A Note on Solid Coloring
of Pure Simplicial Complexes

Joseph O'Rourke[*]

December 17, 2010

Abstract

We establish a simple generalization of a known result in the plane. The simplices in any pure simplicial complex in $\mathbb{R}^d$ may be colored with $d+1$ colors so that no two simplices that share a $(d-1)$-facet have the same color. In $\mathbb{R}^2$ this says that any planar map all of whose faces are triangles may be 3-colored, and in $\mathbb{R}^3$ it says that tetrahedra in a collection may be "solid 4-colored" so that no two glued face-to-face receive the same color.

1   Introduction

The famous 4-color theorem says that the regions of any planar map may be colored with four colors such that no two regions that share a positive-length border receive the same color. A lesser-known special case is that if all the regions are triangles, three colors suffice. For the purposes of generalization, this can be phrased as building a planar object by gluing triangles edge-to-edge, and then 3-coloring the triangles. Because the coloring constraint in this formulation only applies to triangles adjacent the dual graph—whose nodes are triangles and whose arcs join triangle nodes that share a whole edge—slightly more general objects can be 3-colored: *pure* (or *homogenous*)[1] *simplicial complexes* in $\mathbb{R}^2$, whose dual graph may have several components, with independent colorings. See Figure 1.

For simplicity, we will call such a complex a *triangle complex*, its analog in $\mathbb{R}^3$ a *tetrahedron complex*, and the generalization a *d-simplex complex*. We permit these complexes to contain an infinite number of simplices; e.g., tilings of space by simplices are such complexes. The main result of this note is:

Theorem 1 *A d-simplex complex may be $(d+1)$-colored in the sense that each simplex may be colored with one of $d+1$ colors so that any pair of simplices that share a $(d-1)$-facet receive different colors.*

---

[*]Department of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu.

[1] Pure/homogenous means that there are no dangling edges or isolated vertices, and in general, no pieces of dimension less than d that are not part of a simplex of dimension d. So the complex is a collection of d-simplices glued facet-to-facet.

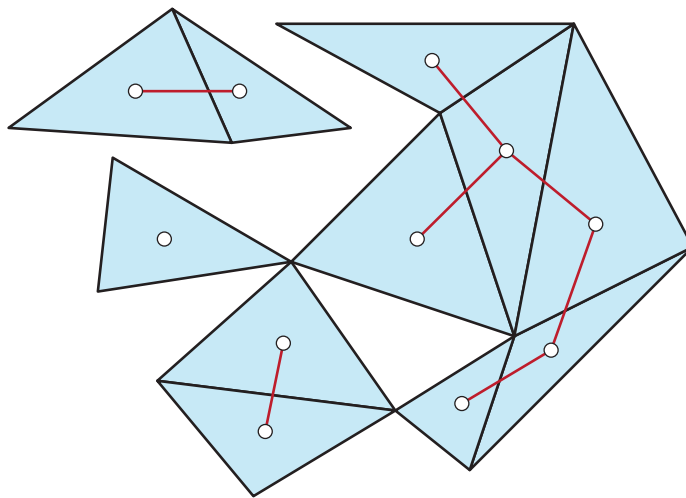Figure 1: A triangle complex and its dual graph G.

One can think of the whole volume of each simplex being colored—so "solid coloring" of tetrahedra in $\mathbb{R}^3$. Although I have not found this result in the literature, it is likely known, as its proof is not difficult—essentially, remove one simplex and induct. Consequently, this note should be considered expository, and I will describe proofs in more detail than in a research announcement. Perhaps more interesting than the result itself are the many related questions in Section 5.

2   Triangle Complexes

Let G be the dual graph of a triangle complex, and let $\Delta(G) = \Delta$ be the maximum degree of nodes of G. For triangle complexes, $\Delta = 3$. Let $\chi(G) = \chi$ be the chromatic number of G. An early result of Brooks [Bro41] says that $\chi \leq \Delta + 1$ for any graph G. For duals of triangle complexes, this theorem only yields $\chi = 4$, the 4-color theorem for triangle complexes. We now proceed to establish $\chi = 3$ in three stages:

1. We first prove it for finite triangle complexes.

2. We then apply a powerful result of deBruijn and Erdős to extend the result to infinite complexes.

3. We formulate a second proof for infinite complexes that does not invoke deBruijn-Erdős.

The primary reason for offering two proofs is that related questions raised in Section 5 may benefit from more than one proof approach.

2

2.1 Finite Triangle Complexes

Let S be a triangle complex containing a finite number of triangles, and G its dual graph. Let $C(S) = C$ be the convex hull of S, i.e., the boundary of the smallest convex polygon enclosing S. The proof is by induction on the number of triangles, with the base case of one triangle trivial.

Case 1. There is a triangle t with at least one edge e on C. Then e is *exposed* (i.e., not glued to another triangle of the complex), and t has at most degree 2 in G. Remove t to produce complex S', 3-color S' by induction, put back t, and color it with a color distinct from the colors of its at most 2 neighbors in G.

Case 2. No triangle has an edge on C. Let v be any vertex of C, and let t be the most counterclockwise (ccw) triangle incident to v. See Figure 2. Then the ccw edge e of t incident to v is exposed. Then—just as in the
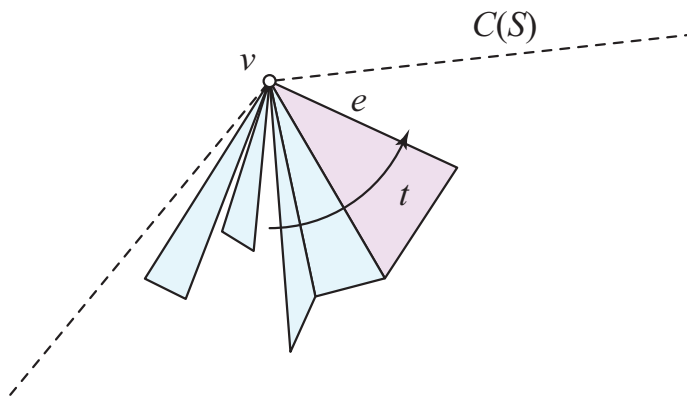

Figure 2: Triangle t has an exposed edge e.

previous case—remove t, 3-color by induction, put t back colored with a color not used by its at most two neighbors.

This simple induction argument establishes $\chi = 3$ for finite triangle complexes.

2.2 deBruijn-Erdős

The result of deBruijn and Erdős is this [EdB51]:

Theorem 2 *If a graph G has the property that any finite subgraph is k-colorable, then G is k-colorable itself.*

This immediately extends the result just proved to infinite triangle complexes. Note that the induction proof presented fails for infinite complexes, because it is possible that every triangle has degree 3 in G for infinite complexes, for example, in a triangular tiling.

2.3 Proof based on K_r

The alternative proof in some sense "explains" why a triangle complex is 3-colorable: because it does not contain K_4 as a subgraph. Of course we could obtain this indirectly by using the above proof and conclude that K_4 could not be a subgraph (because it needs 4 colors), but establishing it directly gives additional insight.

We rely here on this result, obtained independently by several researchers (Borodin and Kostochka, Catlin, and Lawrence, as reported in [Sta02]):

Lemma 1 *If G does not contain any K_r as a subgraph, $4 \leq r \leq \Delta + 1$, then*

$$\chi \leq \frac{r-1}{r}(\Delta + 2) \ .$$

We will now show that K_4 is not a subgraph of G for triangle complexes, which, because $r = 4$ and $\Delta = 3$, then implies

$$\chi \leq \frac{3}{4}(3+2) = 3\frac{3}{4} \ ,$$

and so (because χ is an integer), $\chi \leq 3$.

Lemma 2 $K_4 \not\subseteq G$.

Proof: *Sketch.* We only sketch the argument, because in the Appendix we prove more formally the extension to $\mathbb{R}^d$, including $d = 2$.
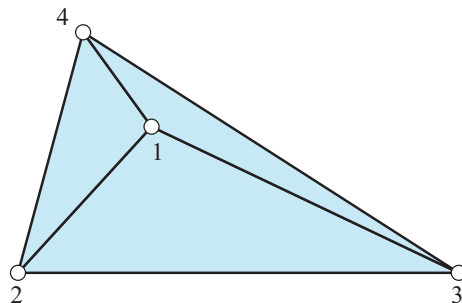

Figure 3: Triangles forming K_3.

If K_4 is a subset of G, then K_3 must be as well. The only configuration of triangles that realizes K_3 is that shown in Figure 3: the three triangles share and surround a vertex (labeled 1 in the figure). Now consider attempting to extend this to K_4 by gluing another triangle to the only uncovered edge of $\triangle\{1, 2, 3\}$, edge $e = \{2, 3\}$. Its apex, call it v_5, must lie below e, but because v_4 lies above e, the new triangle $\triangle\{2, 3, 5\}$ cannot share the edges $\{2, 4\}$ and $\{3, 4\}$, which it

must to be adjacent to the other two triangles. Therefore, K_4 cannot occur in G, and we have established the claim. □

And as we argued above, Lemmas 1 and 2 together imply that $\chi(G) \leq 3$: triangle complexes are 3-colorable.

3 Tetrahedron Complexes

Again we follow the same procedure as above, although we will defer consideration of K_5 to general d-simplex complexes to the Appendix, Section 6. Now S is a finite tetrahedron complex, G its dual graph, and C the convex hull of S, the boundary of a convex polyhedron. Again the proof is by induction. Although we could repeat the structure of the proof for triangle complexes, we opt for an argument that more easily generalizes to d dimensions.

Let v be a vertex of the hull $C = C(S)$, and let S_v be the subset of S of tetrahedra incident to v. Let $C_1 = C(S_v)$ be the convex hull of S_v. If there is a tetrahedron $t \in S_v$ with at least one face f lying on C_1, then t has at most 3 neighbors in S. Remove t, 4-color the smaller complex S', put t back, and color it with a color not used for its at most 3 neighbors. Note that it could well be that the face f lies on $C(S)$ because C_1 and C coincide at f. But having f on C is not the crucial fact; if it is on C_1, it is exposed, and induction then applies.

If no tetrahedron in S_v has a face on C_1, then there must be a tetrahedron t that has an edge e on C_1 (in fact, there must be at least three such tetrahedra). See Figure 4.
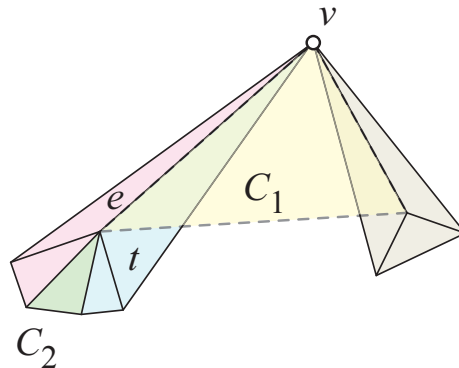

Figure 4: No tetrahedron has a face on C_1.

Let S_e be the subset of those tetrahedra in S_v that share e. Let $C_2 = C(S_e)$ be the convex hull of these tetrahedra. It must be that at least one tetrahedron has a face on C_2. The tetrahedra sharing e are angularly sorted about e, and we can select the most ccw one (which might be the same as the most cw one if $|S_e| = 1$). So we have identified a tetrahedron with an exposed face, and induction applies and establishes the result: finite tetrahedon complexes have

$\chi = 4$. Infinite tetrahedon complexes follow from Theorem 2. And we could now work backward to conclude that G cannot contain K_5 as a subgraph.

4 d-Simplex Complexes

We repeat the outline just employed. The only difficult part is showing that in a finite d-simplex complex S, there must be a simplex with an exposed facet.[2] Then induction goes through just as before.

Say that a convex hull C of points in d dimensions is *full-dimensional* if C is not contained in a $(d-i)$-dimensional flat (hyperplane) for any $i > 0$.

Let v be a vertex of the hull $C = C(S)$, and let S_v be the subset of S of simplices incident to v. Let $C_1 = C(S_v)$ be the convex hull of S_v; this is a d-polytope that contains S_v. If there is a simplex $\sigma \in S_v$ with at least one $(d-1)$-dimensional facet f contained in C_1, then σ has at most d neighbors in S, and induction establishes that S may be $(d+1)$-colored.

So suppose that no simplex in S_v has a $(d-1)$-dimensional facet on C_1. Let $|S_v| = n$. We must have $n > 1$, because otherwise C_1 would bound a single simplex, and all of its facets would be on C_1 and so exposed. We know C_1 is full dimensional because it contains d-simplices. Let $\sigma_1 \in S_v$ be a simplex that has a k-dimensional face f_1 in C_1, such that $k < d - 1$ is maximal among all simplices with faces in C_1. We claim that there must be another simplex $\sigma' \in S_v$ that also has a face f' in C_1, where $f' \neq f_1$. For suppose otherwise, that is, suppose that all simplices in S_v share f_1. Then, because C_1 is full-dimensional, one of these simplices σ'' must have a vertex u not part of f_1 on C_1 (otherwise all simplices lie in the flat containing f_1). But then σ'' has a face (the hull of u and f_1) on C_1 of dimension larger than k, contradicting the choice of σ_1.

So σ' has a face on C_1, and σ' does not share f_1. Let S_{f_1} be all the simplices in S_v that share f_1, and let C_2 be the convex hull of S_{f_1}. Because we know that $\sigma' \notin S_{f_1}$, $|S_{f_1}| < n$.

Now the argument is repeated: C_2 is full-dimensional because it includes at least one d-simplex σ_1. If some simplex in S_{f_1} has a $(d-1)$-dimensional facet on C_2, we have identified an exposed face. Otherwise, we select some simplex σ_2 with a face f_2 on C_2, and separate out into S_{f_2} all the simplices sharing f_2. S_{f_2} must have at least one fewer simplex than does S_{f_1}, following the same reasoning.

Continuing in this manner, we identify smaller and smaller subsets of S:

$$|S| \geq | S_v| > |S_{f_1}| > |S_{f_2}| > \cdots$$

via repeated convex hulls $C_1, C_2, \ldots$, and eventually either identify a simplex with a $(d-1)$-dimensional facet on the corresponding hull C_i, or reach a set of one simplex, which has all of its facets exposed. So there is always a simplex with an exposed facet:

---
[2] We use *facet* for a $(d-1)$-dimensional face, and *face* for any smaller dimensional face.

Lemma 3 *Any finite d-simplex complex contains a simplex with an exposed $(d-1)$-dimensional facet.*

Given the nearly obvious nature of this lemma, it seems likely there is a less labored proof that identifies an exposed simplex more directly.

This lemma then proves Theorem 1 for finite complexes, and deBruijn-Erdős establishes it for infinite complexes. Again we may now conclude that K_{d+2} cannot be a subgraph of $G^{(d)}$, where we use the notation $G^{(d)}$ for the dual graph of a d-simplex complex. A geometric proof of this non-subgraph result is offered in the Appendix. With that, we obtain an alternative proof of Theorem 1, which we restate in slightly different notation:

Theorem 3 *The dual graph $G^{(d)}$ of a d-simplex complex in $\mathbb{R}^d$ has chromatic number $\chi \leq d + 1$.*

Proof: Lemma 7 tells us that K_r is not a subgraph of $G = G^{(d)}$, with $r = d+2$. We have that $\Delta = d + 1$ because each d-simplex has $d + 1$ facets. Therefore we have

$$4 \leq r = d + 2 \leq \Delta + 1 = d + 2$$

for $d \geq 2$. Therefore Lemma 1 applies, and yields

$$\chi \leq \frac{d+1}{d+2}(d+3) \ .$$

Now we can see that

$$\frac{d+1}{d+2}(d+3) < d+2$$

by expanding $(d + 1)(d + 3)$ and $(d + 1)^2$:

$$d^2 + 4d + 3 < d^2 + 4d + 4 \ .$$

Thus χ is strictly less than $d+2$, which, because χ is an integer, implies $\chi \leq d+1$.
□

5 Beyond Simplices

One can ask for analogs of Theorem 1 for complexes composed of shapes beyond simplices. In the plane, a natural generalization is a complex built from convex quadrilaterals glued edge-to-edge. These complexes sometimes need four colors, as the example in Figure 5 shows. One does not need the 4-color theorem for this restricted class, even without the convexity assumption: there must exist a quadrilateral in a quadrilateral complex with an exposed edge, and 4-coloring follows by induction. Complexes built from pentagons can be proved 4-colorable by modifying the Kempe-chain argument;[3] so again the full 4-color theorem is not needed here.

---

[3] I owe this observation to Sergey Norin, `http://mathoverflow.net/questions/49743/4-coloring-maps-of-pentagons`.
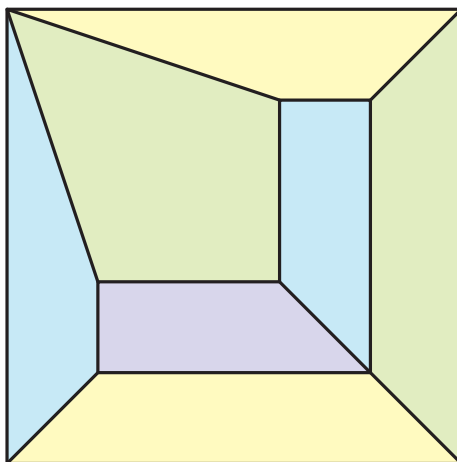
Figure 5: A convex quadrilateral complex that needs four colors [SW00, Fig.3a].

Sibley and Wagon proved in [SW00] the beautiful result: if the convex quadrilaterals are all parallelograms, then three colors suffice (essentially because there must be a parallelogram with two exposed edges). In particular, Penrose rhomb tilings (their original interest) are 3-colorable. Even more restrictive is requiring that the parallelograms be rectangles. Here with a student I proved in [GO03] that such *rectangular brick* complexes of genus 0 are 2-colorable. It is easily seen that complexes of genus 1 or greater might need three colors (surround a hole with an odd cycle).

We also explored generalizations to $\mathbb{R}^3$ in [GO03]. Somewhat surprisingly, genus-0 complexes built from *orthogonal bricks* (rectangular boxes in 3D) are again 2-colorable. We also established that genus-1 orthogonal brick complexes are 3-colorable, and conjectured that the same result holds for arbitrary genus. I am aware of no substantive results on complexes built from parallelopipeds (aside from the observation in [GO03] that four colors are sometimes necessary), a natural generalization of the Sibley-Wagon result.[4] One could also generalize convex quadrilaterals to convex hexahedra (distorted cubes). All of these generalizations seem unexplored.

6   Appendix: $K_{d+2} \not\subseteq G^{(d)}$

Here we establish that $K_{d+2} \not\subseteq G^{(d)}$ without appeal to deBruijn-Erdős. We partition the argument into four lemmas, the first three of which show that there is essentially only one configuration that achieves K_{d+1}, the analog of the configuration in Figure 3. The fourth lemma then shows that K_{d+2} cannot be achieved.

---

[4] Our attempted proof in [GO03] for zonohedra is flawed.

Let σ_1, σ_2, and σ_3 be d-simplices. Suppose σ_1 and σ_2 share a $(d{-}1)$-facet. We will represent each simplex by the set of its vertex labels, with distinct labels representing distinct points in $\mathbb{R}^d$. When specifically referring to the point in space corresponding to label i, we'll use v_i. Let $\sigma_1 = \{1, 2, \ldots, d, (d{+}1)\}$ $\sigma_2 = \{1, 2, \ldots, d, (d{+}2)\}$, with $\sigma_1 \cap \sigma_2 = f_{12} = \{1, 2, \ldots, d\}$ their shared $(d{-}1)$-facet. Under these circumstances, the following lemma holds:

Lemma 4 *If σ_3 shares a $(d{-}1)$-facet with σ_1 and a $(d{-}1)$-facet with σ_2 (and so the three simplices form K_3 in the dual), then the $d+1$ vertices of σ_3 are among the $d+2$ vertices of $\sigma_1 \cup \sigma_2 = \{1, 2, \ldots, d, (d{+}1), (d{+}2)\}$: σ_3 cannot include a vertex that is not a vertex of either σ_1 or σ_2.*

Proof: Suppose to the contrary that σ_3 includes a new vertex labeled $(d{+}3)$. For σ_3 to share a $(d{-}1)$-facet with σ_1, it needs to match d of the $d+1$ vertices of σ_1. But it cannot match the facet $f_{12} = \{1, 2, \ldots, d\}$ because that is already covered by σ_2. Without loss of generality, let us assume that σ_3 includes vertex $(d{+}1)$ but excludes vertex k with $1 \le k \le d$. So the $d+1$ vertices of σ_3 are

$$\sigma_3 = \{(d{+}1), 1, 2, \ldots, (k{-}1), (k{+}1), \ldots, d, (d{+}3)\} \ .$$

Now comparison to σ_2,

$$\sigma_2 = \{1, 2, \ldots, d, (d{+}2)\}$$

shows that it is not possible for σ_3 to match d of the $d+1$ vertices of σ_2 (as it must to share a $(d{-}1)$-facet): the two only share $d-1$ labels:

$$\sigma_2 \cap \sigma_3 = \{1, 2, \ldots, (k{-}1), (k{+}1), \ldots, d\} \ .$$

This contradiction establishes the claim. $\qquad\square$

Lemma 5 *Suppose $d+1$ d-simplices are glued together so that their dual graph is K_{d+1}. Then all the simplices together include only $d+2$ vertices.*

Proof: Let $\sigma_1, \ldots, \sigma_{d+1}$ be the simplices. By Lemma 4, $\sigma_1, \sigma_2, \sigma_3$ together include only $d+2$ vertices, the $d+2$ vertices of $\sigma_1 \cup \sigma_2$. But then repeating the argument for σ_i for each $i = 4, 5, \ldots, d+1$ yields the same conclusion. $\qquad\square$

We continue to study the K_{d+1} configuration in the above lemma. Let us specialize to $d = 3$ to make the situation clear. We have four tetrahedra glued together to form K_4, and Lemma 5 says they have altogether 5 vertices. Because $\binom{5}{4} = 5$, only one of the possible combinations of the labels $\{1, 2, 3, 4, 5\}$ is missing among the four tetrahedra. Without loss of generality, we can say that $\{2, 3, 4, 5\}$ is missing, and that our four tetrahedra have these labels:
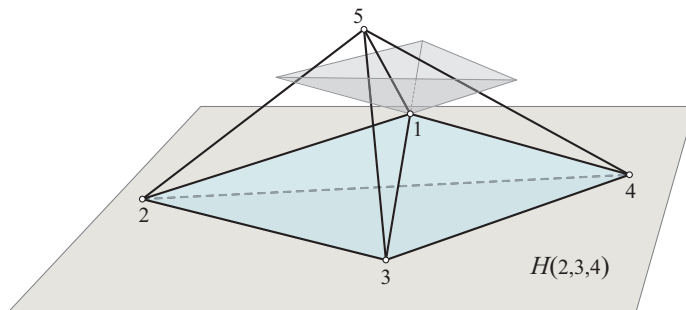
$$\{1, 2, 3, 4\}$$
$$\{1, 2, 3, 5\}$$
$$\{1, 2, 4, 5\}$$

Figure 6: Four tetrahedra whose dual forms K_4.

$$\{1, 3, 4, 5\}$$

Our next claim is that v_5 lies to the same side of the plane determined by the face $\{2, 3, 4\}$ as does v_1. Refer to Figure 6.

Let $H(i, j, k)$ be the plane containing the vertices with labels i, j, and k. Let $H^+(i, j, k; m)$ be the open halfspace bound by $H(i, j, k)$ and exterior to the tetrahedron $\{i, j, k, m\}$, and $H^-(i, j, k; m)$ the analogous open halfspace including tetrahedron $\{i, j, k, m\}$. The claim is that $v_5 \in H^-(2, 3, 4; 1)$. The other three tetrahedra can each be viewed as the hull of v_5 and one of the three faces of the $\{1, 2, 3, 4\}$ tetrahedron above the base: $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 3, 4\}$. Because a tetrahedron can only be formed by a point above each of these faces, we have that

$$v_5 \in H^+(1, 2, 3; 4)$$

$$v_5 \in H^+(1, 2, 4; 3)$$

$$v_5 \in H^+(1, 3, 4; 2)$$

So v_5 must lie in the intersection of these three halfspaces, which is a cone apexed at v_1 that is strictly above the base plane $H(2, 3, 4)$. See again Figure 6. And therefore $v_5 \in H^-(2, 3, 4)$, as claimed.

We now repeat this argument for d-simplices, where the logic is identical but is perhaps obscured by the notation.

The configuration of $d + 1$ d-simplices forming K_{d+1} in Lemma 5 uses only $d+2$ vertices. Because $\binom{d+2}{d+1} = d+2$, only one of the combinations of $d+1$ labels is missing, which we take to be $\{2, 3, \ldots, (d+2)\}$ without loss of generality. So the labels of the $d + 1$ simplices are:

$$\{1, 2, \ldots, d, (d+1)\}$$

$$\{1, 2, \ldots, d, (d+2)\}$$

$$\{1, 2, \ldots, (d+1), (d+2)\}$$

$$\ldots$$

$$\{1, 3, \ldots, d, (d+1), (d+2)\}$$

Lemma 6 *In the configuration of $d+1$ simplices forming K_{d+1} labeled as just detailed above, v_{d+2} lies in $H^- = H^-(2, 3, \ldots, (d+1); 1)$, the same halfspace in which v_1 lies.*

Proof: $H(2, 3, \ldots, (d+1))$ is the flat containing the "base" of the first simplex in the list above, $\sigma_1 = \{1, 2, \ldots, d, (d+1)\}$. The remaining d simplicies in the list share the facets of σ_1 incident to v_1, each including v_{d+2}. Thus v_{d+2} is above each of those facets, i.e., it lies in the corresponding H^+ halfspaces:

$$v_{d+2} \in H^+(1, 2, \ldots, d; (d+1))$$

$$\ldots$$

$$v_{d+2} \in H^+(1, 3, \ldots, d, (d+1); 2)$$

And therefore v_{d+2} lies in the intersection of all these halfspaces, which is a cone apexed at v_1 and lying strictly above $H(2, 3, \ldots, (d+1))$. Therefore v_{d+2} is in H^-. □

Completing the argument is now straightforward.

Lemma 7 $K_{d+2} \not\subseteq G^{(d)}$

Proof: Assume to the contrary that K_{d+2} is a subgraph of $G^{(d)}$. Then K_{d+1} must be also. Using the notation of Lemma 6, that lemma establishes that in a configuration that realizes K_{d+1}, vertex v_{d+2} lies in $H^- = H^-(2, 3, \ldots, (d+1); 1)$. Because $\{2, 3, \ldots, (d+1)\}$ is the only facet of the simplex $\sigma_1 = \{1, 2, \ldots, d, (d+1)\}$ not yet covered by another simplex, the last simplex σ_{d+2} must have labels $\{2, \ldots, d, (d+1), (d+2)\}$. And therefore $v_{d+2} \in H^+(2, 3, \ldots, (d+1); 1)$. But this is a contradiction, as it is saying that v_{d+2} must lie strictly to both sides of $H(2, 3, \ldots, (d+1))$. □

References

[Bro41] R. L. Brooks. On colouring the nodes of a network. In *Mathematical Proceedings of the Cambridge Philosophical Society*, volume 37, pages 194–197. Cambridge Univ Press, 1941.

[EdB51] P. Erdős and N. G. de Bruijn. A colour problem for infinite graphs and a problem in the theory of relations. *Indag. Math.*, 13:371–373, 1951.

[GO03] Suzanne Gallagher and Joseph O'Rourke. Coloring objects built from bricks. In *Proc. 15th Canad. Conf. Comput. Geom.*, pages 56–59, 2003.

[Sta02] L. Stacho. A note on upper bound for chromatic number of a graph. *Acta Math. Univ. Comenianae*, LXXI(1):1–2, 2002.

[SW00] Thomas Sibley and Stan Wagon. Rhombic Penrose tilings can be 3-colored. *American Mathematics Monthly*, 106:251–253, 2000.