Computational Geometry Column 36

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Abstract

Two results in "computational origami" are illustrated.

Computational geometry has recently been applied to solve two open problems in "origami mathematics."

1 One Cut Suffices

The first result is remarkable in its generality:

Theorem 1 *Any planar straight-line drawing may be cut out of one sheet of paper by a single straight cut, after appropriate folding* [DDL98, DDL99].

The drawing need not be connected; it may include adjoining polygons, nested polygons, floating lines segments, and isolated points. The algorithm of Demaine, Demaine, and Lubiw computes a crease pattern whose folding produces a flat origami that aligns all edges of the drawing on the same line L. Removal of L from the paper "cuts out" the drawing.[1]

We illustrate the results of their algorithm applied to a polygon in the shape of the letter `H` in Fig. 1(a1). Fold the sheet first in half along the horizontal bisector of the `H` (a1), and then in half again along the vertical bisector (a2). Now many of the polygon's edges lie on top of one another. Next, fold along the diagonal bisector of the right angle illustrated (a3) to align the adjacent edges. Continuing in this manner, after five folds, all edges lie on the same vertical line, and cutting along the arrow shown in Fig. 1(a6) removes the `H`.

The use of bisectors is a natural technique for overlapping the edges incident to a vertex, and suggests that the medial axis or Voronoi diagram may play a role. In fact the appropriate concept here is the *straight skeleton* [AA96]. For a polygon, this skeleton is defined by the tracks vertices follow when the shape shrinks via inward, parallel movement of the edges. For the `H`-polygon, the skeleton is particularly simple, but more complex shapes lead to shrinking "events" which disconnect the shape; then each is shrunk recursively. The skeleton may be defined for general straight-line plane graphs, developing both interior and exterior to faces. Fig. 1(b) shows the complete skeleton for the `H`-shape.

For this simple shape, all creases lie on lines containing skeleton edges. More complex shapes, for example the butterfly in Fig. 2, require in addition *perpendiculars* incident to skeleton vertices, which (perhaps) recursively generate more perpendiculars based on other cut edges. This recursive phenomena means that the number of creases is unbounded in terms of the number of vertices or minimum "feature" size of the drawing. This flaw has subsequently been circumvented by an algorithm based on disk-packing [BDEH98].

2 Wrapping Polyhedra

Akiyama posed the question of whether any (perhaps nonconvex) polyhedron may be "wrapped" by a single piece of paper [Aki97]. Portions of paper may be hidden by folding and tucking under, but the final result should exactly cover the faces of the polyhedron without requiring the paper to be cut. His question was answered and extended with this theorem:

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[1] It is possible that L will lie along folds, e.g., when the drawing consists of a single line segment.

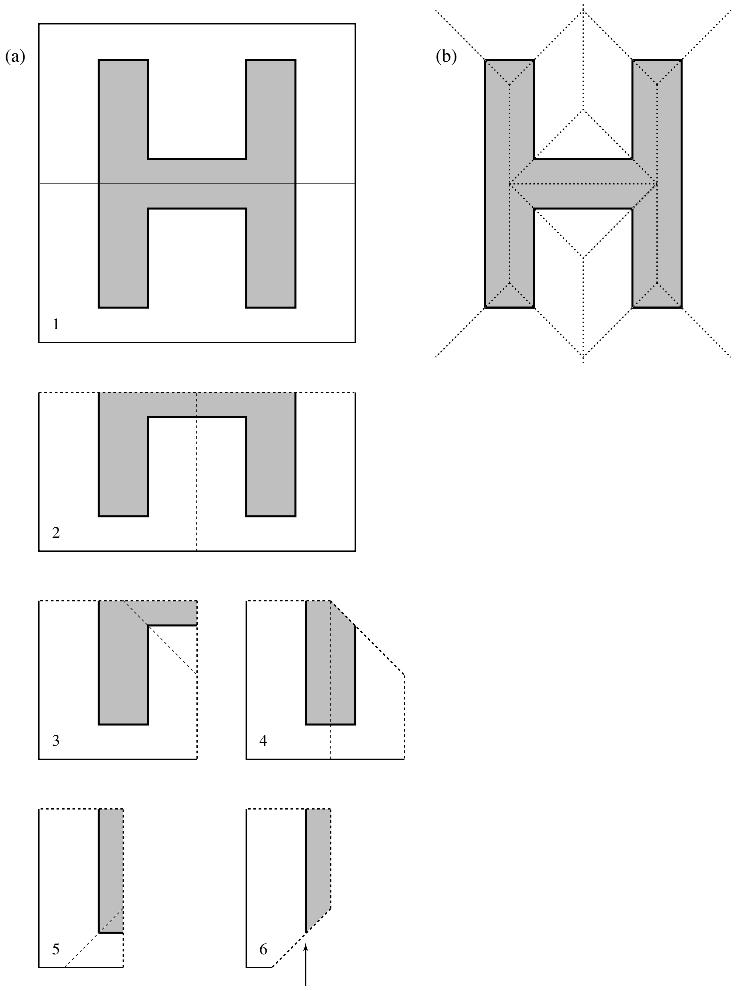

Figure 1: (a) Folding to align edges. The first fold (solid) is a mountain fold; all others (dashed) are valley folds. (b) The straight skeleton of the polygon.

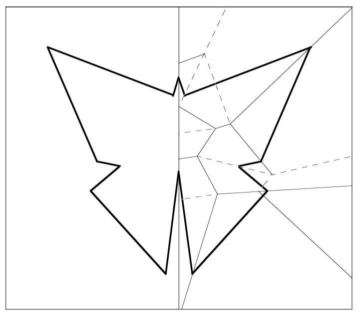

Figure 2: Crease pattern to cut out a butterfly. [Drawing courtesy of Erik Demaine.]

Theorem 2 *Any polyhedron may be wrapped with a sufficiently large square sheet of paper. This implies that any connected, planar, polygonal region may be covered by a flat origami folded from a single square of paper. Moreover, any 2-coloring of the faces may be realized with paper whose two sides are those colors* [DDM99].

Demaine, Demaine, and Mitchell provide three distinct algorithms for achieving such a folding, each with different properties and tradeoffs among desirable quantities. The first is based on Hamiltonian triangulations, the second on straight skeletons, and the third on convex decompositions. I will illustrate the general idea by folding a silhouette for a polygon in the shape of the letter `I`.

All three methods share the same first step: accordian-fold the paper into a strip; see Fig. 3(a). The methods differ on how this strip is used to cover the faces. The convex decomposition method starts with a partition of the faces into convex pieces, and then covers each face in the order determined by a traversal of a spanning tree of the partition dual. An optimized version of their algorithm could achieve the simple covering shown in Fig. 3(b). In this example no particular coloring was sought, but one can see there is freedom on the choice between mountain and valley folds, freedom which ultimately can be exploited to achieve any given 2-coloring.

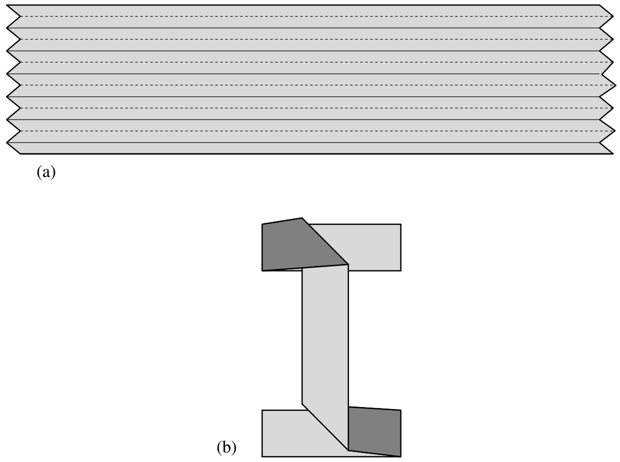

Figure 3: Folding a square to cover the shape `I`: (a) accordian-fold to strip; (b) strip folding.

Although one method in [DDM99], the "zig-zag milling" method that follows a Hamiltonian triangulation, hides an arbitrarily small fraction of the strip's area, the accordian-fold step wastes much of the original square paper. It remains open to achieve the same universality with a more efficient wrapping.

References

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