Computational Geometry Column 51

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Abstract

Can a simple spherical polygon always be triangulated? The answer depends on the definitions of "polygon" and "triangulate."

Define a *spherical polygon* to be a simple, closed curve on a sphere S composed of a finite number of great circle arcs (also known as *geodesic arcs*) meeting at vertices. Can every spherical polygon be triangulated? Figure 1 shows an example of what is intended.[1] The planar analog is well-known and a cornerstone of computational geometry: the interior

Figure 1: A triangulated spherical polygon of n=14 vertices. The polygon edges are blue/dark, the diagonals are red/light.

of every planar simple polygon can be triangulated (and efficiently so). The situation for spherical polygons is not so straightforward. There are three complications.

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[1]See the Acknowledgments for how the figures in this note were created.

1 Two Sides

First, a planar polygon has a bounded interior and an unbounded exterior, and it is the interior that is triangulated. A spherical polygon partitions S into two bounded regions that are on an equal footing. If we ask, "Can every spherical polygon be triangulated to both sides?" the answer is NO, as Figure 2 illustrates. Here the regular pentagon (a, b, c, d, e) can be

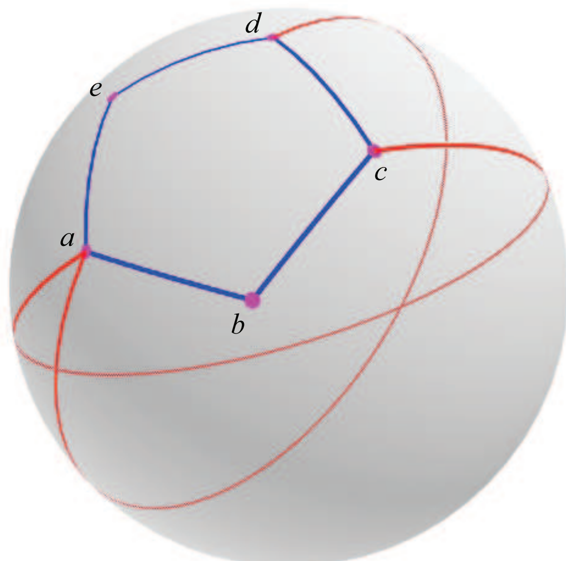

Figure 2: The nonconvex side of this pentagon, which contains a great circle, cannot be triangulated.

triangulated to its convex side, but not to the other side, as can be seen as follows. There is only one triangulation of a regular pentagon up to rotational symmetry, so we can just examine one to check, say the one including diagonals ac and ad. As the figure shows, these two diagonals cross on the far side of the sphere. So there is no triangulation by noncrossing vertex-to-vertex diagonals, the usual definition of a triangulation in the plane. So the question should be, "Can every spherical polygon be triangulated to one side or the other?"

2 Spherical Triangles

The second complication is that the answer here is also NO, under the most natural definition of what constitutes a triangulation on a sphere S. Define a *spherical triangle* to be a region bounded by three geodesic arcs such that the internal angle at all three vertices is strictly less than π. The convex side of the triangle is then distinguished as its *inside*. Define a *triangulation* of a spherical polygon $\mathcal{P}$ as a partition of one of the two regions of S bound by $\mathcal{P}$ into spherical triangles via noncrossing vertex-to-vertex diagonals. Under this definition, the quadrilateral $Q = (a, b, c, d)$ in Figure 3 has no triangulation to either side.[2] Call the

[2]This example was first shown to me by Boris Springborn (Technische Universität Berlin, Institut für Mathematik), Sep. 2007. The equivalent version illustrated here is due to Ophir Setter (School of Computer

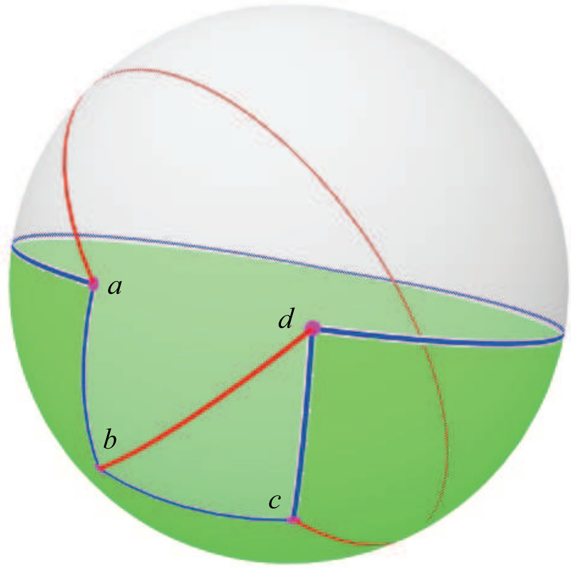

Figure 3: Springborn/Setter untriangulable quadrilateral. The long diagonal ac crosses edge ad. The short diagonal bd leaves the exterior angles at $\{a, d\}$ greater than π.

lower shaded region of S bounded by Q its "inside." The long diagonal ac, which is interior to Q in a neighborhood of c, crosses ad on the far side of S, as does, symmetrically, the long diagonal bd. So the interior of Q has no diagonal. The other (short) portions of the great circles through ac and bd are intersection-free diagonals exterior to Q, but either one leaves angles $> \pi$ at vertices a and d, and so do not partition the exterior into spherical triangles as defined.

3 Short and Long Arcs

The third complication is that this situation changes if we restrict the polygon arcs to be less than a semi- great circle. For a unit-radius sphere, this condition is equivalent to insisting that all arcs of $\mathcal{P}$ are $< \pi$ in length. Then the following theorem holds:

Theorem 1 *Let $\mathcal{P}$ be a spherical polygon on a unit sphere S each of whose edges have length strictly less than π. If a side of $\mathcal{P}$ does not contain a great circle, then that side has a spherical triangulation (in the sense defined above).*

Note that it cannot be that both sides contain a great circle, because two great circles intersect twice, and those intersection points would have to lie on both sides. Reviewing our previous examples: the polygon in Figure 1 does not contain a great circle to either side, and so has a triangulation to both sides; the pentagon in Figure 2 can only be triangulated to the convex side, which fits in a hemisphere, and the ad edge of Q in Figure 3 is longer than π, and so falls outside the scope of this theorem.

Science, Tel Aviv University), Jun. 2008.

The theorem is proved as a lemma in a paper by Brehm and Kühnel [BK82, Lem. 4] in pursuit of a rather different goal. Their proof is essentially the same as the standard proof for planar polygons that shows that there is always an internal diagonal: identify consecutive vertices a, b, c with b convex; then either the internal diagonal ac cuts off spherical triangle $\triangle abc$, or $\triangle abc$ contains a vertex x such that bx is a diagonal. Repeating this process results in a triangulation.

We've seen that the restriction to edges of length $< \pi$ is necessary for this theorem to hold. This restriction is also natural in that the projection of a straight segment inside S from its center x to the surface always produces an arc that satisfies this restriction (if it does not include x). This plays a role in triangulating 3D straight-segment polygons. It was shown in [BDE98] that not all such 3D polygons P can be triangulated. But if there is a point x from which every point of P is visible (i.e., for each $p \in P$, $xp \cap P = \{p\}$), then P is triangulable: projecting P to a bounding sphere S centered on x leads to a simple spherical polygon $\mathcal{P}$, at least one side of which can be triangulated by Theorem 1. The spherical triangles can then be "pulled back" to flat triangles triangulating P.

If a spherical polygon $\mathcal{P}$ fits strictly in a hemisphere of S, say, the southern hemisphere, then there is a simple algorithm for triangulating $\mathcal{P}$ [BDE98]: Place the south pole on a plane Π, and project $\mathcal{P}$ from the center of S to Π. This "gnomonic" projection maps each arc of $\mathcal{P}$ to a straight segment on Π. Triangulating the resulting planar simple polygon then yields a triangulation of $\mathcal{P}$ on S.

When $\mathcal{P}$ does not fit in a hemisphere, mimicking the proof of Theorem 1 yields a quadratic algorithm. Although I have not seen this addressed in the literature, it seems likely that the efficient planar algorithm complexities can be matched on the sphere.

Although we have seen in Figure 2 that the great circle condition of Theorem 1 is necessary, it was recently established that, again for spherical polygons $\mathcal{P}$ with all edge lengths strictly less than π, $\mathcal{P}$ can be triangulated with the addition of at most one extra ("Steiner") point [Hal08]. For example, if the pentagon in Figure 2 contains the north pole, a Steiner point at the south pole suffices to triangulate the side that contains the equator.

4 3-gons?

Finally, let us reconsider the definition of a triangulation. Define a *3-gon* to be a region of S bounded by three geodesic arcs, without the $< \pi$ angle restriction used in the definition of a spherical triangle. Then the quadrilateral Q in Figure 3 can be partitioned to one side into 3-gons by either of the short, exterior diagonals ac or bd. Can every spherical polygon be partitioned by noncrossing vertex-to-vertex diagonals, to one side or the other, into 3-gons? It appears that this question has not been previously addressed.

Acknowledgments. I thank Boris Springborn for discussions and for identifying [BK82]. And I thank Eli Fogel, Dan Halperin, and Ophir Setter for employing the new CGAL package they implemented for arrangements of geodesic arcs on a sphere [FSH08] to produce Figures 1-3.

References

[BDE98] Gill Barequet, Matthew Dickerson, and David Eppstein. On triangulating three-dimensional polygons. *Comput. Geom. Theory Appl.*, 10(3):155–170, 1998.

[BK82] Ulrich Brehm and Wolfgang Kühnel. Smooth approximation of polyhedral surfaces regarding curvatures. *Geom. Dedicata*, 12:435–461, 1982.

[FSH08] Eli Fogel, Ophir Setter, and Dan Halperin. Exact implementation of arrangements of geodesic arcs on the sphere with applications. In *Proc. 24th European Workshop Comput. Geom.*, pages 83–86. March 2008.

[Hal08] Dan Halperin. Every spherical polygon is triangulable using at most one Steiner point. Manuscript, July 2008.