Computational Geometry Column 39

Joseph O'Rourke*

Abstract

The resolution of a decades-old open problem is described: polygonal chains cannot lock in the plane.

A *polygonal chain* is a connected series of line segments. Chains may be open, or closed to form a polygon. A *simple* chain is one that does not self-intersect: only segments adjacent in the chain intersect, and then only at their shared endpoint. If the segments of a polygonal chain are viewed as rigid bars, and the vertices as universal joints, natural questions are whether every open chain can be *straightened*—reconfigured to lie on a straight line—and whether every closed chain can be *convexified*—reconfigured to form a planar convex polygon. In both cases, the chains are to remain simple throughout the motion. If a chain cannot be so reconfigured, it is called *locked*.

These questions were raised by several researchers independently since the 1970's,[1] and were the subject of intense investigation by the late 1990's. It was first established that chains can lock in three dimensions (3D): both locked open chains, and locked closed but unknotted chains are possible [CJ98, BDD$^+$99]. In 4D, neither open nor closed chains can lock [CO99]. But aside from some special cases (e.g., star-shaped polygons cannot lock [ELR$^+$98], polygonal trees can lock [BDD$^+$98]), the problems remained unresolved for chains in 2D.

Connelly, Demaine, and Rote have now settled the questions, establishing that neither open nor closed chains can lock in 2D [CDR00]. Their result is even more general: no collection of disjoint simple chains are locked (although of course a chain nested inside a polygon is forever confined). They prove that every such collection has an *expansive motion*: one during which the distance between every pair of vertices increases or stays the same. An expansive motion automatically maintains simplicity, for the distance between some pair of vertices would have to decrease to reach self-intersection. Their proof first establishes that any configuration of chains has an infinitesimal expansive motion, and then combines these motions into a global expansive motion. Although there are different ways to stitch the infinitesimal motions together, one produces especially natural movements. This motion minimizes the squared lengths of the velocity vectors v_i applied to each vertex i at each time: minimize $\sum_i \|v_i\|^2$, subject to constraints maintaining segment lengths and forcing expansivity.[2] The objective function is strictly convex and the system can be solved by quadratic programming. Figs. 1 and 2 illustrate the solution so obtained for a collection of three closed chains (triangles) entangled with one open chain.

Let us simplify the discussion to a single open chain and offer a crude sketch of their key argument. View nonadjacent vertices of the chain as connected by a *strut*, which is permitted to increase in length or stay the same, but never decrease. If this bar-and-strut framework has an infinitesimal motion, then the motion is necessarily expansive. An *equilibrium stress* in an assignment of weights (*stresses*) to each bar and strut so that (a) the stress on each strut is nonnegative, and (b) the stress vectors (vectors along the bars/struts scaled by the stresses) are in equilibrium at each vertex. Bar stresses may be positive (compression) or negative (tension). If a framework is "stuck," or, more formally, "first-order rigid" [CW96, Whi97], then it has a nonzero equilibrium stress. So if a framework only has the trivial zero equilibrium stress, then it possesses an infinitesimal expansive motion.

In order to apply a theorem that holds for planar frameworks, every intersection point of the framework is used to divide the bars/struts, which produces a planar framework equivalent to the original in terms of equilibrium stresses. Now the century-old Maxwell-Cremona theorem is applied: if the framework has a nonzero

*Dept. of Computer Science, Smith College, Northampton, MA 01063, USA. orourke@cs.smith.edu. Supported by NSF Grant CCR-9731804.

[1] G. Bergman, U. Grenander, W. Lenhart, J. Mitchell, S. Schanuel, and S. Whitesides. It seems to have first appeared in print in 1995: [LW95] and [Kir95, p. 270].

[2] Their proof employs a further nonquadratic "barrier" penalty term for smoothness.

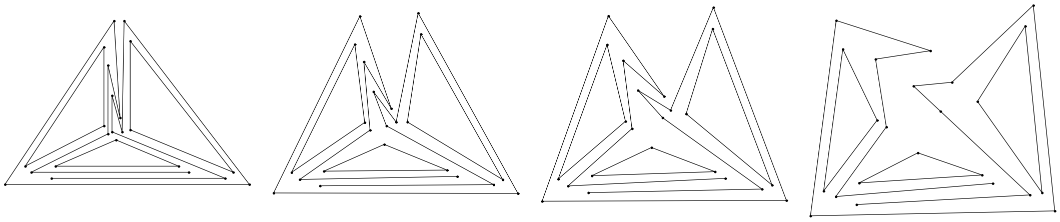

Figure 1: Four frames in the unlocking of four polygonal chains. The initial disjoint collection is shown in the leftmost frame. [Animation courtesy of Erik Demaine.]

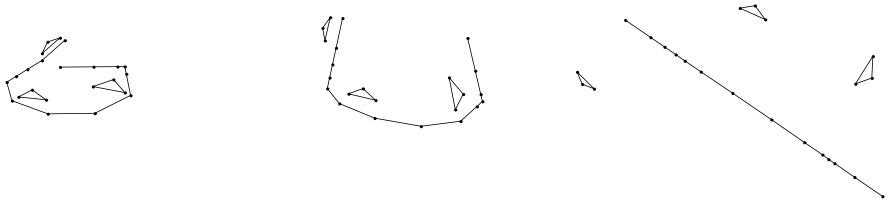

Figure 2: Three frames toward the end of the unlocking motion, continuing Fig. 1 (at a different scale).

equilibrium stress, then it can be "lifted" to a nonflat polyhedral terrain that projects to the framework, with positive-stress edges (always bars) lifting to "mountains," and negative-stress edges to "valleys." Finally it is shown that such a lifting is impossible, by concentrating on its maximum. For example, if this maximum is a single point, then it must lie at the junction of at least three mountain-edges, which violates the fact that every vertex of the chain is incident to at most two bars. A case analysis on the topologically possible maxima proves that only the degenerate flat lifting is possible, which implies that the original framework has only the zero equilibrium stress, which implies that it has an infinitesimal expansive motion, which implies that it has a global expansive motion, i.e., it is not locked.

Building on this work, Streinu has found a way to decompose the expansive global motion for chains of n vertices into $O(n^2)$ sections, each of which is the motion of a one degree-of-freedom mechanism [Str00]. This mechanism is constructed by removing one hull edge from a *pseudotriangulation* of the chains, a partition by diagonal bars into regions each bounded by three reflex chains (where a single segment counts as a reflex chain). The mechanism is opened following its only free trajectory in configuration space until two adjacent edges align, at which point the pseudotrianguation is revised locally, and the expansion process continued.

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