Computational Geometry Column 40

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Abstract

It has recently been established by Below, De Loera, and Richter-Gebert that finding a minimum size (or even just a small) triangulation of a convex polyhedron is NP-complete. Their 3SAT-reduction proof is discussed.

All triangulations of a polygon of n vertices use the same number of triangles, $n-2$, but the same does not hold in higher dimensions, even for convex polytopes. We will only discuss 3-polytopes, i.e., convex polyhedra P, where a *triangulation* is a collection of tetrahedra whose vertices are drawn from the vertices of P, whose union is P, and such that the intersection of any two of the tetrahedra is either empty or a vertex, edge, or face common to the two tetrahedra. The number of tetrahedra in a triangulation of a convex polyhedron of n vertices might be as small as $n-3$ or as large as $\binom{n}{2} - 2n + 3$. It is easy to obtain a linear-size triangulation of a *simplicial* polyhedron (all faces triangles) by the following *starring* procedure ([Ber97, p. 424]). Select one vertex v and include the tetrahedron formed by the convex hull of v and each face f not incident to v. Because a simplicial polyhedron has $F = 2n - 4$ faces, this method yields at most $2n - 7$ tetrahedra (at least three tetrahedra are incident to v). For nonsimplicial polyhedra, it could be better. For example, applied to a cube, starring results in a triangulation by 6 tetrahedra (Fig. 1a). So this method provides a triangulation using at most twice the minimum number. But getting closer to the optimum (in the case of a cube, 5 (e.g., Fig. 1b)) has proven difficult. Now we know why: Below, De Loera, and Richter-Gebert (BDR) proved that deciding whether a convex polyhedron can be triangulated with fewer than k tetrahedra is NP-complete [BDR00].

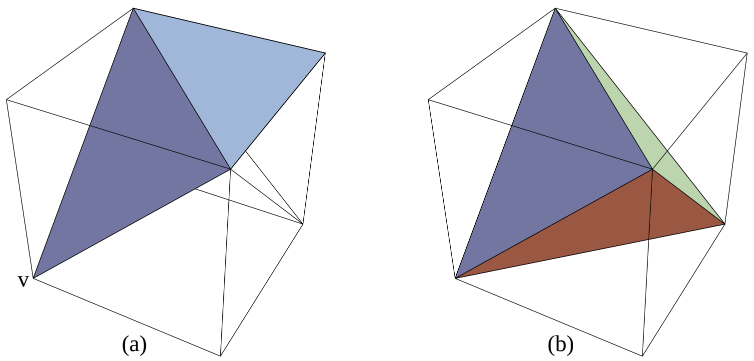

Figure 1: (a) One tetrahedron in a triangulation of six tetrahedra starred from v. Here 6 of the 12 faces are incident to v; $6 = 2n - 10$. (b) A triangulation using five tetrahedra; the central tetrahedron is shown.

Their proof follows the structure of Ruppert and Seidel's similar proof that the same question for nonconvex polyhedra is intractable [RS92]. But the latter authors showed that even deciding whether a polyhedron could be triangulated is hard, whereas we've seen all convex polyhedra are easily triangulated. Here I will sketch just one aspect of the proof in [BDR00].

Both proofs rely on Schönhardt's untriangulable polyhedron ([O'R87, p. 254]; [Ber97, p. 423]), shown in Fig. 2(b). The three reflex diagonals (a) block triangulation. The polyhedron is first transformed by enlarging the base B (c). Now if B is glued to a larger "frame" polyhedron, its top face A, which BDR call the "skylight," must be connected through B to a vertex below to form the tetrahedron that includes A. It is this "visibility"

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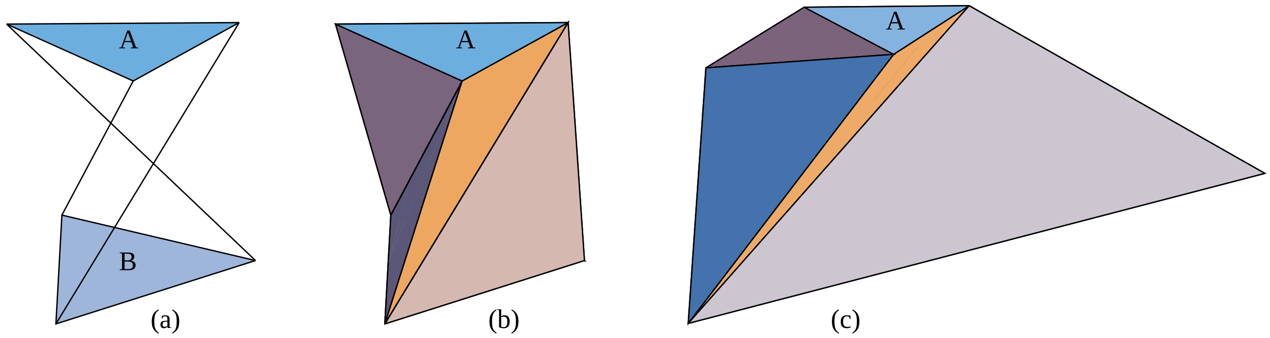

Figure 2: Schönhardt polyehdron. (a) Reflex diagonals; (b) Polyhedron; (c) Base B enlarged with respect to "skylight" A.

constraint that Ruppert and Seidel exploited to arrange for their 3-SAT reduction.

BDR convexify the attached Schönhardt polyhedra through the following strategy. They prove (in [BB$^+$00]) that a fan-shaped polyhedron like that shown in Fig. 3 (embedded in a larger polyhedron) is efficiently triangulated by employing the internal axis diagonal ab, but any triangulation that avoids that diagonal uses many more tetrahedra. So they string a shallow arc of points exterior to the three reflex diagonals of each Schönhardt polyhedron, to achieve two goals: (1) to convexify each; and (2) to heavily penalize any triangulation that does not employ the reflex diagonals. This forces the inclusion of the Schönhardt diagonals, which forces skylight visiblity constraints.

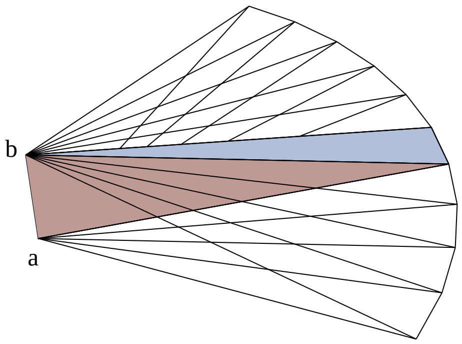

Figure 3: A fan is best triangulated using diagonal ab; one tetrahedron is shown.

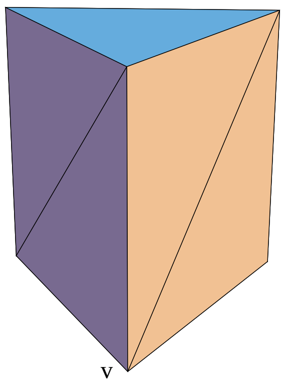

Figure 4: Starring from v is incompatible with surface triangulation.

The remainder (and majority) of their proof exploits these constraints to construct variable and clause gadgets on the frame polyhedron, carefully arranging lines of sight to result in a convex polyhedron that can be triangulated with few tetrahedra iff a particular logical formula is satisfiable. Beside the intricacy of the logical structure, two delicate issues are retaining convexity, and assuring the vertex coordinates remain singly-exponential, and so polynomially representable.

At least two interesting open questions remain. The first is determining the complexity of finding a maximum size (or just a large) triangulation. Perhaps surprisingly, large triangulations of d-polytopes do have application, to algebraic geometry and to integer programming.

The second problem has practical significance in geometric modeling, in particular, to meshing. For a nonsimplicial polyhedron, i.e., one with faces of more than three sides, the surface may be triangulated in several different ways. Typically solid models are given with a particular surface triangulation. It is unknown how difficult it is to decide whether there exists a triangulation of the polyhedron into tetrahedra that is *compatible* with a given surface triangulation, in the sense that each triangle on the surface is a face of a tetrahedron. For example, no starring of the triangular prism shown in Fig. 4 is compatible with the displayed

(Schönhardt-like) surface triangulation, for starring from v demands all faces incident to v be also starred. In fact no triangulation of this prism is compatible without adding an interior "Steiner" point.

References

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