Computational Geometry Column 34

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Abstract

Problems presented at the open-problem session of the 14th Annual ACM Symposium on Computational Geometry are listed.

Jack Snoeyink, University of British Columbia:

1. Given a set P of points and a set S of disjoint line segments in the plane, does there always exist a spanning tree of P that, when embedded with straight edges, has the property that no segment in S is cut by more than two edges?
2. If the weight of an edge of the spanning tree is the number of segments of S that it crosses, then does the minimum spanning tree have weight at most $2|S|$?

   One can ask the same questions for paths instead of trees.

Note that an affirmative answer for the first problem implies an affirmative answer for the second. While a few constructions give spanning trees with $O(|S| \log |S|)$ weight, the best known lower bound is $2|S|$. One can assume that the segments of S induce a triangulation; in general, the application is to locate many points in a triangulation by walking along some "nice" path. Disjointness of segments is important; otherwise spanning trees with low stabbing number give upper and lower bounds of $\Theta(|S|\sqrt{|S|})$.

Vadim Shapiro, University of Wisconsin: Given two smooth real algebraic hypersurfaces S_1 and S_2 defined by polynomials of degree at most d that are tangent along the curve C (i.e., the locus of second order contact is the curve C), is there a (smooth?) algebraic hypersurface S that is:

1. the zero set of a polynomial of degree at most $d/2$,
2. contains C, and
3. at each point of C, S meets both S_1 and S_2 transversally.

If the degree of S must be greater than $d/2$, what is the lowest possible degree of S satisfying the other requirements?

This problem arises in construction of Boolean set representations for curved polyhedra; the practical algorithms for constructing such surfaces are of particular interest. More details are in [SV].

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[SV] V. Shapiro and D. L. Vossler, Separation for boundary to CSG conversion, *ACM Transactions on Graphics* **12** (1993), 35–55.

Joseph O'Rourke, Smith College: The cover of *Gödel, Escher, Bach* (by D. R. Hofstadter) shows a solid piece of carved wood, which casts the letters `G E B` as shadows in three orthogonal directions. This suggests two questions:

1. Are there simple conditions on three shapes to be realizable as shadows of a single, connected solid object in 3-space? An example of realizable shapes is shown in Figure 1. An example of unrealizable shapes is provided by the three letters `X X O`. Are there any interesting classes of shapes such that any three in the class are mutually realizable?
2. Extend this question to d dimensions. For example, can the letters `J O S E P H` be realized as the six 2-dimensional shadows of a 4-dimensional object?

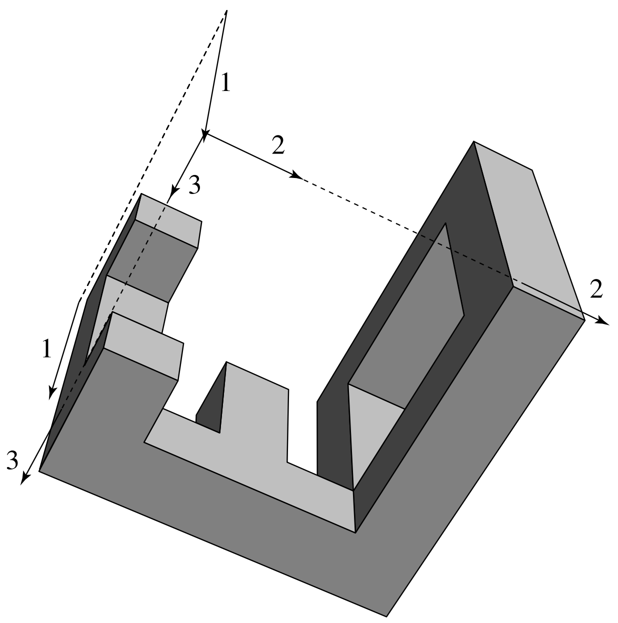

Figure 1: An orthogonal polyhedron whose shadow in each of the three labeled directions is an orthogonally polygonal letter of the alphabet. Fig. 4.22 in [OR].

[OR] J. O'Rourke, *Computational Geometry in C (Second Edition)*, Cambridge University Press, Cambridge, 1998.

Micha Sharir, Tel Aviv University: Let S be a set of n points in the plane, each moving with a fixed velocity.

How many times does the combinatorial structure of the Euclidean Voronoi diagram of S change over time?

The best known lower bound is quadratic, and the best known upper bound is cubic; see [SA, Sec. 8.6.3]. If the distance is measured in the L_1-metric, then the structure of the Voronoi diagram changes only $O(n^2\alpha(n))$ times [Ch], where $\alpha(\cdot)$ is the inverse Ackermann function.

[Ch] L. P. Chew, Near-quadratic bounds for the L_1 Voronoi diagram of moving points, *Proc. 5th Canad. Conf. Comput. Geom.*, 1993, pp. 364–369.

[SA] M. Sharir and P. K. Agarwal, *Davenport-Schinzel Sequences and Their Geometric Applications*, Cambridge University Press, New York, NY, 1995.

Subhash Suri, Washington University: Let S be a set of n points in the plane. For a given point c in the plane, the *star* of S is the set of edges $\{cp \mid p \in S\}$. The *weight* of a star is the sum of the lengths of its edges. Let $MS(S)$ denote the minimum-weight star of S, where the minimum is taken over all points in the plane, and let $MWT(S)$ denote the maximum-weight Euclidean matching of S.

Obtain an upper bound on the ratio $MS(S)/MWT(S)$.

It can be proved that $MS(S)/MWT(S) \leq 2$, but the conjecture is that the ratio is bounded by $2/\sqrt{3}$. The latter bound can be achieved by many examples.

This problem arises in analyzing the performance of a network design heuristic. $MS(S)$ and $MWT(S)$ provide upper and lower bounds, respectively, on the cost of traffic handling. See [FST] for details.

[FST] A. Fingerhut, S. Suri, and J. S. Turner, Designing minimum cost nonblocking communication networks, *J. Algorithms* **24** (1997), 287–309.

John Sullivan, University of Illinois: A knot K in $\mathbb{R}^3$ can be represented by its projection on the xy-plane. The xy-projection is a self-intersecting curve K^*. At each such intersection point of K^*, we describe which of the two corresponding portions of K lies above in the z-direction. Given such a combinatorial description of a knot K with n intersection points in K^*, what is the minimum length of a rope of width 1 that can realize K.

Some knots may require the length of the rope to be $\Omega(n)$, but is this an upper bound? It can be achieved with $O(n^2)$.

Julien Basch, Stanford University: Let S be a set of n segments in the plane. For a given x-coordinate t, let ℓ_t be the line $x = t$, and let $S_t \subseteq S$ be the set of segments intersected by the line ℓ_t. Suppose we construct a heap on S_t, using the y-coordinates of the intersection points of S_t with ℓ_t as the keys. We maintain this heap as the value of t varies continuously from $-\infty$ to $+\infty$. That is, whenever ℓ_t passes through an endpoint, the associated segment is inserted into or deleted from the heap in the usual way, and when ℓ_t passes an intersection point between two segments $p, q \in S_t$ so that p is the parent of q, the two elements are swapped to maintain the heap property.

Obtain an upper bound on the number of swaps required to maintain the heap.

If S is a set of lines, the best known upper bound on the number of swaps is $O(n\log^2 n)$. For segments, the best known upper bound is $O(n^{3/2}\log n)$ [BGR].

[BGR] J. Basch, L. Guibas, and G. Ramkumar, Sweeping lines and line segments with a heap, *Proc. 13th Annu. ACM Symp. Comput. Geom.*, 1997, pp. 469–472.

Victor Milenkovic, University of Miami: Let P be a simple polygon in the plane. Let o be a reference point in P, which coincides with the origin in the standard placement of P. Let ρ be a ray emanting from o attached to P. Let $P(\theta)$ denote the placement of P at which o lies at the origin and the angle between the ρ and the x-axis is θ. Given a function $t : [\alpha, \beta] \to \mathbb{R}^2$, where $0 \leq \alpha \leq \beta \leq 2\pi$, define

$$\mathcal{I}(\alpha, \beta) = \bigcap_{\alpha \leq \theta \leq \beta} P(\theta) + t(\theta).$$

Given α, β, find a function $t(\theta)$ so that the area of $\mathcal{I}(\alpha, \beta)$ is maximized. Can $t(\theta)$ be computed efficiently?

What if we impose certain restrictions on $t(\theta)$? For example, $t(\theta)$ can be restricted to be rotation with respect to a point in P.

This problem arises in packing a family of polygons inside another polygon; see [Mi].

[Mi] V. J. Milenkovic, Rotational polygon containment and minimum enclosure, *Proc. 14th Annual ACM Symp. Comput. Geom.*, 1998, 1–8.

Joseph Mitchell, SUNY Stony Brook: Let S be a set of points in convex position in $\mathbb{R}^3$. Can the convex hull of S always be triangulated (partitioned into tetrahedra) so that the dual graph of the triangulation has a Hamiltonian path? The same question applies to points in convex position in $\mathbb{R}^d$, $d \geq 4$.

It is obviously true for points in convex position in the plane.

Once such a triangulation is given for points in convex position, then points can be added in the interior of the convex hull and can be triangulated so that the new triangulation also admits a Hamiltonian path. See [AHMS].

[AHMS] E.M. Arkin, M. Held, J.S.B. Mitchell, and S.S. Skiena, Hamiltonian triangulations for fast rendering, *The Visual Computer* **12** (1996), 429–444.

Steve Vavasis, Cornell University: Let P be a polyhedron in $\mathbb{R}^3$, and let Σ be a simplicial complex. Can one check in near linear time whether Σ is a valid mesh for P, that is, whether the simplices in Σ have disjoint interiors and their union is P?

Jack Snoeyink, University of British Columbia: Let Γ be a set of n x-monotone arcs with a total of k intersection points. Every pair in Γ intersects at most once. The following two operations are allowed on Γ:

(i) Given an arc, return its left and right endpoints.

(ii) Given an endpoint p of an arc and another arc γ, determine whether p lies above or below γ, or whether the x-projections of p and γ are disjoint.

Using these primitives, how fast can one report all pairs of intersecting arcs in Γ?

No subquadratic algorithm is known. An $\Omega(n\sqrt{k})$ lower bound is not difficult to prove. Is there an algorithm that achieves this complexity? A related question is how fast can one find a maximal set of nonintersecting arcs.

This problem arises in developing an efficient, robust algorithm for computing the intersection points of segments.

Herbert Edelsbrunner, University of Illinois: Let S be the set of vertices of a strictly convex n-gon in $\mathbb{R}^2$. Let $u(S)$ be the number of pairs of vertices p, q in s at distance 1 from each other. Prove or disprove that $u(S) = O(n)$.

Füredi [Fu] proved that that $u(S) = O(n \log n)$, and Edelsbrunner and Hajnal [EH] proved that there are strictly convex n-gons in the plane for which $u(S) \geq 2n - 7$.

[EH] H. Edelsbrunner and P. Hajnal, A lower bound on the number of unit distances between the vertices of a convex polygon, *J. Combin. Theory Ser. A* **56** (1991), 312–316.

[Fu] Z. Füredi, The maximum number of unit distances in a convex n-gon, *J. Combin. Theory Ser. A* **55** (1990), 316–320.

Herbert Edelsbrunner, University of Illinois: An *unfolding* of the boundary of a convex polytope P is a polygon obtained by cutting a tree on the surface that spans the vertices. The resulting flattened polygon has no polytope vertices in its interior, and all interior "fold lines," i.e., polytope edges, leave no trace in the polygon. However, we retain the gluing pattern by recording which pairs of polygon edges were generated by a surface cut. See Figure 2.

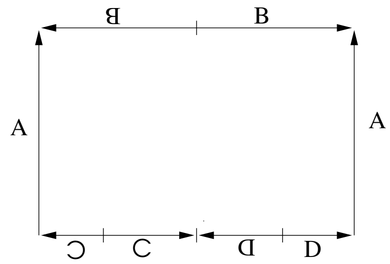

Figure 2: Unfolding of a tetrahedron.

Alexandrov [A] showed that if the total angle is at most 2π at every vertex and the gluing pattern results in a region homeomorphic to $\mathbb{S}^2$, then the polygon is an unfolding of a unique convex polytope.

Give an algorithm that constructs the convex polytope from the polygon.

[A] A. D. Aleksandrov, *Konvexe Polyeder*, Akademie Verlag, Berlin, 1958.

Joseph Mitchell, SUNY Stony Brook: Let S be a set of points in the plane so that the diameter of S is at most 2.

Is there a unit radius circle that passes through exactly two points of S (an *ordinary circle*)?

The classic result of Sylvester shows that for n points in the plane, not all on a common line, there exists an *ordinary line* (a line through exactly two points). See [BM, CS] for a summary of known results on this problem. This question about unit circles [A. Bezdek, personal comm.] is a natural generalization.

If diameter of S is at most $\sqrt{2}$, then there is always such a unit circle.

[BM] P. Borwein and W. Moser, A survey of Sylvester's problem and its generalizations, *Aequationes Mathematicae* **40** (1990), 111-135.

[CS] J. Csima and E. T. Sawyer, The $6n/13$ theorem revisited, *Graph Theory, combinatorics and Applications: Proc. 7th Quadrennial Intl. Conf. Theory and Appls. of Graphs*, vol. 1, (Y. Alavi and A. Schwenk, eds.), John Wiley and Sons, Inc. 1995, pp. 235–249.

Pankaj K. Agarwal, Duke University: Let S be a set of segments in the plane. Suppose S contains a subset A of k pairwise disjoint segments. Describe a polynomial-time algorithm to find a large subset of pairwise disjoint segments of S.

A few approximation algorithms are described in [AKS,DMMMZ] if S is a set of orthogonal rectangles or circles.

[AKS] P. K. Agarwal, M. van Kreveld, and S. Suri, Label placement by maximum independent set in rectangles, to appear in *Comput. Geom.: Theory and Appls.*

[DMMMZ] S. Doddi, M. Marathe, A. Mirzaian, B. Moret, and B. Zhu, Map labeling and its generalization, *Proc. 8th ACM-SIAM Sympos. Discrete Algorithms*, pp. 148–157, 1997.