound. Consider a placement of guards in P such that there are no 2-dark points in P. Our goal is to prove that there are at most $4n-2$ guards. Let C be the convex hull of the guards. We will show in Lemma 4 that the number of guards on ∂C, not counting collinear guards interior to P, is at most $2n$. Triangulating C leads to at most $2n-2$ triangles. Lemma 3 then shows that there is at most one extra guard inside each triangle, which leads to the $4n-2$ upper bound. To make this rigorous, we must take into account collinear guards and the exceptional case of Lemma 3.

We first shrink P so that it maximally touches C, as follows. Move each edge of P parallel to itself toward the interior until it hits a guard. If an edge e only has a guard at one endpoint, then rotate e about that endpoint toward the interior until it hits another guard. The reduced polygon contains all the guards, has no 2-dark point, and has at most n vertices, so it suffices to prove the bound on the number of guards for the reduced polygon. Henceforth we assume every edge of P has either one or more guards in its interior, or a guard at its endpoint (or at both endpoints).

The proof requires careful handling of collinear guards: a guard is called ***collinear*** if it lies on a line between two other guards.

Define $\boldsymbol{G^*}$ as the set of guards on ∂C, but excluding those guards that are collinear and not on ∂P. So collinear guards on ∂P are in G^*, but collinear guards on ∂C and internal to P are excluded from G^*. See Fig. 2. Equivalently, G^* consists of the guards on ∂P together with any guard that is a corner of C in the interior of P. Define $\boldsymbol{g^* = |G^*|}$. This is the key count that is needed to complete the upper-bound proof.

Lemma 4 *The number of guards g^* as defined above is at most $2n$.*

Proof. Let g^P be the number of guards on ∂P and let c be the number of guards that are corners of C in the interior of P. As noted above, $g^* = g^P + c$. We will bound g^P and c separately. Both bounds are in terms of the number of darkened vertices, where a vertex v of P is ***darkened*** if guards on ∂P generate a dark ray through v.

We first bound g^P. The constraint that limits g^P is that a vertex v cannot be darkened from both incident edges, as that would render v a 2-dark point.

The idea is to count guards and darkened vertices per edge. A guard internal to an edge counts towards the edge, and a vertex guard counts half towards each incident edge. More precisely, for an edge e, let $g(e)$ be the number of guards internal to e plus half the number of vertex guards on e. Then $g^P = \sum_e g(e)$.

Fig. 3 shows the possibilities: $g(e) = 2$, either from two internal guards, or one internal guard and two endpoint guards; $g(e) = 1\frac{1}{2}$ from one endpoint guard and one internal guard; or $g(e) = 1$ from one internal guard or two endpoint guards.

These are the only possibilities: (a) An edge cannot have four or more guards, as then the extreme points would be at least 2-dark. (b) And an edge can only have three guards when two are at the endpoints of the edge: an endpoint without a guard would be rendered 2-dark by the three guards on the edge. (c) An edge cannot have just a guard at one endpoint, because the shrinking procedure would rotate that edge about the endpoint until it hit another guard.

Next we observe from Fig. 3 a relationship between $g(e)$ and $d(e)$, the number of dark rays on edge e generated by guards on e: if $g(e) = 2$ then $d(e) = 2$; if $g(e) = 1\frac{1}{2}$ then $d(e) = 1$; and if $g(e) = 1$ then $d(e) = 0$. Equivalently, $d(e) = 2(g(e) - 1)$.

Finally, we note that d, the number of darkened vertices, is $\sum_e d(e)$, since each dark ray on e darkens an endpoint of e, and no vertex can be darkened from both incident edges.

Putting these together,

$$d = \sum_e d(e) = \sum_e 2(g(e)-1) = 2\sum_e g(e)-2n = 2g^P-2n$$

which gives $g^P = n + \frac{1}{2}d$. For example, for even n, placing a guard at every vertex and a guard in the interior of every other edge darkens every vertex, so $g^P = \frac{3}{2}n$.

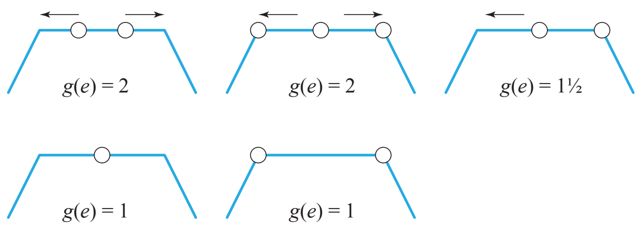

Figure 3: Edge counts. Arrows indicate darkened vertices.

We next bound c, the number of guards strictly internal to P that are corners of C. Let g_0 be such a corner guard. Moving left and right on C, let g_1 and g_2 be the first guards that are on ∂P, say on edges e_1 and e_2. Note that there cannot be another vertex of C internal to P between g_1 and g_2, as then two dark rays would cross inside P: see Fig. 4(a). Also note that g_0 is not collinear with g_1 and g_2, because we are counting g^*, which excludes collinear guards on C. Since every edge has a guard, edges e_1 and e_2 must be incident at a vertex v of P, and v has no guard (because otherwise g_0 would be internal to C). The dark rays incident to g_0 from g_1 and g_2 cross e_1 and e_2 as shown in Fig. 4(b). So v cannot be darkened by the guards on e_1 or e_2 otherwise again two dark rays would cross.

Thus each guard g_0 counted in c corresponds to a non-darkened vertex, so $c \leq n - d$.

In total,

$$g^* = g^P + c \leq n + \frac{1}{2}d + (n-d) = 2n - \frac{1}{2}d \leq 2n \ .$$

Equality is achieved when there is one guard internal to each edge, and one guard inside P between each consecutive pair, and no collinear guards nor darkened vertices of P. See Fig. 4(c). □

Theorem 5 *The number of guards g that can be placed in a convex n-gon so that no two dark rays intersect inside is at most $g = 4n - 2$.*

Proof. Consider a placement of guards inside P that avoids 2-dark points. We use G^* and g^* as defined above. By Lemma 4, $g^* \leq 2n$. Triangulate the guards in G^*. By definition of G^*, this includes collinear guards on ∂P but excludes collinear guards internal to P.

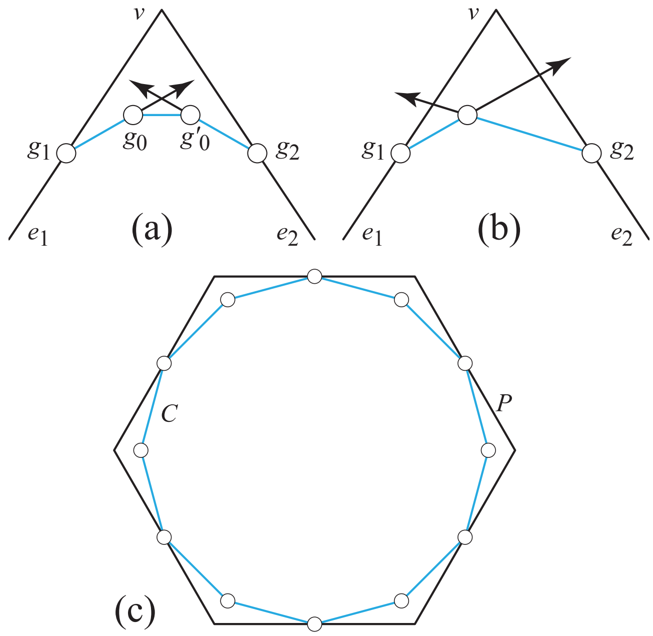

Figure 4: (a) g_0 and g_0' create intersecting dark rays in P. (b) v cannot be a darkened vertex. (c) The upper bound $g^* = 2n$ can be achieved.

There are at most $2n-2$ triangles in this triangulation. By Lemma 3, there is at most one extra guard in each triangle, for a total of at most $2n+(2n-2) = 4n-2$ guards, *so long as* we rule out the exceptional case of Lemma 3 where a triangle of guards can contain two extra guards. But that exception only happens when one of the extra guards is on ∂P, and all the guards on ∂P were already included in G^*. □

3 Lower Bound

The challenge is to locate $g = 4n-2$ guards so that there are no 2-dark points in P, thus proving the lower bound of Theorem 2.

We first illustrate a placement in a triangle of $g = 10$ guards without 2-dark points, i.e., so that no two dark rays intersect inside the triangle. We then introduce the general strategy for the triangle, and hint at the strategy for convex n-gons, but proofs are deferred to Appendix A.3.

3.1 $g = 4n - 2$ guards achievable for triangle

Fig. 5 illustrates a placement of 10 guards in a triangle P such that all dark-ray intersections are strictly exterior to P. Although it is difficult to verify visually, even enlarged, a calculation described in the Appendix verifies that all dark-ray intersections lie strictly exterior to the triangle. This demonstrates $g = 4n-2$ is achievable for triangles.

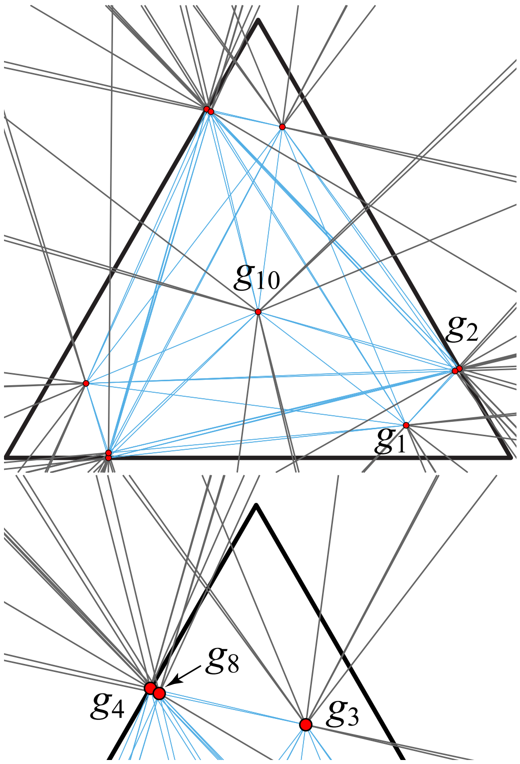

Figure 5: $g = 10$ guards 9-covering a triangle. Apex enlargement below. Indexing follows Fig. 6.

Several features of this construction will repeat for general n-gons:

(1) n guards are on edges of P.

(2) $2n$ guards are on the hull ∂C (the maximum by Lemma 4).

(3) Three guards are placed near each vertex,

(4) Two of the three guards near a vertex are nearly co-located.

(5) There is one extra guard in each triangle of a triangulation of P (this is g_{10} in Fig. 5).

This construction leads to 3 guards near each of P's n vertices, plus $n-2$ guards in the triangles of a triangulation, yielding $g = 4n-2$. Note that the triangulation is of the n-gon P, not the $2n$-gon convex hull C used in the proof of Theorem 5.

Idea of the construction in Fig. 5. Before turning to the general construction, we first provide intuition for the triangle construction, illustrated in Fig. 6. The triangle is partitioned into six sectors with g_{10} in the center. Three guards are placed in the yellow sectors near each vertex, so that the dark rays they generate at g_{10} exit through the empty white sectors. First, two of three guards are placed as illustrated: g_2, g_4, g_6 on triangle edges, and g_1, g_3, g_5 slightly inside the adjacent edges. The final three guards will be placed inside the convex hull of $g_1, \ldots, g_6$, but their locations are tightly constrained. The guards placed so far define three dark wedges apexed at guards g_1, g_3, g_5, where the wedge apexed at g_i contains all the dark rays at g_i. The last three guards g_7, g_8, g_9 are placed quite close to the even-index guards g_2, g_4, g_6 so that none of their dark rays enter the dark wedges. For further explanation, see Section A.3. The construction works for any triangle: there are no shape assumptions.

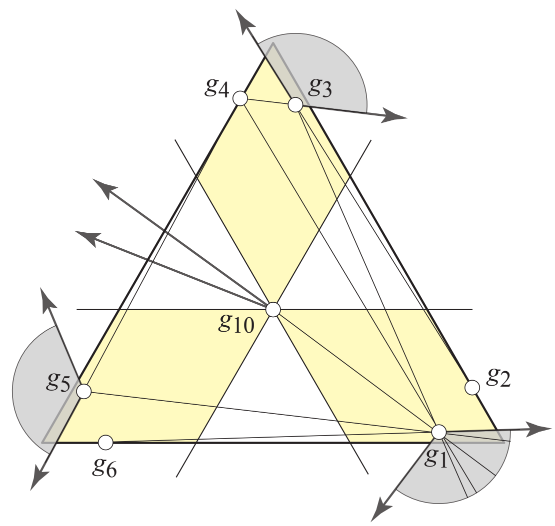

Figure 6: Dark rays from g_{10} exit through empty white sectors. Dark wedges apexed at g_1, g_3, g_5 contain the dark rays from all other guards, illustrated for the g_1 wedge.

The conclusion of the lower bound construction in the Appendix (Section A.3) is this theorem:

Theorem 6 *It is possible to place $4n-2$ guards in a convex n-gon P so that all dark-ray intersections lie strictly exterior to P.*

Theorems 5 and 6 establish the tight bounds in Theorem 2.

4 Simple Polygon

We mentioned in the Introduction that the variant we are exploring—multiple coverage and guards-blocking-guards—is not a natural fit for arbitrary simple polygons. In a convex polygon P, each pair of guards sees all of P except for their dark rays, whereas in an arbitrary polygon, guard visibility is also blocked by reflexivities of ∂P.

4.1 Necessity

The comb example that establishes necessity of $\lfloor n/3 \rfloor$ guards to cover a simple polygon of n vertices, also shows the necessity of $k\lfloor n/3 \rfloor$ guards to cover to depth k—since no guard can see into more than one spike of the comb, each of the $\lfloor n/3 \rfloor$ spikes needs at least k distinct guards. In fact, if the comb has at least two spikes, then $k\lfloor n/3 \rfloor$ guards also suffice. The general construction for $k \geq 2$ is illustrated in Fig. 7 for depth $k = 4$ and $n = 9$.

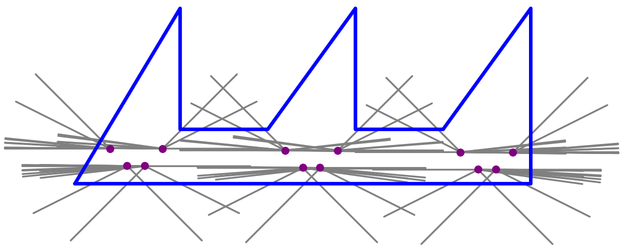

Figure 7: $4 \cdot 3 = 12$ guards suffice to 4-cover the comb of 9 vertices.

Place k guards in a convex arc below each spike of the comb so that none of the dark rays generated by these guards enters any spike. Points in a spike are covered to depth k by the k guards below it. Although many dark rays cross in the base corridor of the comb, slight vertical staggering of the convex arcs of k guards ensures that no corridor point is at the intersection of three dark rays, which ensures coverage to depth k for $k \geq 2$ and at least two spikes.

4.2 Sufficiency

For sufficiency, we have not obtained a tight bound: To cover a simple polygon P of n vertices to depth k, we show that $g = (k+2)\lfloor n/3 \rfloor$ guards suffice. First triangulate P, 3-color, and choose the smallest color class, which has cardinality at most $\lfloor n/3 \rfloor$ [Fis78]. In Fig. 8, say we select color 1. If a color-1 vertex v is convex, then define a cone C apexed at v bounded by the edges incident to v. If a color-1 vertex v is reflex, then define C to be the "anticone" at v: the cone apexed at v and bound by the extensions of the incident edges into the interior.

To cover P to depth k, place $k+2$ guards along a convex arc near a color-1 vertex v, and inside v's cone. In the figure, we aim to 3-cover and so place 5 guards in each cone. Now it is clear that the $k+2$ guards at color-1 vertex v see into all the triangles incident to v. These guards generate crossing dark rays, but by perturbing the locations of the guards we can avoid three dark rays meeting in P. The result is coverage to depth 2 less than the number of guards at each color-1 vertex:

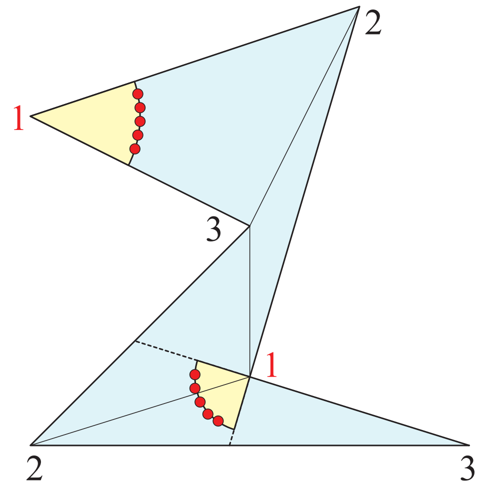

Figure 8: Cones at the color-1 reflex vertices each contain $k+2$ guards. Here the 5 guards achieve a 3-cover.

Theorem 7 *To cover a simple polygon of n vertices to depth k, $g = k\lfloor n/3 \rfloor$ guards are sometimes necessary, and $g = (k+2)\lfloor n/3 \rfloor$ guards always suffice.*

5 10 Guards in a Wedge

Finally, in Appendix A.5 we establish a tight bound for a wedge, which can be viewed as an unbounded 2-sided convex polygon with one vertex and two rays:

Lemma 8 *Covering a wedge to depth k requires the same number of guards as it does to cover a triangle to depth k, except that to 3-cover requires 4 guards. In particular, $g = 10$ guards can cover to depth 9.*

The surprising part of this result is that 10 guards can be placed in a wedge without creating 2-dark points—despite the fact that our triangle construction (see Fig. 6) fails for a wedge because it has 2-dark points just outside each triangle edge.

6 Open Problems

1. Investigate bounds or the complexity (NP-hard?) of placing points in a simple polygon so that no two dark rays intersect. (As noted in Section 4, the connection between this problem and k-guarding fails for non-convex polygons.)

2. Close the simple polygon gap in Theorem 7.

3. Can the tight bound for a wedge in Lemma 8 be generalized to tight bounds for unbounded convex polygons with two rays joined by a chain of $n-1$ vertices and $n-2$ edges?

Acknowledgements. We benefited from suggestions of three referees.

References

[AAM21] Mikkel Abrahamsen, Anna Adamaszek, and Tillmann Miltzow. The art gallery problem is ∃ℝ-complete. *J. ACM*, 69(1), 2021.

[BBC+94] Patrice Belleville, Prosenjit Bose, Jurek Czyzowicz, Jorge Urrutia, and Joseph Zaks. *K*-guarding polygons on the plane. In *Proc. 6th Canad. Conf. Comput. Geom.*, pages 381–386, 1994.

[BEK13] Daniel Busto, William S Evans, and David G Kirkpatrick. On *k*-guarding polygons. In *Proc. 25th Canad. Conf. Comput. Geom.*, 2013.

[DO11] Satyan Devadoss and Joseph O'Rourke. *Discrete and Computational Geometry*. Princeton University Press, 2011.

[Fis78] Stephen Fisk. A short proof of Chvátal's watchman theorem. *J. Combin. Theory Ser. B*, 24:374, 1978.

[O'R87] Joseph O'Rourke. *Art Gallery Theorems and Algorithms*. Oxford University Press, New York, NY, 1987. `http://cs.smith.edu/~jorourke/books/ArtGalleryTheorems/`.

[Sal09] Ihsan Salleh. *K*-vertex guarding simple polygons. *Comput. Geom. Theory Appl*, 42(4):352–361, 2009.

A Appendix

A.1 4-guarding a Triangle

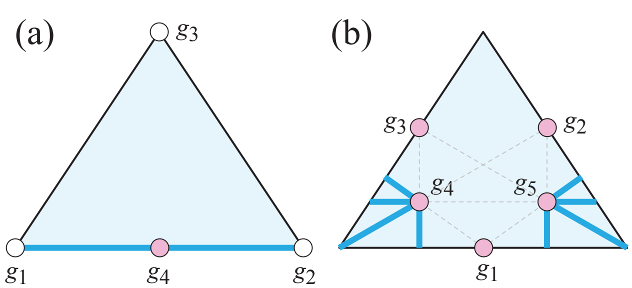

Figure 9: Five guards needed to 4-cover. (a) All strictly interior points are 4-covered, but the blue segments to either side of g_4 are only 3-covered. (b) Points on the dark rays (blue segments) incident to g_4 and g_5 are 4-covered; all other points are 5-covered.

A.2 Regime (2) Lemma

Lemma 9 *Any placement of $n+1$ guards in a convex n-gon P results in a dark point in P.*

Proof. If a guard g_0 is strictly internal to P, then there is a dark ray at g_0 generated by every other guard. So it must be that all guards are on ∂P.

View each edge of P as half-open, including its clockwise endpoint but not its counterclockwise endpoint. So the edges are disjoint and their union is ∂P. Every edge e can contain at most one guard: If e contains two or more, one, g_1, is interior to e and so there is a dark ray at g_1 along e. So there can be at most n guards while avoiding dark points. □

A.3 General Lower Bound Construction

Example: Square. Before commencing with the general construction, we illustrate it with a square. Placing $4n-2=14$ guards in a square without any 2-dark points follows the same construction as with the triangle in Fig. 5: 3 guards near each vertex, and $n-2=2$ "elbow" guards ℓ_i determined by a special triangulation, in this case just a diagonal of the square. See Fig. 10. Coordinates may be found in the Appendix (Section A.6).

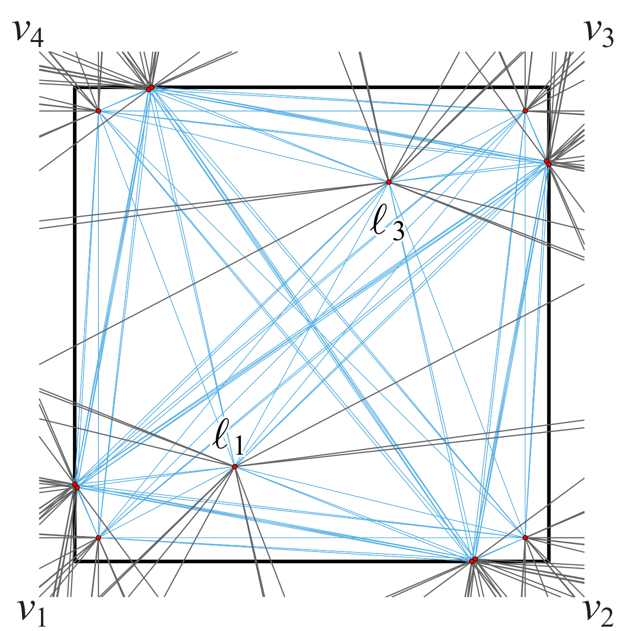

Figure 10: 14 guards covering to depth 13. Triangulation diagonal is v_1v_3. Elbow guards ℓ_1, ℓ_3. Vertex guards x_i, y_i, z_i near the four corners.

Overall Construction. The overall plan of the construction is the same as for a triangle and a square:

$3n$ guards, 3 near each vertex, plus one guard per triangle in a triangulation of P of $n-2$ triangles. The three guards to be placed near v_i will be called ***vertex guards***. The triangulation is a ***serpentine*** triangulation formed by a ***zigzag path*** that visits all the vertices, as illustrated in Fig. 11. The single guard in each triangle will be called an ***elbow*** guard.

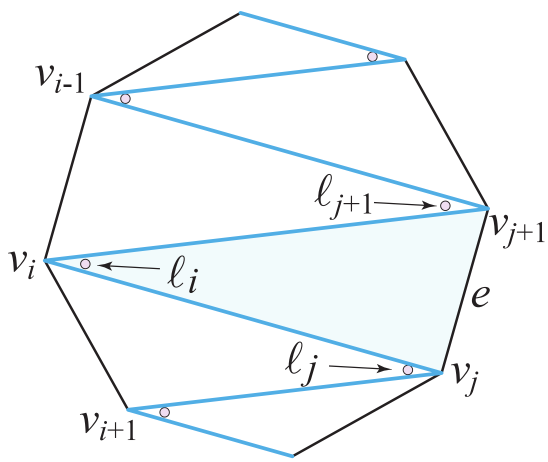

Figure 11: Zigzag triangulation and elbow guards ℓ_i.

Notation. We label the vertices in counterclockwise (ccw) order: $v_0, \ldots, v_{n-1}$ with index arithmetic modulo n. Thus "before" means clockwise (cw) and "after" means ccw. Let v_i be one of the $n-2$ internal vertices of the zigzag path. Then v_i is the apex of a triangle T_i bounded by two edges of the zigzag path plus a ***base*** that is an edge of the polygon. The elbow guard of T_i, which we denote ℓ_i, will be placed close to vertex v_i. For ease of notation, we will focus on one triangle with apex v_i and base $v_j v_{j+1}$. In each edge of P we place two "dividing points" that are used to separate wedges of dark rays. The dividing points adjacent to v_i are labeled m_i (on the minus (cw) side) and p_i (on the plus (ccw) side). See Fig. 12.

Note that there are two vertices of P with no elbow guard, and consequently either ℓ_j or ℓ_{j+1} (or both) might not exist. For example, in Fig. 10, neither ℓ_2 nor ℓ_4 exist.

Dark-ray Wedges. The elbow guard ℓ_i will be located close to v_i, and v_i's three vertex guards even closer to v_i. We first place the elbow guards and define "safe regions" for vertex guards so that the dark rays incident to elbow guards lie in disjoint "dark ray wedges." Exact placement of vertex guards will be described later.

Let e be the base edge of T_i, $e = v_j v_{j+1}$. Then the three portions of e demarcated by p_j, m_{j+1} each are crossed by wedges of dark rays incident to elbow guards. The central portion of e is crossed by rays generated by v_i's vertex guards through ℓ_i (blue). The $v_j p_j$ segment of e is crossed by the rays at ℓ_j, generated by all the vertex guards and elbow guards associated with vertices ccw from v_{i+1} to v_{j-1}, and symmetrically the $m_{j+1} v_{j+1}$ segment of e is crossed by dark rays at ℓ_{j+1}, generated by all the vertex guards and elbow guards associated with vertices ccw from v_{j+2} to v_{i-1}.

From the viewpoint of ℓ_i, there are three dark wedges emanating from it, one crossing $p_j m_{j+1}$ and two (shown in pink) crossing $v_i m_i$ and $v_i p_i$, before and after v_i.

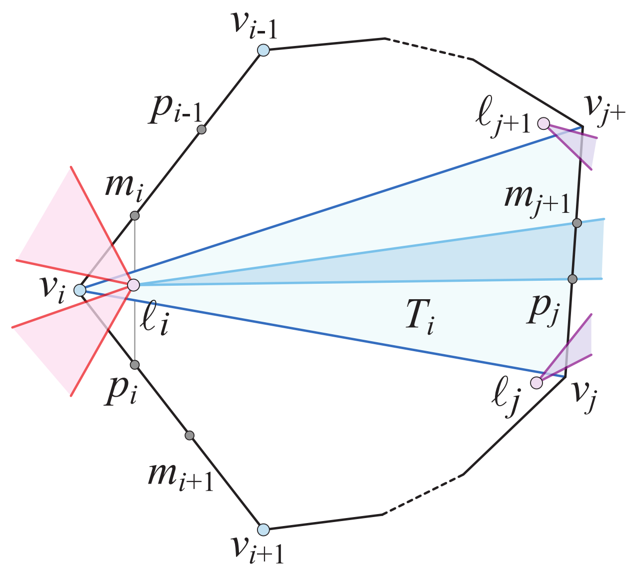

Figure 12: The dark-ray wedges that cross $e = v_j v_{j+1}$ and the dark-ray wedges emanating from ℓ_i.

Locating ℓ_i. We now describe how to place each ℓ_i so that the dark-ray wedges illustrated in Fig. 12 indeed contain the claimed rays, and create a "safe region" for v_i's vertex guards.

Place ℓ_i at the intersection of two lines: the line $m_i p_i$, and the line through v_i and the midpoint of $p_j m_{j+1}$.

Let b_i be the point where the line through p_j and ℓ_i exits P. Observe that b_i lies in the segment $v_i m_i$. Our mnemonic is that b_i is just "before" v_i. Let a_i be the point where the line through m_{j+1} and ℓ_i exits P. Then a_i lies in the segment $v_i p_i$, just after v_i.

For a vertex v_i that has an elbow guard, define its ***safe region*** R_i to be the convex quadrilateral $b_i v_i a_i \ell_i$, which is contained in the triangle $m_i v_i p_i$. For a vertex v_i without an elbow guard (the first and last vertices of the zigzag path), its safe region is the triangle $m_i v_i p_i$. Observe that the safe regions are pairwise disjoint.

Claim 1 *If vertex guards for v_i are placed in R_i then the dark rays incident with elbow guards lie in the wedges as specified above and do not enter the safe regions.*

Proof. Consider the dark rays incident to ℓ_i. Since v_i's vertex guards lie in the wedge $a_i\ell_ib_i$, they generate dark rays at ℓ_i that lie in the complementary wedge $m_{j+1}\ell_ip_j$. Vertex guards and elbow guards associated with vertices ccw from v_{i+1} to v_j lie in the wedge $p_i\ell_ip_j$ so they generate dark rays at ℓ_i that lie in the complementary wedge $m_i\ell_ib_i$ (yellow wedges in Fig. 13). Similarly vertex and elbow guards associated with vertices ccw from v_{j+1} to v_{i-1} lie in the wedge $m_{j+1}\ell_im_i$ so they generate dark rays at ℓ_i that lie in the complementary wedge $a_i\ell_ip_i$. (green wedges in Fig. 13). □

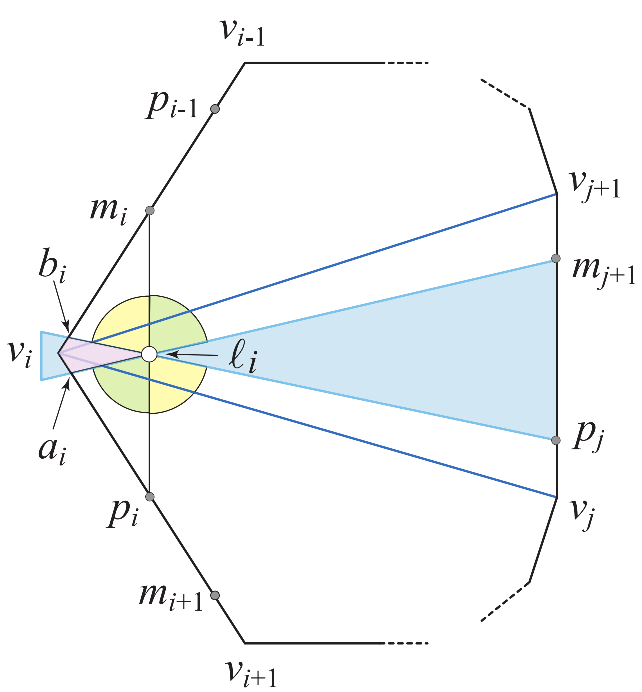

Figure 13: Constraints on locating ℓ_i, and for locating vertex guards in a safe region $R_i = b_iv_ia_i\ell_i$.

Locating 3 vertex guards. Call the three v_i vertex guards x_i, y_i, z_i. We will place them in that order, inside the safe region R_i. x_i will be placed on an edge of P, and x_i and y_i will be on the convex hull C of the guards, with z_i strictly inside C.

The following construction references a_i and b_i so it applies to the case when ℓ_i exists. But for a vertex v_i without an elbow guard, the same construction works with m_i and p_i in place of b_i and a_i.

Construct a triangle with apex v_i and two points on ∂P strictly inside the safe region R_i. Place x_i at the corner of this triangle on edge v_iv_{i-1}, and place y_i on the base of the triangle and on the p_i side of the line $v_i\ell_i$. Observe that all the elbow guards are inside the resulting hull C. Because x_i is the only guard on its edge, there are no dark rays incident to x_i inside P.

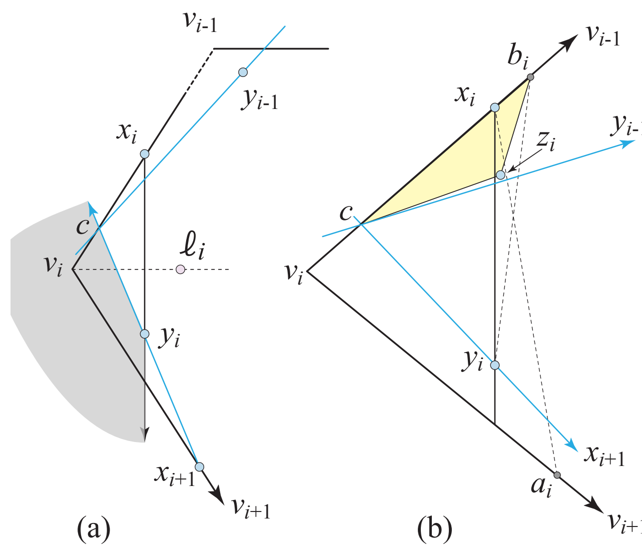

Figure 14: (a) Locating x_i and y_i. Wedge of dark rays apexed at y_i shaded. (b) Locating z_i so that dark rays incident to z_i exit P safely.

Because y_i lies on C with neighbours x_i and x_{i+1}, all the dark rays incident to y_i lie in the complementary wedge bounded by the lines y_ix_i and y_ix_{i+1}, and including v_i (gray in Fig. 14(a)). Note that no other dark rays intersect this wedge because it lies inside the safe region.

We now place z_i. Let c be the point where the line $x_{i+1}y_i$ intersects the edge v_iv_{i-1}. See Fig. 14(a).

We will ensure that the dark rays incident to z_i—except for the one generated by x_i—lie in the wedge cz_ib_i (yellow in Fig. 14(b)). This implies that these rays do not intersect any other dark rays.

We place z_i:

1. inside C,
2. on the x_i side of lines y_ib_i and $y_{i-1}c$,
3. on the y_i side of line x_ia_i.

Observe that these constraints determine a non-empty region for z_i.

Conditions 1 and 3 ensure that the dark ray incident to z_i generated by x_i hits the edge v_iv_{i+1} in the segment between y_i's dark wedge and a_i, so it intersects no other dark ray.

Conditions 1 and 2 ensure that, if we ignore x_i, then z_i lies on the convex hull C' of the guards, with neighbours y_i and y_{i-1}. Therefore the dark rays incident to z_i lie in the complementary wedge—apexed at z_i and exterior to C'—which lies inside the wedge b_iz_ic, as required.

We note that, although our construction places guards quite close together, the coordinates have polynomially-bounded bit complexity, since we used a finite sequence of linear constraints. By contrast, irrational coordinates may be required for the conventional art gallery problem in a simple polygon [AAM21].

Note that at no point do we rely on the metrical properties of P, so the construction works for all convex polygons:

Theorem 6 *It is possible to place $4n-2$ guards in a convex n-gon P so that all dark-ray intersections lie strictly exterior to P.*

To repeat our earlier claim, Theorems 5 and 6 establish the tight bounds in Theorem 1.

A.4 General Position Guards

At several junctures we claimed we can avoid 3-dark points inside P by perturbing the guard locations to be in "general position." Although this follows from general perturbation results, we give a straightforward inductive construction.

We show how to place g guards in a specified open region of the plane (a convex polygon in regime (3), or near the vertex of a vertex cone in the situation of Section 4) while avoiding 3-dark points anywhere in the plane.

Place the guards sequentially. After placing i guards, let $\mathcal{A}_i$ be the arrangement of lines determined by: (a) pairs of guard points; and (b) a guard point and a 2-dark point at the intersection of two dark rays. (For $i \le 3$ noncollinear guards, there are no 2-dark points.) Place the $(i+1)$-st guard at any point in the open region not on a line of $\mathcal{A}_i$. This is possible since the region is open. Note that this avoids three collinear guards and also avoids three dark rays crossing. Now update the arrangement to $\mathcal{A}_{i+1}$ and repeat.

A.5 10 Guards in a Wedge

Define a ***wedge*** as the region of the plane bounded by two rays from a convex vertex a, i.e., a cone with apex a. The connection between k-guarding and dark points (Observation 1) still holds, and the main issue is the analogue of Theorem 2—what is the maximum number of guards that can be placed in a wedge without creating 2-dark points? For a triangle, the bound is $4n-2 = 10$. In this section we prove that the same bound holds for a wedge.

The upper bound of 10 is easy: If we could place 11 guards in a wedge without 2-dark points, then we could simply cut off the empty part of the wedge to create a triangle with 11 guards and no 2-dark points, a contradiction to the Theorem 5 upperbound.

However, the lower bound of 10, i.e., a placement of 10 guards without 2-dark points, does not carry over from our triangle construction, because there were dark ray intersections beyond every edge of the triangle. Nevertheless, we now show this bound is tight, with the example illustrated in Figs. 15 and 16. We number the guards from bottom to top. Here is a description of the construction:

- g_1 is directly below the apex a, and far below.
- g_2 is slightly left of g_1, so that the upward dark ray at g_2 exits the wedge at a particular "safe" spot between g_7 and g_{10}.
- Guard pairs g_3, g_4, g_5, g_6, g_7, g_8 are symmetrically placed with respect to a vertical line L through a.
- Guards g_7, g_8 are located on the two edges of the wedge.
- g_{10} is on L near a, while g_9 is right of L.
- There are six guards on the convex hull C of the guards: $\{g_1, g_3, g_7, g_{10}, g_8, g_4\}$.
- g_5, g_6 are just slightly inside C.

We provide coordinates for the guards in Appendix A.6, and have verified that there are no 2-dark points in the wedge.

Note that this construction provides an alternative arrangement of guards for a triangle: Introduce a triangle edge bc below g_1, and apply an affine transformation to $\triangle abc$ to match Fig. 15.

We summarize the implications for k-guarding a wedge in this lemma.

Lemma 8 *Covering a wedge to depth k requires the same number of guards as it does to cover a triangle to depth k, except that to 3-cover requires 4 guards. In particular, $g = 10$ guards can cover to depth 9.*

Proof. If $k \le 2$, a guard at the one vertex, or one guard on the interior of each edge, suffices. However, any placement of 3 guards creates a dark point in the wedge, so for $k \ge 3$, at least $k+1$ guards are needed to k-guard. For $k \le 9$, the configuration just described shows that $k+1$ guards suffice—this covers the middle regime. For $k \ge 10$, $g = k+2$ guards are needed and suffice, from Observation (3) in Section 1.1 and its explanation in Section A.4. □

A.6 Guard Coordinates

We include here explicit coordinates for guards in a triangle, a square, and a wedge. In all cases, Mathematica code has verified that dark-ray intersections are strictly exterior.

Coordinates for 10 guards in an equilateral triangle, Fig. 5. Triangle corners are $(0, 200), (\pm 100\sqrt{3}, -100)$. Guard locations for the other g_i are symmetrical placements following Fig. 6.

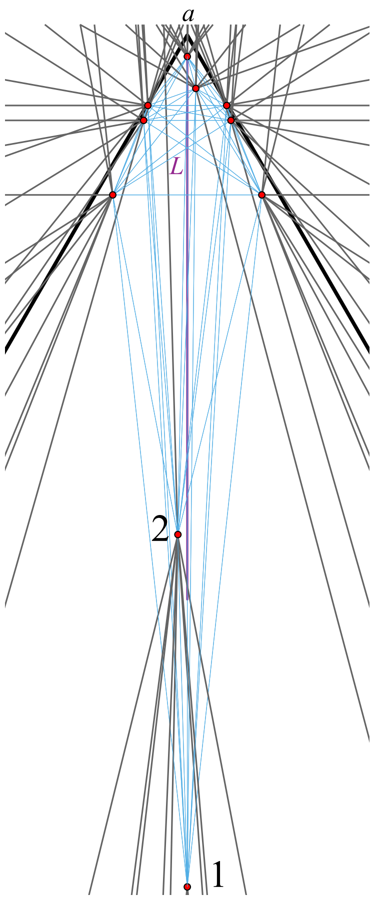

Figure 15: Wedge apex a, 10 guards with no 2-dark points.

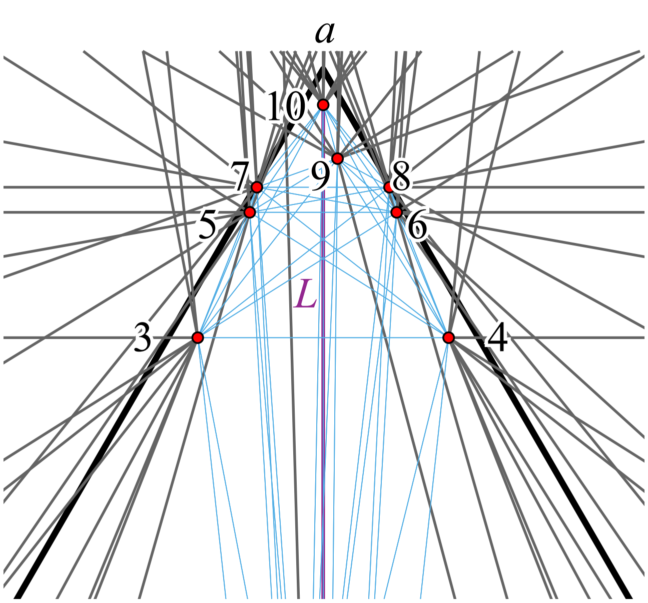

Figure 16: Closeup of upper portion of Fig.15.

g_i	$x,$	y
5	$-102.57,$	-96
6	$-102.6,$	-100
7	$-118,$	-49
10	$0,$	0

Coordinates for 14 guards in a square, Fig. 10. Square corner coordinates $(\pm 200, \pm 200)$. Guard locations $g_6, \ldots, g_{14}$ are symmetrical placements of g_3, g_4, g_5.

g_i	$x,$	y
1	$-65,$	-120
2	$65,$	120
3	$-180,$	-180
4	$-198,$	-137.7
5	$-200,$	-135

Coordinates for 10 guards in a wedge, Figs. 15 and 16. Apex at $(0, 200)$, apex angle $\pi/3$. Guard locations g_4, g_6, g_8 are symmetrical placements of g_3, g_5, g_7.

g_i	$x,$	y
1	$0,$	-600
2	$-9,$	-270
3	$-70,$	50
4	$70,$	50
5	$-41,$	120
6	$41,$	120
7	$-38.1,$	134
8	$38.1,$	134
9	$8,$	150
10	$0,$	180