Computational Geometry Column 47

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Abstract

A remarkable theorem is described: "It is possible to tile the plane with non-overlapping squares using exactly one square of each integral dimension." Thus, one can "square the plane."

More than thirty years ago, Solomon Golomb posed [Gol75] the question of whether or not the infinite plane could be tiled using one copy of each square of integer side length: one copy of the squares (1×1), (2×2), (3×3), and so on. The problem was subsequently discussed in a Martin Gardner column, and in Grünbaum and Shephard's *Tiling and Patterns* [GS87, p. 79], but remained unsolved. The challenge is to avoid repeating any square, at the same time as using all of them. Golomb's "heterogeneous tiling conjecture" has now been established by Henle and Henle [HH06]. Here we sketch the main idea of the proof, which is both elementary and subtle.

Define a figure to be *perfect* if it is composed entirely of non-overlapping squares of different sizes. Define an $\mathcal{L}$ to be any six-sided orthogonal polygon, i.e., one whose edges meet at right angles. (An example is in Fig. 3a.)

Their proof is constructive. It starts with any perfect $\mathcal{L}$, for example, the join of (4×4) and (3×3) squares. We'll call this the $(4,3)-\mathcal{L}$, and name other $\mathcal{L}$'s similarly. The heart of their proof is a procedure for "puffing" a perfect $\mathcal{L}$ to a perfect rectangle R, by surrounding it with a number of distinct squares not used before. It is important that growth occurs in all four directions. Then the smallest square not yet employed in the construction is attached to R, forming a new $\mathcal{L}$. And the process is repeated. "And finally, infinity being what it is, we are guaranteed to incorporate in the tiling a square of each integral dimension"!

Here are two examples of the puffing procedure. Starting from the $(4,3)-\mathcal{L}$, their algorithm surrounds it with squares of size

$$7, 11, 15, 26, 41, 44, 54, 57,$$

producing a 98×111 perfect rectangle; see Fig. 1. Starting instead from the $(5,3)-\mathcal{L}$, a 431×497 rectangle (Fig. 2) is created by surrounding it with squares of size

$$8, 13, 18, 31, 49, 52, 132, 181, 184, 247, 250.$$

One can see the rectangularization is sensitive to the dimensions of the starting $\mathcal{L}$. In fact,

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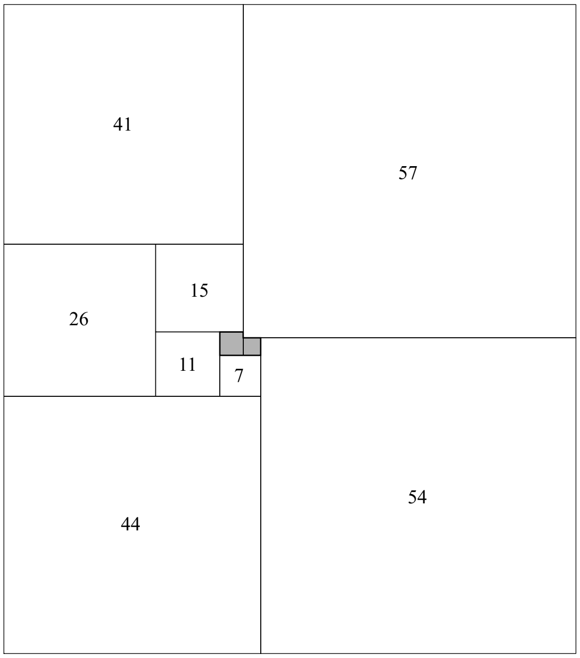

Figure 1: Start: $(4,3)-\mathcal{L}$. End: 98×111 rectangle.

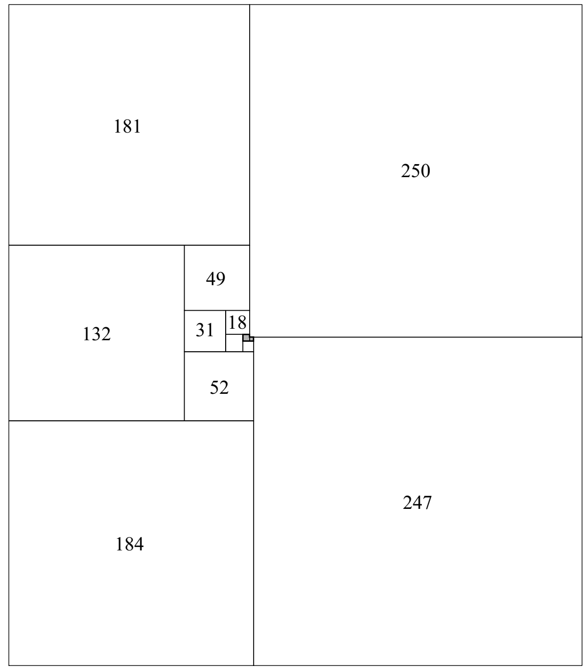

Figure 2: Start: $(5,3)-\mathcal{L}$. End: 431×497 rectangle.

starting from the $(5,2)-\mathcal{L}$, leads, after the addition of 2613 squares, to a rectangle with dimensions approximately $(2.3 \times 10^{498}) \times (2.7 \times 10^{498})$.

We now give a hint of the proof details. First, the procedure is slightly different for the first step than for later steps. To explain this, we need some more notation. An $\mathcal{L}$ in *standard position* is as shown in Fig. 3a, with its six edges labeled as indicated. Their algorithm always adds squares that match the full length of an edge of the figure. *Move* A is placing a square on side A, and so on. Move AB is placing a square on side A, and then on side B of the new figure. Etc. Now, these moves cannot be made to the start $\mathcal{L}$'s we illustrated: for example, move A for the $(4,3)-\mathcal{L}$ repeats the (4×4) rectangle. So a first step is to "regularize" the $\mathcal{L}$. A *regular* $\mathcal{L}$ is a perfect $\mathcal{L}$ for which each of the moves B, F, and ED results in either a perfect $\mathcal{L}$ in standard position, or a perfect rectangle. See Fig. 3bcd for these moves. Note that the $\mathcal{L}$ in Fig. 3a is not regular, because the move ED results in an $\mathcal{L}$ in nonstandard position.

Henle & Henle show that a perfect $\mathcal{L}$ in standard position can be puffed to form a regular $\mathcal{L}$ (with some special properties), and that every regular $\mathcal{L}$ can be puffed to a rectangle. The proof of the first claim is not obvious; the proof of the second claim is more intricate, involving an analysis of the lengths $(C-E) \mod D$. So the proof path is:

$$(\text{perfect } \mathcal{L} \text{ in standard position}) \rightarrow (\text{regular } \mathcal{L}) \rightarrow (\text{perfect rectangle } R)$$

Attaching the smallest unused square to the right of R and repeating completes the proof.

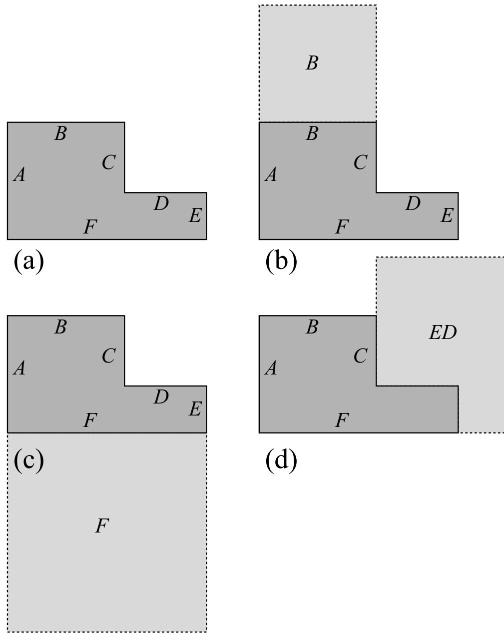

Figure 3: (a) An $\mathcal{L}$ in standard position. (b,c,d): After moves B, F, and ED.

Many open problems remain, including whether or not it is possible to "cube space": tile 3-space with non-overlapping cubes, exactly one of each integer side length.

References

[Gol75] S. Golomb. The heterogeneous tiling conjecture. *J. Recreational Math.*, 8(2):138–9, 1975.

[GS87] B. Grünbaum and G. C. Shephard. *Tilings and Patterns*. W. H. Freeman, New York, NY, 1987.

[HH06] F. V. Henle and J. M. Henle. Squaring the plane. Unpublished manuscript. Presented at the *Joint Mathematics Meetings*, San Antonio, TX, January 14, 2006, Abstract 1014-52-1524.